Filomat 38:3 (2024), 1043–1054
https://doi.org/10.2298/FIL2403043N

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Anisotropic nonlinear elliptic equations with variable exponents and two weighted first order terms

Mokhtar Naceri[a]

[a] *ENS of Laghouat; Box 4033 Station post avenue of Martyrs, Laghouat, Algeria.*

Abstract. This paper is devoted to studying the existence of distributional solutions for a boundary value problems associated to a class of anisotropic nonlinear elliptic equations with variable exponents characterized by two strictly positive$-\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ first order terms (the weight functions belong to the anisotropic variable exponents Sobolev space with zero boundary), and this is in bounded open Lipschitz domain (with Lipschitz boundary) of $\mathbb{R}^N$ $(N \geq 2)$. The functional setting involves anisotropic varible exponents Lebesgue-Sobolev spaces.

1. Introduction

Let Ω be a bounded open set in $\mathbb{R}^N$ $(N \geq 2)$ with Lipschitz boundary $\partial\Omega$.
Our goal is to prove the existence of distributional solution to the anisotropic nonlinear elliptic problems of the form

$$\begin{aligned} -\sum_{i=1}^{N} D_i\Big(A(x)\sigma_i(x, D_i u)\Big) = -\sum_{i=1}^{N} D_i\Big(B(x)u|u|^{p_i(x)-2}\Big) + f(x), \quad & \text{in } \Omega. \\ u = 0, \quad & \text{on } \partial\Omega, \end{aligned} \tag{1}$$

Where,
•) f is in $L^1(\Omega)$, and $A(\cdot)$, $B(\cdot)$ two strictly positive $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, such that

$$A(x) \geq \alpha, \tag{2}$$

where, $\alpha > 0$.
•) $\sigma_i : \Omega \times \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, N$, are Carathéodory functions and satisfying;

2020 *Mathematics Subject Classification*. Primary 35J60, 35J66 ; Secondary 35J67.
Keywords. Nonlinear elliptic Equations; Anisotropic Sobolev spaces; Variable exponents; Distributional solution; Weighted first order terms; Existence.
Received: 22 March 2023; Revised: 02 August 2023; Accepted: 04 August 2023
Communicated by Maria Alessandra Ragusa
Email address: nasrimokhtar@gmail.com (Mokhtar Naceri)

a.e. $x \in \Omega$ and for all $\eta, \eta' \in \mathbb{R}$ $((\eta, \eta') \neq (0,0))$, the following :

$$\sigma_i(x,\eta)\eta \geq c_1|\eta|^{p_i(x)}, \tag{3}$$

$$|\sigma_i(x,\eta)| \leq c_2\left(\sum_{j=1}^{N}|\eta|^{p_j(x)}+|h|\right)^{1-\frac{1}{p_i(x)}}, \quad h \in L^1(\Omega) \tag{4}$$

$$\left(\sigma_i(x,\eta)-\sigma_i(x,\eta')\right)(\eta-\eta') \geq \begin{cases} c_3|\eta-\eta'|^{p_i(x)}, & \text{if } p_i(x) \geq 2 \\ c_4\frac{|\eta-\eta'|^2}{\left(|\eta|+|\eta'|\right)^{2-p_i(x)}}, & \text{if } 1 < p_i(x) < 2 \end{cases} \tag{5}$$

where $c_l,\ l = 1, \ldots, 4$ are positive constants.

This paper is concerned with the study of the existence results of distributional solutions concerning a class of anisotropic nonlinear elliptic equations with variable exponents and characterized by two first order terms with strictly positive$-\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ coefficients where the weight functions belong to the anisotropic variable exponents Sobolev space with zero boundary, and the datum $f \in L^1(\Omega)$. The existence results of this type of equations with various data in the isotropic scalar case, is proven in [1–7]. The existence of distributional solutions for anisotropic nonlinear weighted elliptic equations with variable exponents it was studied in [8, 9].

The proof requires a priori estimates for a sequence of suitable approximate solutions (u_n), which in turn is proving its existence by Leray-Schauder's fixed point Theorem. We then prove the boundedness of u_n in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ and the a.e. convergence of the partial derivatives $D_iu_n,\ i = 1, \ldots, N$ in $\overline{\Omega}$, which can be turned into strong L^1-convergence. Equipped with this convergence we pass to the limit in the strong L^1 sense in $A_n(x)\sigma_i(x, D_iu_n)$, and in $B_n(x)u_n|u_n|^{p_i(x)-2}$, and finally conclude that the approximate solutions u_n converge to the solution of (1).

The work has been organized in the following form:
Section 2 for some mathematical preliminaries, where here we reminded the isotropic and anisotropic variable exponent Lebesgue-Sobolev spaces, then some embedding theorems . The main theorem and its proof come in section 3.

2. Preliminaries

In this section we need to provide some basics definitions and properties about isotropic and anisotropic variable exponent Lebesgue-Sobolev spaces (see [14–19]).

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 2)$, we denote

$$C_+(\overline{\Omega}) = \{\text{continuous function} \quad p(\cdot) : \overline{\Omega} \longmapsto \mathbb{R} \quad \text{such that} \quad 1 < p^- \leq p^+ < \infty\},$$

where

$$p^+ = \max_{x\in\overline{\Omega}} p(x) \qquad \text{and} \qquad p^- = \min_{x\in\overline{\Omega}} p(x).$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ by

$$L^{p(\cdot)}(\Omega) := \{\text{measurable functions } u : \Omega \mapsto \mathbb{R};\ \rho_{p(\cdot)}(u) < \infty\}$$

where

$$\rho_{p(\cdot)}(u) := \int_\Omega |u(x)|^{p(x)}dx, \quad \text{the convex modular.}$$

The space $L^{p(\cdot)}(\Omega)$ equipped with the norm

$$\|f\|_{p(\cdot)} := \|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 \mid \rho_{p(\cdot)}(f/\lambda) \leq 1\right\}$$

becomes a Banach space. Moreover, is reflexive if $p^- > 1$.
$L^{p'(\cdot)}(\Omega)$ symbolize to the dual of $L^{p(\cdot)}(\Omega)$ where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.
$\forall u \in L^{p(\cdot)}(\Omega)$, $\forall v \in L^{p'(\cdot)}(\Omega)$ the Hölder type inequality:

$$\left|\int_\Omega uv\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)} \leq 2\|u\|_{p(\cdot)}\|v\|_{p'(\cdot)},$$

holds true.
We define also the Banach space $W_0^{1,p(\cdot)}(\Omega)$ by

$$W_0^{1,p(\cdot)}(\Omega) := \left\{f \in L^{p(\cdot)}(\Omega) : |Df| \in L^{p(\cdot)}(\Omega) \text{ and } f = 0 \text{ on } \partial\Omega\right\}$$

endowed with the norm $\|f\|_{W_0^{1,p(\cdot)}(\Omega)} = \|Df\|_{p(\cdot)}$. Moreover, is reflexive and separable if $p(\cdot) \in C_+(\overline{\Omega})$.
The following Lemma will be used later.

Lemma 2.1 ([15, 16]). *If* (u_n), $u \in L^{p(\cdot)}(\Omega)$, *then the following relations hold*

(i) $\|u\|_{p(\cdot)} < 1$ (*respectively* $= 1$, > 1) $\Longleftrightarrow \rho_{p(\cdot)}(u) < 1$ (*respectively* $= 1$, > 1),

(ii) $\min\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right) \leq \|u\|_{p(\cdot)} \leq \max\left(\rho_{p(\cdot)}(u)^{\frac{1}{p^+}}, \rho_{p(\cdot)}(u)^{\frac{1}{p^-}}\right)$,

(iii) $\min\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right) \leq \rho_{p(\cdot)}(u) \leq \max\left(\|u\|_{p(\cdot)}^{p^-}, \|u\|_{p(\cdot)}^{p^+}\right)$,

(iv) $\|u\|_{p(\cdot)} \leq \rho_{p(\cdot)}(u) + 1$,

(v) $\|u_n - u\|_{p(\cdot)} \to 0 \Longleftrightarrow \rho_{p(\cdot)}(u_n - u) \to 0$.

Now, we present the anisotropic Sobolev space with variable exponent which is used for the study of problems (1).
First of all, let $p_i(\cdot) : \overline{\Omega} \to [1, +\infty)$ for all $i = 1, \ldots, N$ be a continuous functions, we set $\forall x \in \overline{\Omega}$

$$\overrightarrow{p}(\cdot) = (p_1(x), \ldots, p_N(x)), \quad p_+(x) = \max_{1\leq i\leq N} p_i(x),\ p_-(x) = \min_{1\leq i\leq N} p_i(x),$$

$$\overline{p}(x) = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i(x)}}, \quad p_+(x) = \max_{1\leq i\leq N} p_i(x),$$

$$p_+^+ = \max_{x\in\overline{\Omega}} p_+(x), \quad p_-(x) = \min_{1\leq i\leq N} p_i(x),$$

$$p_-^- = \min_{x\in\overline{\Omega}} p_-(x), \quad \overline{p}^\star(x) = \begin{cases} \frac{N\overline{p}(x)}{N-\overline{p}(x)}, & \text{for } \overline{p}(x) < N, \\ +\infty, & \text{for } \overline{p}(x) \geq N. \end{cases}$$

The anisotropic variable exponent Sobolev space $W^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is defined as follow

$$W^{1,\overrightarrow{p}(\cdot)}(\Omega) = \left\{u \in L^{p_+(\cdot)}(\Omega), D_i u \in L^{p_i(\cdot)}(\Omega),\ i = 1, \ldots, N\right\},$$

which is Banach space with respect to the norm

$$\|u\|_{W^{1,\overrightarrow{p}(\cdot)}(\Omega)} = \|u\|_{p_+(\cdot)} + \sum_{i=1}^{N} \|D_i u\|_{p_i(\cdot)}.$$

We define the spaces $W_0^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ as follow

$$W_0^{1,\overrightarrow{p}(\cdot)}(\Omega) = \overline{C_0^\infty(\Omega)}^{W^{1,\overrightarrow{p}(\cdot)}(\Omega)},$$
$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) = W^{1,\overrightarrow{p}(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega).$$

Remark 2.2. *([13]) If Ω is a bounded open set with Lipschitz boundary $\partial\Omega$, then*

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) = \left\{u \in W^{1,\overrightarrow{p}(\cdot)}(\Omega),\ u_{|\partial\Omega} = 0\right\},$$

where, $u_{|\partial\Omega}$ denotes the trace on $\partial\Omega$ of u in $W^{1,1}(\Omega)$.

We have the following embedding results.

Lemma 2.3 ([13, 14]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(\cdot) \in (C_+(\overline{\Omega}))^N$. If $r \in C_+(\overline{\Omega})$ and $\forall x \in \overline{\Omega}$, $r(x) < \max(p_+(x),\ \overline{p}^{\star}(x))$. Then the embedding*

$$\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{r(\cdot)}(\Omega) \text{ is compact}. \tag{6}$$

Lemma 2.4 ([13, 14]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overrightarrow{p}(\cdot) \in (C_+(\overline{\Omega}))^N$. Suppose that*

$$\forall x \in \overline{\Omega},\ p_+(x) < \overline{p}^{\star}(x). \tag{7}$$

Then the following Poincaré-type inequality holds

$$\|u\|_{L^{p_+(\cdot)}(\Omega)} \leq C \sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)},\ \forall u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega), \tag{8}$$

where C is a positive constant independent of u.
Thus $\sum_{i=1}^{N} \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}$ is an equivalent norm on $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$.

In this paper, we use the scalar truncation function $\mathcal{T}_k : \mathbb{R} \longrightarrow \mathbb{R}$ at levels $\pm k$ defined as for all $s \in \mathbb{R}$ as;

$$\mathcal{T}_k(s) := \max(-k, \min(k, s)). \tag{9}$$

In addition to this, we use the standard scalar truncation function $T_t : \mathbb{R} \longrightarrow \mathbb{R}$ (at height $t > 0$) defined for all $s \in \mathbb{R}$ as;

$$T_t(s) = \frac{1}{2}(|s+t| - |s-t|) = \begin{cases} s, & \text{if } |s| \leq t, \\ \frac{s}{|s|}t, & \text{if } |s| > t. \end{cases} \tag{10}$$

We also need its derivative (see [10–12]);

$$DT_t(s) = \begin{cases} 1, & |s| < t, \\ 0, & |s| > t. \end{cases} \tag{11}$$

We need further the following function defined for $s \in \mathbb{R}$ by

$$G_t(s) = \begin{cases} 0, & if\ |s| \leq t, \\ s - t, & if\ s > t, \quad t > 0 \\ s + t, & if\ s < -t, \end{cases}$$

as a test function in the approximate weak formulation.

3. Statement of Results

Definition 3.1. *We say that u is a distributional solution for problem* (1) *if* $u \in W_0^{1,1}(\Omega)$, *and for all* $\varphi \in C_c^\infty(\Omega)$,

$$\sum_{i=1}^{N} \int_\Omega A(x)\sigma_i(x, D_i u) D_i\varphi \, dx = \int_\Omega B(x) u \sum_{i=1}^{N} |u|^{p_i(x)-2} D_i\varphi \, dx + \int_\Omega f(x)\varphi \, dx.$$

Our main result is the following.

Theorem 3.2. *Let* $p_i(\cdot) > 1$, $i = 1, \dots, N$, *are continuous functions on* Ω *such that* (7) *holds and* $\overline{p} < N$, *and let* f *is in* $L^1(\Omega)$, *and* $A(\cdot)$, $B(\cdot)$ *two strictly positive* $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ *such that* (2) *holds. Let* σ_i, $i = 1, \dots, N$ *be Carathéodory functions satisfying* (3), (4), (5). *Then the problem* (1) *has at least one solution* $u \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ *in the sense of distributions.*

3.1. Approximate solutions

We are going to prove the existence of solution to problem (1).
We define

$$A_n(x) = \frac{A(x)}{1 + \frac{A(x)}{n}}, \; B_n(x) = \frac{B(x)}{1 + \frac{B(x)}{n}}, \; f_n(x) = \frac{f(x)}{1 + \frac{|f(x)|}{n}} \quad n \in \mathbf{N}^*. \tag{12}$$

We must first notice that :
Since $\Theta(x) = \frac{x}{1+\frac{x}{n}}$ is increasing, we deduce by (2) that, for all $x \in \overline{\Omega}$

$$\frac{\alpha}{1+\alpha} \leq A_n(x) \leq n. \tag{13}$$

Lemma 3.3. *Let* $p_i(\cdot) > 1$, $i = 1, \dots, N$, *are continuous functions on* Ω *such that* (7) *holds and* $\overline{p} < N$, *and let* f *is in* $L^1(\Omega)$, *and* $A(\cdot)$, $B(\cdot)$ *two strictly positive* $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ *such that* (2) *holds. Let* σ_i, $i = 1, \dots, N$ *be Carathéodory functions satisfying* (3), (4), (5).
Then, there exists at least one weak solution $u_n \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ *to the approximated problems*

$$\begin{aligned} -\sum_{i=1}^{N} D_i\Big(A_n(x)\sigma_i(x, D_i u_n)\Big) &= -\sum_{i=1}^{N} D_i\Big(B_n(x) u_n |u_n|^{p_i(x)-2}\Big) + f_n(x), \quad \text{in } \Omega, \\ u_n &= 0, \quad \text{on } \partial\Omega, \end{aligned} \tag{14}$$

in the sense that

$$\sum_{i=1}^{N} \int_\Omega A_n(x)\sigma_i(x, D_i u_n) D_i\varphi \, dx = \int_\Omega B_n(x) u_n \sum_{i=1}^{N} |u_n|^{p_i(x)-2} D_i\varphi \, dx + \int_\Omega f_n(x)\varphi \, dx, \tag{15}$$

for every $\varphi \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$.

Proof. This proof derived from Leray-Schauder fixed point Theorem.
We consider for $X = L^{P_+(\cdot)}(\Omega)$ the operator

$$\begin{aligned} \psi : & X \times [0,1] \longrightarrow X \\ & (v_n, \delta) \longmapsto u_n = \psi(v_n, \delta), \end{aligned}$$

where u_n is the only weak solution of the problem

$$\begin{cases} -\sum\limits_{i=1}^{N} D_i\left(A_n(x)\sigma_i(x, D_i u_n)\right) = \delta\left(f_n - \sum\limits_{i=1}^{N} D_i\Big(B_n v_n |v_n|^{p_i(x)-2}\Big)\right) \text{in}\Omega, \\ u_n = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{16}$$

verify, for all $\varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, the weak formulation

$$\sum_{i=1}^{N}\int_{\Omega} A_n\sigma_i(x, D_iu_n)D_i\varphi\, dx = \delta\left(\int_{\Omega} f_n\varphi\, dx + \int_{\Omega} B_nv_n \sum_{i=1}^{N}|v_n|^{p_i(x)-2}D_i\varphi\, dx\right). \tag{17}$$

The existence of the weak solution u of the problem (16) in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ is directly produced by the main Theorem on pseudo-monotone operators. Let's prove the uniqueness of this solution.
Let u_1, $u_2 \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ be two weak solutions of (16). Considering the weak formulation of u_1 and u_2, by choosing $\varphi = u_1 - u_2$ as a test function, we have

$$\sum_{i=1}^{N}\int_{\Omega} A_n\sigma_i(x, D_iu_1)D_i(u_1-u_2)\, dx = \delta\left(\int_{\Omega} f_n(u_1-u_2)\, dx + \int_{\Omega} B_nv_n \sum_{i=1}^{N}|v_n|^{p_i(x)-2}D_i(u_1-u_2)\, dx\right), \tag{18}$$

and

$$\sum_{i=1}^{N}\int_{\Omega} A_n\sigma_i(x, D_iu_2)D_i(u_1-u_2)\, dx = \delta\left(\int_{\Omega} f_n(u_1-u_2)\, dx + \int_{\Omega} B_nv_n \sum_{i=1}^{N}|v_n|^{p_i(x)-2}D_i(u_1-u_2)\, dx\right). \tag{19}$$

By subtracting (19) from (18), we get that

$$\sum_{i=1}^{N}\int_{\Omega} A_n\left(\sigma_i(x, D_iu_1) - \sigma_i(x, D_iu_2)\right)D_i(u_1-u_2)\, dx = 0. \tag{20}$$

Putting for all $i = 1, \dots, N$,

$$I_i = \int_{\Omega}\left(\sigma_i(x, D_iu_1) - \sigma_i(x, D_iu_2)\right)(D_iu_1 - D_iu_2)\, dx.$$

Then, By using (13), the fact that $(\sigma_i(x, D_iu_1) - \sigma_i(x, D_iu_2))(D_iu_1 - D_iu_2) \geq 0$ (due (5)), and (20), we get for all $i = 1, \dots, N$,

$$I_i = 0. \tag{21}$$

Right now, we put for all $i = 1, \dots, N$,

$$\Omega_i^1 = \{x \in \Omega, p_i(x) \geq 2\}, \text{ and } \Omega_i^2 = \{x \in \Omega, 1 < p_i(x) < 2\}.$$

Then, By (5) we have, for all $i = 1, \dots, N$

$$I_i \geq c_3 \int_{\Omega_i^1} |D_i(u_1-u_2)|^{p_i(x)}. \tag{22}$$

On the other hand, by Hölder inequality, (5), Lemma 2.1, and since u_1, $u_2 \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, we have

$$\begin{aligned}
\int_{\Omega_i^2} |D_i(u_1-u_2)|^{p_i(x)}\, dx &\leq 2\left\|\frac{|D_i(u_1-u_2)|^{p_i(x)}}{(|D_iu_1| + |D_iu_2|)^{\frac{p_i(x)(2-p_i(x))}{2}}}\right\|_{L^{\frac{2}{p_i(\cdot)}}(\Omega_i^2)} \times \left\|(|D_iu_1| + |D_iu_2|)^{\frac{p_i(x)(2-p_i(x))}{2}}\right\|_{L^{\frac{2}{2-p_i(\cdot)}}(\Omega_i^2)} \\
&\leq 2\max\left\{\left(\int_{\Omega_i^2}\frac{|D_i(u_1-u_2)|^2}{\left(|D_iu_1| + |D_iu_2|\right)^{2-p_i(x)}}\, dx\right)^{\frac{p_i^-}{2}}, \left(\int_{\Omega_i^2}\frac{|D_i(u_1-u_2)|^2}{(|D_iu_1| + |D_iu_2|)^{2-p_i(x)}}\, dx\right)^{\frac{p_i^+}{2}}\right\} \\
&\quad \times \max\left\{\left(\int_{\Omega}\left(|D_iu_1| + |D_iu_2|\right)^{p_i(x)}\, dx\right)^{\frac{2-p_i^+}{2}}, \left(\int_{\Omega}\left(|D_iu_1| + |D_iu_2|\right)^{p_i(x)}\, dx\right)^{\frac{2-p_i^-}{2}}\right\} \\
&\leq 2c\max\left\{\left(I_i\right)^{\frac{p_i^-}{2}}, \left(I_i\right)^{\frac{p_i^+}{2}}\right\}\left(1 + \rho_{p_i}(|D_iu_1| + |D_iu_2|)\right)^{\frac{2-p_-^-}{2}} \\
&\leq c'\max\left\{\left(I_i\right)^{\frac{p_i^-}{2}}, \left(I_i\right)^{\frac{p_i^+}{2}}\right\}.
\end{aligned} \tag{23}$$

By combining (22), (23), and (21), we obtain

$$\int_\Omega |D_i(u_1-u_2)|^{p_i(x)}\,dx = 0, \quad i=1,\dots,N. \tag{24}$$

Then, from (24) and (*iii*) of Lemma 2.1 we conclude that

$$\|D_i(u_1-u_2)\|_{p_i(\cdot)} = 0, \quad i=1,\dots,N. \tag{25}$$

By using (7) and (25) we get

$$\|u_1-u_2\|_{\overrightarrow{p}(\cdot)} = 0, \quad i=1,\dots,N. \tag{26}$$

Then, (26) implies that $u_1 = u_2$ and so the solution of (16) is unique.

It is clear that $\psi(v_n, 0) = 0$ for all $v_n \in X$, because $u_n = 0 \in L^{p_+}(\Omega)$ is the only weak solution of the problem

$$\begin{cases} -\sum_{i=1}^N D_i\left(A_n(x)\sigma_i(x, D_i u_n)\right) = 0 & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

Now we'll give an estimate of the solution to the problem (16), for that taking $\varphi = u_n$ as test function in (17), and using (3), (13), (8), Hölder inequality, and Young's inequality we have

$$\frac{c_1\alpha}{1+\alpha}\sum_{i=1}^N\int_\Omega |D_i u_n|^{p_i(x)}\,dx \le c\left\|f_n\right\|_{p_i'(\cdot)}\left\|u_n\right\|_{p_i(\cdot)} + nc'\left(C(\varepsilon)\sum_{i=1}^N\int_\Omega |v_n|^{p_i(x)}\,dx + \varepsilon\sum_{i=1}^N\int_\Omega |D_i u_n|^{p_i(x)}\,dx\right) \tag{27}$$

Choosing $\varepsilon = \frac{c_1\alpha}{2nc'(1+\alpha)}$ in (27) and using the boundedness of f_n in $L^{p_i'(\cdot)}(\Omega)$, the fact that $\rho_{p_i(\cdot)}(v_n) \le |\Omega| + \rho_{p_+(\cdot)}(v_n)$, we obtain

$$\frac{c_1\alpha}{2(1+\alpha)}\sum_{i=1}^N\int_\Omega |D_i u_n|^{p_i(x)}\,dx \le c(n)\left\|u_n\right\|_{p_+(\cdot)} + c''nN\rho_{p_+(\cdot)}(v_n) + c'(n). \tag{28}$$

Then, we have

$$\frac{c_1\alpha}{2(1+\alpha)}\sum_{i=1}^N\int_\Omega |D_i u_n|^{p_i(x)}\,dx \le C(n)\left\|u_n\right\|_{\overrightarrow{p}(\cdot)} + C'(n). \tag{29}$$

On the other hand, we have $\sum_{i=1}^N\int_\Omega |D_i u_n|^{p_i(x)}\,dx \ge \sum_{i=1}^N \min\{\left\|D_i u_n\right\|^{p_i^-}_{p_i(x)}, \left\|D_i u_n\right\|^{p_i^+}_{p_i(x)}\}$.

We define for all $i = 1,\dots,N$; $\xi_i = \begin{cases} p_+^+, & \text{si } \left\|D_i u_n\right\|_{p_i(\cdot)} < 1 \\ p_-^-, & \text{si } \left\|D_i u_n\right\|_{p_i(\cdot)} \ge 1 \end{cases}$, we obtain

$$\begin{aligned} \sum_{i=1}^N \min\{\left\|D_i u_n\right\|^{p_i^-}_{p_i(.)}, \left\|D_i u_n\right\|^{p_i^+}_{p_i(.)}\} &\ge \sum_{i=1}^N \left\|D_i u_n\right\|^{\xi_i}_{p_i(.)} \\ &\ge \sum_{i=1}^N \left\|D_i u_n\right\|^{p_-^-}_{p_i(.)} - \sum_{\{i,\xi_i=p_+^+\}}\left(\left\|D_i u_n\right\|^{p_-^-}_{p_i(.)} - \left\|D_i u_n\right\|^{p_+^+}_{p_i(.)}\right) \\ &\ge \sum_{i=1}^N \left\|D_i u_n\right\|^{p_-^-}_{p_i(.)} - \sum_{\{i,\xi_i=p_+^+\}}\left\|D_i u_n\right\|^{p_-^-}_{p_i(.)} \ge \left(\frac{1}{N}\sum_{i=1}^N \left\|D_i u_n\right\|_{p_i(.)}\right)^{p_-^-} - N. \end{aligned}$$

Then, we get

$$\sum_{i=1}^{N}\int_{\Omega}|D_iu_n|^{p_i(x)}\,dx \geq \left(\frac{1}{N}\left\|u_n\right\|_{\overrightarrow{p}(\cdot)}\right)^{p_-^-} - N. \tag{30}$$

From (29) and (30), we conclude

$$\frac{c_1\alpha}{2(1+\alpha)N^{p_-^-}}\left\|u_n\right\|_{\overrightarrow{p}(\cdot)}^{p_-^-} \leq C(n)\left\|u_n\right\|_{\overrightarrow{p}(\cdot)} + C''(n). \tag{31}$$

Si $\left\|u_n\right\|_{\overrightarrow{p}(\cdot)} \leq 1$, we have

$$\left\|u_n\right\|_{\overrightarrow{p}(\cdot)} \leq 1. \tag{32}$$

Si $\left\|u_n\right\|_{\overrightarrow{p}(\cdot)} > 1$, from (31) we have

$$\left\|u_n\right\|_{\overrightarrow{p}}^{p_-^- -1} \leq c(n). \tag{33}$$

Then, there exists $c'(n) > 0$ such that

$$\left\|u_n\right\|_{\overrightarrow{p}(\cdot)} \leq c'(n). \tag{34}$$

Compactness of ψ: Let $\tilde{B}$ be a bounded of $L^{p_+(\cdot)}(\Omega)\times[0,1]$. Thus $\tilde{B}$ is contained in a product of the type $B\times[0,1]$ with B a bounded of $L^{p_+(\cdot)}(\Omega)$, which can be assumed to be a ball of center O and of radius $r>0$. For $u \in \psi(\tilde{B})$, we have, thanks to (34):

$$\left\|u\right\|_{\overrightarrow{p}(\cdot)} \leq \rho.$$

For $u = \psi(v,\delta)$ with $(v,\delta) \in B\times[0,1]$ ($\left\|v\right\|_{p_+(\cdot)} \leq r$). This proves that ψ applies $\tilde{B}$ in the closed ball of center O and radius ρ (ρ depend on n and r due (28)) in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \hookrightarrow L^{p_+(\cdot)}(\Omega)$ compactly due (7) and (6).
Let u_n be a sequence of elements of $\psi(\tilde{B})$, therefore $u_n = \psi(v_n,\delta_n)$ with $(v_n,\delta_n) \in \tilde{B}$. Since u_n remains in a bounded of $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, it is possible to extract a sub-sequence which converges strongly to an element u of $L^{p_+(\cdot)}(\Omega)$. This proves that $\overline{\psi(\tilde{B})}^{L^{p_+(\cdot)}(\Omega)}$ is compact. So ψ is compact.
Now, let's prove that; $\exists M>0$,

$$\forall(v_n,\delta) \in X\times[0,1] : v_n = \psi(v_n,\delta) \Rightarrow \left\|v_n\right\|_X \leq M.$$

For that, we give the estimate of elements of $L^{p_+(\cdot)}(\Omega)$ such that $v_n = \psi(v_n,\delta)$, then we have,

$$\sum_{i=1}^{N}\int_{\Omega}A_n\sigma_i(x,D_iv_n)D_i\varphi\,dx = \delta\left(\int_{\Omega}f_n\varphi\,dx + \int_{\Omega}B_nv_n\sum_{i=1}^{N}|v_n|^{p_i(x)-2}D_i\varphi\,dx\right), \text{ for all } \varphi \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{35}$$

We use in the weak formulation (35) the test function $\varphi = v_n$, and use (3), (13), Young's inequality, the boundedness of f_n in $L^{p_i'(\cdot)}(\Omega)$, the fact that $\rho_{p_i(\cdot)}(v_n) \leq |\Omega| + \rho_{p_+(\cdot)}(v_n)$, we obtain for all $\varepsilon > 0$, $\varepsilon' > 0$:

$$\begin{aligned}\frac{c_1\alpha}{1+\alpha}\sum_{i=1}^{N}\int_{\Omega}|D_iv_n|^{p_i(x)}\,dx &\leq \int_{\Omega}|f_n||v_n|\,dx + n\int_{\Omega}\sum_{i=1}^{N}|v_n|^{p_i(x)-1}|D_iv_n|\,dx\\ &\leq C(\varepsilon')\int_{\Omega}|f_n|^{p_i'(x)}\,dx + \varepsilon'\int_{\Omega}|v_n|^{p_i(x)}\,dx + n\left(C(\varepsilon)\sum_{i=1}^{N}\int_{\Omega}|v_n|^{p_i(x)}\,dx + \varepsilon\sum_{i=1}^{N}\int_{\Omega}|D_iv_n|^{p_i(x)}\,dx\right)\\ &\leq C'(\varepsilon') + \varepsilon'(|\Omega| + \rho_{p_+(\cdot)}(v_n)) + n\left(NC(\varepsilon)(|\Omega| + \rho_{p_+(\cdot)}(v_n)) + \varepsilon\sum_{i=1}^{N}\int_{\Omega}|D_iv_n|^{p_i(x)}\,dx\right).\end{aligned} \tag{36}$$

Choosing $\varepsilon = \varepsilon_0 = \frac{c_1\alpha}{2n(1+\alpha)}$ in (36), we get

$$\frac{c_1\alpha}{2(1+\alpha)}\sum_{i=1}^{N}\int_\Omega |D_i v_n|^{p_i(x)}\,dx \le \Big(\varepsilon' + nNC(\varepsilon_0)\Big)\rho_{p_+(\cdot)}(v_n) + C'(\varepsilon') + \Big(\varepsilon' + nNC(\varepsilon_0)\Big)|\Omega|. \tag{37}$$

Then, we obtain

$$\frac{c_1\alpha}{2(1+\alpha)}\sum_{i=1}^{N}\rho_{p_i(\cdot)}(D_i v_n) \le \Big(\varepsilon' + nNC(\varepsilon_0)\Big)\rho_{p_+(\cdot)}(v_n) + C'(\varepsilon') + \Big(\varepsilon' + nNC(\varepsilon_0)\Big)|\Omega|. \tag{38}$$

By using (*iv*) from Lemma 2.1, (8) (since (7)), the fact that $\rho_{p_+(\cdot)}(v_n) < +\infty$ (since $v_n \in L^{p_+(x)}(\Omega)$), and for any fixed choice of $\varepsilon' > 0$, we obtain

$$\left\|v_n\right\|_{p_+(\cdot)} \le c(n). \tag{39}$$

It then follows from the Leray-Schauder's Theorem that the operator $\psi_1 : X \longrightarrow X$ defined by $\psi_1(u) = \psi(u,1)$ has a fixed point, which shows the existence of a solution of (14) in the sense of (15). □

3.1.1. A priori estimates

Lemma 3.4. *Let $\{u_n\} \subset \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ be the sequence of approximating solutions of* (15). *Assume f, A, B and p_i, σ_i, $i = 1,\dots,N$ be restricted as in Theorem 3.2. Then*

$$u_n \text{ is bounded in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{40}$$

Proof. After chousing $\varphi = u_n$ in the weak formulation (15), and the same technique as in the proof of (39) we can get (37) (Of course with replacement v_n by u_n), and on the other hand with the use of (30), we obtain

$$\left\|u_n\right\|_{\overrightarrow{p}(\cdot)} \le C(n). \tag{41}$$

□

Lemma 3.5. *Let $\{u_n\} \subset \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ be the sequence of approximating solutions of* (15). *Assume f, A, B and p_i, σ_i, $i = 1,\dots,N$ be restricted as in Theorem 3.2. Then there exists a subsequence (still denoted (u_n)) and $u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$, such that*

$$u_n \rightharpoonup u \quad \text{weakly in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega) \text{ and a.e in } \Omega, \tag{42}$$

Hence, up to a further subsequence, for all $i = 1,\dots,N$

$$D_i u_n \longrightarrow D_i u \quad \text{a.e. in } \overline{\Omega}. \tag{43}$$

Proof. From (41) the sequence (u_n) is bounded in $\mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$.
So, there exists a function $u \in \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega)$ and a subsequence (still denoted by (u_n)) where for them, we find that (42) is fulfilled.

Taking $T_t(u_n)$ as test function in (15) and using (3), (13), and Young inequality it follows that for any $\varepsilon > 0$

$$\frac{c_1\alpha}{1+\alpha}\sum_{i=1}^{N}\int_\Omega |D_i(T_t(u_n))|^{p_i(x)}\,dx \le t\left\|f\right\|_{L^1(\Omega)} + C(\varepsilon)(1+t^{p_+^+})\sum_{i=1}^{N}\int_\Omega |B_n|^{p_i'(x)}\,dx + \varepsilon\sum_{i=1}^{N}\int_\Omega |D_i(T_t(u_n))|^{p_i(x)}\,dx. \tag{44}$$

Thanks to (44) we deduce that for all $t > 0$ and all $i = 1,\dots,N$

$$D_i(T_t(u_n)) \in L^{p_i(x)}(\Omega) \text{ and } T_t(u_n) \rightharpoonup T_t(u) \quad \text{weakly in } \mathring{W}^{1,\overrightarrow{p}(\cdot)}(\Omega). \tag{45}$$

Now, let us define for all $i = 1, \dots, N$ and $t > 0$ fixed

$$I^t_{i,n}(x) = \Big(\sigma_i(x, D_i(T_t(u_n))) - \sigma_i(x, D_i(T_t(u)))\Big)\Big(D_i(T_t(u_n)) - D_i(T_t(u))\Big)$$

For $0 < \theta < 1$, $0 < h < t$, and all $i = 1, \dots, N$, we can get

$$\begin{aligned}\int_\Omega \left(I^t_{i,n}(x)\right)^\theta dx &= \int_{\{|T_t(u_n)-T_t(u)|>h\}} \left(I^t_{i,n}(x)\right)^\theta dx + \int_{\{|T_t(u_n)-T_t(u)|\le h\}} \left(I^t_{i,n}(x)\right)^\theta dx\\ &\le \left(\int_\Omega I^t_{i,n}(x)\,dx\right)^\theta \Big|\{|T_t(u_n) - T_t(u)| > h\}\Big|^{1-\theta}\\ &+ \left(\int_{\{|T_t(u_n)-T_t(u)|\le h\}} I^t_{i,n}(x)\,dx\right)^\theta |\Omega|^{1-\theta}. \end{aligned} \tag{46}$$

Then, we can write

$$\int_\Omega \left(I^t_{i,n}(x)\right)^\theta dx \le J_1 + J_2. \tag{47}$$

Where,

$$J_1 = \left(\int_\Omega I^t_{i,n}(x)\,dx\right)^\theta \Big|\{|T_t(u_n) - T_t(u)| > h\}\Big|^{1-\theta}$$

$$J_2 = \left(\int_{\{|T_t(u_n)-T_t(u)|\le h\}} I^t_{i,n}(x)\,dx\right)^\theta |\Omega|^{1-\theta}.$$

For every fixed h, thanks to (45), and the convergence in measure of $T_t(u_n)$, we can get

$$\lim_{n\longrightarrow+\infty} J_1 = 0 \tag{48}$$

Now, choosing $T_h(u_n - T_t(u))$ (with $0 < h < t$) as test function in (15) and using (3), (13), we obtain

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega \sigma_i(x, D_i(T_t(u_n)))D_i\Big(T_h(T_t(u_n) - T_t(u))\Big)\,dx - \sum_{i=1}^N \int_{\{|u_n-T_t(u)|<h\}} \sigma_i(x, D_i(G_t(u_n)))D_i\Big(T_t(u)\Big)\,dx\\ &\quad\le ch + c' \sum_{i=1}^N \int_\Omega B_n u_n|u_n|^{p_i(x)-2} D_i\Big(T_h(u_n - T_t(u))\Big)\,dx. \end{aligned} \tag{49}$$

Then, using (49) and the fact that $I^t_{i,n}(x) \ge 0$ (since (5)), we deduce

$$\begin{aligned}0 \le \sum_{i=1}^N \int_{\{|T_t(u_n)-T_t(u)|\le h\}} I^t_{i,n}(x)\,dx &= \sum_{i=1}^N \int_\Omega \Big(\sigma_i(x, D_i(T_t(u_n))) - \sigma_i(x, D_i(T_t(u)))\Big)D_i\Big(T_h(T_t(u_n) - T_t(u))\Big)\,dx\\ &\le ch + c' \sum_{i=1}^N \int_\Omega B_n u_n|u_n|^{p_i(x)-2} D_i\Big(T_h(u_n - T_t(u))\Big)\,dx\\ &+ \sum_{i=1}^N \int_{\{|u_n|>t\}\cap\{|u_n-T_t(u)|<h\}} \sigma_i(x, D_i u_n)D_i\Big(T_t(u)\Big)\,dx\\ &- \sum_{i=1}^N \int_\Omega \sigma_i(x, D_i(T_t(u)))D_i\Big(T_h(T_t(u_n) - T_t(u))\Big)\,dx. \end{aligned} \tag{50}$$

By noticing that $\{|u_n - T_t(u)| < h\} \subset \{|u_n| \leq h+t\} \subset \{|u_n| \leq 2t\}$, and that $(\sigma_i(x, D_i(T_{2t}(u_n))))$ is bounded in $L^{p_i'(x)}(\Omega)$, and (45), we can pass to the limit with respect to n in (50) when $n \longrightarrow +\infty$, we get

$$\limsup_{n\longrightarrow+\infty} \sum_{i=1}^{N} \int_{\{|T_t(u_n)-T_t(u)|\leq h\}} I_{i,n}^t(x)\,dx \leq ch + c' \sum_{i=1}^{N} \int_\Omega Bu|u|^{p_i(x)-2} D_i\Big(T_h(G_t(u))\Big)\,dx + \sum_{i=1}^{N} \int_{\{t<|u|<t+h\}} \tau_t D_i\Big(T_t(u)\Big), \tag{51}$$

where $\tau_t \in L^{p_i'(x)}(\Omega)$ is the weak limit of $\sigma_i(x, D_i(T_{2t}(u_n)))$.
After letting $h \longrightarrow 0$ in (51), we obtain

$$\lim_{n\longrightarrow+\infty} J_2 = 0. \tag{52}$$

We combine (47), (48), (52), and using (5), we get

$$\lim_{n\longrightarrow+\infty} \int_\Omega \Big(I_{i,n}^t(x)\Big)^\theta\,dx = 0. \tag{53}$$

From (53) we deduce, like in [1], that: for every $t > 0$ and every $i = 1, \ldots, N$

$$D_i(T_t(u_n)) \longrightarrow D_i(T_t(u)), \text{ almost everywhere in } \overline{\Omega}. \tag{54}$$

And through the results obtained in [2] we can get (43). □

Lemma 3.6. *Let $A(\cdot)$, $B(\cdot)$ are in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$, and $A_n(\cdot)$, $B_nde(\cdot)$ be defined in (12). Then $A_n(\cdot)$, $B_nde(\cdot)$ are bounded in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ and*

$$A_n \longrightarrow A, \text{ Strongly in } \mathring{W}^{1,\vec{p}(\cdot)}(\Omega) \tag{55}$$

$$B_n \longrightarrow B, \text{ Strongly in } \mathring{W}^{1,\vec{p}(\cdot)}(\Omega). \tag{56}$$

Proof. Since, for all $x \in \overline{\Omega}$

$$D_iA_n(x) = \frac{D_iA(x)}{\Big(1 + \frac{A(x)}{n}\Big)^2},\ i = 1, \ldots, N,$$

we have that $|D_iA_n(x)| \leq |D_iA(x)|$, and therefore $A_n(\cdot) \in \mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$, due to $0 < A_n(x) \leq A(x)$, we obtain that, A_n is bounded in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$, and (55). In a similar way we get the boundedness of B_n in $\mathring{W}^{1,\vec{p}(\cdot)}(\Omega)$ and (56). □

3.2. Proof of the Theorem 3.2 :

From (4) and (40), we get for all $i = 1, \ldots, N$

$$\int_\Omega |\sigma_i(x, D_iu_n)|^{p_i'(\cdot)}\,dx \leq (1 + c_2^{p_+'^+}) \int_\Omega \left(\sum_{j=1}^{N} |D_iu_n|^{p_j(x)} + |h|\right) dx$$
$$\leq (1 + c_2^{p_+'^+}) \int_\Omega \left(N \sum_{j=1}^{N} |D_ju_n|^{p_j(x)} + |h|\right) dx \leq C\|u_n\|_{\vec{p}(\cdot)}^{p_+^+} + C' \leq C''.$$

And therefore

$$\sigma_i(x, D_iu_n) \text{ is bounded in } L^{p_i'(\cdot)}(\Omega), \quad i = 1, \ldots, N. \tag{57}$$

By (43) and (57) we have, for all $i = 1, \dots, N$

$$\sigma_i(x, D_i u_n) \rightharpoonup \sigma_i(x, D_i u) \quad \text{weakly in } L^{p_i'(\cdot)}(\Omega),\ p_i'(\cdot) = \frac{p_i(\cdot)}{p_i(\cdot) - 1}. \tag{58}$$

Now, we have

$$\int_\Omega \left(|u_n|^{p_i(\cdot)-1}\right)^{p_i'(\cdot)} dx = \int_\Omega |u_n|^{p_i(\cdot)} dx \leq C.$$

Then, we obtain

$$(u_n |u_n|^{p_i(\cdot)-2}) \text{ is bounded in } L^{p_i'(\cdot)}(\Omega), \quad i = 1, \dots, N. \tag{59}$$

By (42) and (59) we have, for all $i = 1, \dots, N$

$$u_n |u_n|^{p_i(\cdot)-2} \rightharpoonup u|u|^{p_i(\cdot)-2} \quad \text{weakly in } L^{p_i'(\cdot)}(\Omega). \tag{60}$$

Then, from (58), (60), (55), and (56) , we can pass to the limit in the weak formulation (15). This proves Theorem 3.2.

References

[1] L. Boccardo, *Some nonlinear Dirichlet problems in $L^1(\Omega)$ involving lower order terms in divergence form*, Progress in elliptic and parabolic partial differential equations (Capri, 1994) Pitman Res. Notes Math. Ser. **350** (1996) 43–57.

[2] A. Porretta, *Some remarks on the regularity of solutions for a class of elliptic equations with measure data*, Huston J. Math. **26** (2000) 183-213.

[3] S. Buccheri, *Gradient estimates for nonlinear elliptic equations with first order terms.* manuscripta math. **165**, 191–225 (2021)

[4] M.F. Betta, A. Mercaldo, F. Murat, M.M. *Porzio, Existence and uniqueness results for nonlinear elliptic problems with a lower order term and measure data.* C. R. Acad. Sci. Paris Sér. I **334** (2002), 757–762.

[5] M.F. Betta, A. Mercaldo, F. Murat, M.M. Porzio,*Uniqueness of renormalized solutions to nonlinear elliptic equations with lower order term and right-hand side in $L^1(\Omega)$.* ESAIM Control Optim. Calc. Var. **8** (2002), 239–272 (special issue dedicated to the memory of Jacques-Louis Lions).

[6] M.F. Betta, A. Mercaldo, F. Murat, M.M. Porzio, *Existence of renormalized solutions to nonlinear elliptic equations with a lower-order term and righthand side a measure.* J. Math. Pures Appl. **82** (2003), 90–124.

[7] M.F. Betta, A. Mercaldo, F. Murat, M.M. Porzio, *Uniqueness results for nonlinear elliptic equations with a lower order term.* Nonlinear Anal. **63** (2005) 153–170.

[8] M. Naceri, *Existence results for anisotropic nonlinear weighted elliptic equations with variable exponents and L^1 data*, Proceedings of the Romanian Academy Series A-Mathematics Physics Technical Sciences Information Science, **23 (4)**, pp. 337–346, 2022.

[9] M. Naceri, *Anisotropic nonlinear weighted elliptic equations with variable exponents*, Georgian Mathematical Journal, **vol. 30, no. 2**, 2023, pp. 277–285.

[10] M. Naceri, F. Mokhtari, *Anisotropic nonlinear elliptic systems with variable exponents and degenerate coercivity*, Appl. Anal. **100(11)**, 2347–2367 (2021).

[11] M. Naceri, M.B. Benboubker, *Distributional solutions of anisotropic nonlinear elliptic systems with variable exponents: existence and regularity*, Advances in Operator Theory. **7(2)**, 1–34 (2022).

[12] M. Naceri, *Anisotropic nonlinear elliptic systems with variable exponents, degenerate coercivity and $L^{q}(\cdot)$ data*, Ann. Acad. Rom. Sci. Ser. Math. Appl. **14(1-2/2022)**:107-140.

[13] X. Fan, *Anisotropic variable exponent Sobolev spaces and $\overrightarrow{p}(x)$-Laplacian equations.* Complex Var Elliptic Equ., **56** (2011), 623–642.

[14] X. Fan, D. Zhao, *On the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$.* J. Math. Anal. Appl., **263** (2001), 424–446.

[15] D. Cruz-Uribe, A. Fiorenza, M. Ruzhansky, J. Wirth, *Variable Lebesgue Spaces and Hyperbolic Systems.* Advanced Courses in Mathematics - CRM Barcelona. Birkhäuser, Basel, 2014.

[16] L. Diening, P. Harjulehto, P. Hästö, M. Ruzicka, *Lebesgue and Sobolev Spaces with Variable Exponents, Lecture Notes in Mathematics.* Springer.vol. 2017, New York (2011).

[17] A. Aberqi, J. Bennouna, O. Benslimane, M.A. Ragusa, *On p(z)-Laplacian System Involving Critical Nonlinearities*, Journal of Function Spaces, **vol.2022, art.n.6685771**, (2022).

[18] B. Aharrouch, A. Aberqi, J. Bennouna, *Existence and regularity of solutions to unilateral nonlinear elliptic equation in Marcinkiewicz space with variable exponent*, Filomat, **37 (17)**, 5785–5797, (2022).

[19] N.C. Eddine, P.D. Nguyen, M.A. Ragusa, *Existence and multiplicity of solutions for a class of critical anisotropic elliptic equations of Schrodinger-Kirchhoff-type*, Mathematical Methods in the Applied Sciences, doi: 10.1002/mma.9474, (2023).