Anisotropic nonlinear elliptic equations with variable exponents and two weighted first order terms

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Abstract. This paper is devoted to studying the existence of distributional solutions for a boundary value problems associated to a class of anisotropic nonlinear elliptic equations with variable exponents characterized by two strictly positive $W^{1,p_i}(\Omega)$ first order terms (the weight functions belong to the anisotropic variable exponents Sobolev space with zero boundary), and this is in bounded open Lipschitz domain (with Lipschitz boundary) of $\mathbb{R}^N$ ($N \geq 2$). The functional setting involves anisotropic variable exponents Lebesgue-Sobolev spaces.

1. Introduction

Let $\Omega$ be a bounded open set in $\mathbb{R}^N$ ($N \geq 2$) with Lipschitz boundary $\partial \Omega$. Our goal is to prove the existence of distributional solution to the anisotropic nonlinear elliptic problems of the form

$$
-\sum_{i=1}^{N}D_i\left(A(x)\sigma_i(x,D_iu)\right) = -\sum_{i=1}^{N}D_i\left(B(x)u|u|^{p_i(x)-2}\right) + f(x), \quad \text{in } \Omega.
$$

$$
u = 0, \quad \text{on } \partial \Omega,
$$

Where,

1) $f$ is in $L^1(\Omega)$, and $A()$, $B()$ two strictly positive $W^{1,\overline{p}(\cdot)}(\Omega)$, such that

$$A(x) \geq \alpha,$$

where, $\alpha > 0$.

*1) $\sigma_i: \Omega \times \mathbb{R} \rightarrow \mathbb{R}$, $i = 1, \ldots, N$, are Carathéodory functions and satisfying;*
a.e. \( x \in \Omega \) and for all \( \eta, \eta' \in \mathbb{R} ((\eta, \eta') \neq (0, 0)) \), the following:

\[
\sigma_i(x, \eta) \eta \geq c_i |\eta|^{p_i(x)},
\]
\[
|\sigma_i(x, \eta)| \leq c_2 \left( \sum_{i=1}^{N} |\eta|^{p_i(x)} + |h| \right)^{1-\frac{1}{p_i^m}}, \quad h \in L^1(\Omega)
\]
\[
(\sigma_i(x, \eta) - \sigma_i(x, \eta')) (\eta - \eta') \geq \begin{cases} 
  c_3 |\eta - \eta'|^{p_i(x)}, & \text{if } p_i(x) \geq 2 \\
  c_4 \frac{|\eta - \eta'|^2}{(|\eta| + |\eta'|)^{2m}}, & \text{if } 1 < p_i(x) < 2
\end{cases}
\]

where \( c_i, i = 1, \ldots, 4 \) are positive constants.

This paper is concerned with the study of the existence results of distributional solutions concerning a class of anisotropic nonlinear elliptic equations with variable exponents and characterized by two first order terms with strictly positive–\( W^{1,\overline{r}(x)}(\Omega) \) coefficients where the weight functions belong to the anisotropic variable exponents Sobolev space with zero boundary, and the datum \( f \in L^1(\Omega) \). The existence results of this type of equations with various data in the isotropic scalar case, is proven in [1–7]. The existence of distributional solutions for anisotropic nonlinear weighted elliptic equations with variable exponents it was studied in [8, 9].

The proof requires a priori estimates for a sequence of suitable approximate solutions \( (u_n) \), which in turn is proving its existence by Leray-Schauder’s fixed point Theorem. We then prove the boundedness of \( u_n \) in \( W^{1,\overline{r}(x)}(\Omega) \) and the a.e. convergence of the partial derivatives \( D_i u_n, i = 1, \ldots, N \) in \( \overline{\Omega} \), which can be turned into strong \( L^1 \)-convergence. Equipped with this convergence we pass to the limit in the strong \( L^1 \)-convergence. Equipped with this convergence we pass to the limit in the strong \( L^1 \)-convergence. Equipped with this convergence we pass to the limit in the strong \( L^1 \)-convergence.

The work has been organized in the following form:

Section 2 for some mathematical preliminaries, where here we reminded the isotropic and anisotropic variable exponent Lebesgue-Sobolev spaces, then some embedding theorems. The main theorem and its proof come in section 3.

2. Preliminaries

In this section we need to provide some basics definitions and properties about isotropic and anisotropic variable exponent Lebesgue-Sobolev spaces (see [14–19]).

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \) \((N \geq 2)\), we denote

\[
C_{*}(\overline{\Omega}) = \{ \text{continuous function } p(\cdot) : \overline{\Omega} \rightarrow \mathbb{R} \text{ such that } 1 < p^- \leq p^+ < \infty \},
\]

where

\[
p^+ = \max_{x \in \overline{\Omega}} p(x) \quad \text{and} \quad p^- = \min_{x \in \overline{\Omega}} p(x).
\]

We define the Lebesgue space with variable exponent \( L^{p(\cdot)}(\Omega) \) by

\[
L^{p(\cdot)}(\Omega) := \{ \text{measurable functions } u : \Omega \rightarrow \mathbb{R}; \rho_{p(\cdot)}(u) < \infty \}
\]

where

\[
\rho_{p(\cdot)}(u) := \int_{\Omega} |u(x)|^{p(x)} dx, \quad \text{the convex modular.}
\]

The space \( L^{p(\cdot)}(\Omega) \) equipped with the norm

\[
\|f\|_{p(\cdot)} := \|f\|_{L^{p(\cdot)}(\Omega)} = \inf \{ \lambda > 0 \mid \rho_{p(\cdot)}(f/\lambda) \leq 1 \}
\]
becomes a Banach space. Moreover, is reflexive if $p^- > 1$.

$L^p(\Omega)$ symbolize to the dual of $L^{p'}(\Omega)$ where $\frac{1}{p} + \frac{1}{p'} = 1$.

$\forall u \in L^p(\Omega), \forall v \in L^{p'}(\Omega)$ the Hölder type inequality:

$$\left| \int_{\Omega} uv \, dx \right| \leq \left( \frac{1}{p^-} + \frac{1}{p^-} \right) \|u\|_{p^-} \|v\|_{p^-} \leq 2\|u\|_{p^-} \|v\|_{p^-},$$

holds true.

We define also the Banach space $W^{1,p(-)}_0(\Omega)$ by

$$W^{1,p(-)}_0(\Omega) := \left\{ f \in L^{p(-)}(\Omega) : |Df| \in L^{p(-)}(\Omega) \text{ and } f = 0 \text{ on } \partial \Omega \right\}$$

endowed with the norm $\|f\|_{W^{1,p(-)}_0(\Omega)} = \|Df\|_{p(-)}$. Moreover, is reflexive and separable if $p(-) \in C_+(\overline{\Omega})$.

The following Lemma will be used later.

**Lemma 2.1** ([15, 16]). If $(u_n), u \in L^{p(-)}(\Omega)$, then the following relations hold

(i) $\|u\|_{p(-)} < 1$ (respectively $= 1, > 1$) $\iff$ $p(-)u < 1$ (respectively $= 1, > 1$),

(ii) $\min \left( p(-)u^+, p(+)u^+ \right) \leq \|u\|_{p(-)} \leq \max \left( p(-)u^+, p(+)u^+ \right),$

(iii) $\min \left( p(-)u^+, \|u\|_{p(-)}^p \right) \leq p(-)u \leq \max \left( p(-)u^+, \|u\|_{p(-)}^p \right),$

(iv) $\|u\|_{p(-)} \leq p(-)u + 1,$

(v) $\|u_n - u\|_{p(-)} \to 0 \iff p(-)(u_n - u) \to 0.$

Now, we present the anisotropic Sobolev space with variable exponent which is used for the study of problems (1).

First of all, let $p_i(-) : \overline{\Omega} \to [1, +\infty)$ for all $i = 1, \ldots, N$ be a continuous functions, we set $\forall x \in \overline{\Omega}$

$$\overline{p}(x) = (p_1(x), \ldots, p_N(x)), \quad p_+(x) = \max_{1 \leq i \leq N} p_i(x), \quad p_-(x) = \min_{1 \leq i \leq N} p_i(x),$$

$$\overline{p}(x) = \frac{N}{\sum_{i=1}^{N} \frac{1}{p_i(x)}}, \quad p_+(x) = \max_{x \in \overline{\Omega}} p_i(x),$$

$$p_+^x = \max_{x \in \overline{\Omega}} p_+(x), \quad p_-(x) = \min_{1 \leq i \leq N} p_i(x),$$

$$p_+^x = \min_{x \in \overline{\Omega}} p_-(x), \quad \overline{p}^x(x) = \left\{ \begin{array}{ll}
\frac{N p(x)}{N - \overline{p}(x)} & \text{for } \overline{p}^x(x) < N, \\
+\infty & \text{for } \overline{p}^x(x) \geq N.
\end{array} \right.$$}

The anisotropic variable exponent Sobolev space $W^{1,\overline{p}(-)}(\Omega)$ is defined as follow

$$W^{1,\overline{p}(-)}(\Omega) = \left\{ u \in L^{p(-)}(\Omega), D_iu \in L^{p(-)}(\Omega), i = 1, \ldots, N \right\},$$

which is Banach space with respect to the norm

$$\|u\|_{W^{1,\overline{p}(-)}(\Omega)} = \|u\|_{p(-)} + \sum_{i=1}^{N} \|D_iu\|_{p_i(-)},$$

We define the spaces $W^{1,\overline{p}(-)}_0(\Omega)$ and $W^{1,\overline{p}(-)}(\Omega)$ as follow

$$W^{1,\overline{p}(-)}_0(\Omega) = C^0_0(\Omega) \cap W^{1,\overline{p}(-)}(\Omega),$$

$$W^{1,\overline{p}(-)}(\Omega) = W^{1,\overline{p}(-)}(\Omega) \cap W^{1,1}_0(\Omega).$$
Remark 2.2. ([13]) If $\Omega$ is a bounded open set with Lipschitz boundary $\partial \Omega$, then
\[
\hat{W}^{1, \overline{\rho}}(\Omega) = \left\{ u \in W^{1, \overline{\rho}}(\Omega), \ u_{|\partial \Omega} = 0 \right\},
\]
where, $u_{|\partial \Omega}$ denotes the trace on $\partial \Omega$ of $u$ in $W^{1,1}(\Omega)$.

We have the following embedding results.

Lemma 2.3 ([13, 14]). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overline{\rho}(\cdot) \in (C_+ (\overline{\Omega}))^N$. If $r \in C_+ (\overline{\Omega})$ and $\forall x \in \overline{\Omega}$, $r(x) < \max(p_+(x), \overline{\rho}^*(x))$. Then the embedding
\[
\hat{W}^{1, \overline{\rho}}(\Omega) \hookrightarrow L^r(\Omega)
\]
is compact.

Lemma 2.4 ([13, 14]). Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $\overline{\rho}(\cdot) \in (C_+ (\overline{\Omega}))^N$. Suppose that $\forall x \in \overline{\Omega}$, $p_+(x) < \overline{\rho}^*(x)$. Then the following Poincaré-type inequality holds
\[
\|u\|_{L^{p_+}(\Omega)} \leq C \sum_{i=1}^N \|D_i u\|_{L^{p_+}(\Omega)}, \ \forall u \in \hat{W}^{1, \overline{\rho}}(\Omega),
\]
where $C$ is a positive constant independent of $u$.

In this paper, we use the scalar truncation function $T_k : \mathbb{R} \rightarrow \mathbb{R}$ at levels $\pm k$ defined as for all $s \in \mathbb{R}$ as;
\[
T_k(s) := \max(-k, \min(k, s)).
\]

In addition to this, we use the standard scalar truncation function $T_t : \mathbb{R} \rightarrow \mathbb{R}$ (at height $t > 0$) defined for all $s \in \mathbb{R}$ as;
\[
T_t(s) = \frac{1}{2}(|s + t| - |s - t|) = \begin{cases} s, & \text{if } |s| \leq t, \\ \frac{s}{n}, & \text{if } |s| > t. \end{cases}
\]

We also need its derivative (see [10–12]);
\[
DT_t(s) = \begin{cases} 1, & |s| < t, \\ 0, & |s| > t. \end{cases}
\]

We need further the following function defined for $s \in \mathbb{R}$ by
\[
G_t(s) = \begin{cases} 0, & \text{if } |s| \leq t, \\ s - t, & \text{if } s > t, \ t > 0 \\ s + t, & \text{if } s < -t, \end{cases}
\]
as a test function in the approximate weak formulation.
3. Statement of Results

Definition 3.1. We say that $u$ is a distributional solution for problem (1) if $u \in W^{1,1}_0(\Omega)$, and for all $\varphi \in C_c^\infty(\Omega)$,

$$\sum_{i=1}^N \int_\Omega A(x)\sigma_i(x, Du)D_i \varphi \, dx = \int_\Omega B(x)u \sum_{i=1}^N |u_i|^{p_i(x)-2}D_i \varphi \, dx + \int_\Omega f(x)\varphi \, dx.$$

Our main result is the following.

Theorem 3.2. Let $p_i(\cdot) > 1$, $i = 1, \ldots, N$, are continuous functions on $\Omega$ such that (7) holds and $\bar{p} < N$, and let $f$ is in $L^1(\Omega)$, and $A(\cdot)$, $B(\cdot)$ two strictly positive $W^{1,\bar{p}}(\Omega)$ such that (2) holds. Let $\sigma_i$, $i = 1, \ldots, N$ be Carathéodory functions satisfying (3), (4), (5). Then the problem (1) has at least one solution $u \in W^{1,\bar{p}}(\Omega)$ in the sense of distributions.

3.1. Approximate solutions

We are going to prove the existence of solution to problem (1).

We define

$$A_n(x) = \frac{A(x)}{1 + \frac{\bar{p} - 1}{n}} \quad B_n(x) = \frac{B(x)}{1 + \frac{\bar{p} - 1}{n}} \quad f_n(x) = \frac{f(x)}{1 + \frac{\bar{p} - 1}{n}} \quad n \in \mathbb{N}. \quad (12)$$

We must first notice that:

Since $\Theta(x) = \frac{x}{1 + x}$ is increasing, we deduce by (2) that, for all $x \in \overline{\Omega}$

$$\frac{\alpha}{1 + \alpha} \leq A_n(x) \leq n. \quad (13)$$

Lemma 3.3. Let $p_i(\cdot) > 1$, $i = 1, \ldots, N$, are continuous functions on $\Omega$ such that (7) holds and $\bar{p} < N$, and let $f$ is in $L^1(\Omega)$, and $A(\cdot)$, $B(\cdot)$ two strictly positive $W^{1,\bar{p}}(\Omega)$ such that (2) holds. Let $\sigma_i$, $i = 1, \ldots, N$ be Carathéodory functions satisfying (3), (4), (5). Then, there exists at least one weak solution $u_n \in W^{1,\bar{p}}(\Omega)$ to the approximated problems

$$-\sum_{i=1}^N D_i\left(A_n(x)\sigma_i(x, Du_n)\right) = -\sum_{i=1}^N D_i\left(B_n(x)u_n|u_n|^{p_i(x)-2}\right) + f_n(x), \quad \text{in } \Omega,$n = 0, \quad \text{on } \partial \Omega, \quad (14)$$

in the sense that

$$\sum_{i=1}^N \int_\Omega A_n(x)\sigma_i(x, Du_n)D_i \varphi \, dx = \int_\Omega B_n(x)u_n \sum_{i=1}^N |u_n|^{p_i(x)-2}D_i \varphi \, dx + \int_\Omega f_n(x)\varphi \, dx, \quad (15)$$

for every $\varphi \in W^{1,\bar{p}}(\Omega) \cap L^\infty(\Omega)$.

Proof. This proof derived from Leray-Schauder fixed point Theorem.

We consider for $X = L^{p_i(\cdot)}(\Omega)$ the operator

$$\psi : X \times [0, 1] \to X \quad (v_n, \delta) \mapsto u_n = \psi(v_n, \delta),$$

where $u_n$ is the only weak solution of the problem

$$\begin{cases}
-\sum_{i=1}^N D_i\left(A_n(x)\sigma_i(x, Du_n)\right) = \delta \left(f_n - \sum_{i=1}^N D_i\left(B_n v_n|v_n|^{p_i(x)-2}\right)\right) \text{in } \Omega, \\
u_n = 0 \quad \text{on } \partial \Omega,
\end{cases} \quad (16)$$

Proof of (16). This proof derived from Leray-Schauder fixed point Theorem.
verify, for all $\phi \in \dot{W}^{1,\pi}(\Omega)$, the weak formulation
\begin{equation}
\sum_{i=1}^{N} \int_{\Omega} A_{n} \sigma_{1}(x, D_{n}u) D_{n} \phi \, dx = \delta \left( \int_{\Omega} f_{n} \phi \, dx + \int_{\Omega} B_{n} v_{n} \sum_{i=1}^{N} |v_{i}|^{p_{1}} - 2 D_{i} \phi \, dx \right).
\end{equation}

The existence of the weak solution $u$ of the problem (16) in $\dot{W}^{1,\pi}(\Omega)$ is directly produced by the main Theorem on pseudo-monotone operators. Let’s prove the uniqueness of this solution.

Let $u_{1}, u_{2} \in \dot{W}^{1,\pi}(\Omega)$ be two weak solutions of (16). Considering the weak formulation of $u_{1}$ and $u_{2}$, by choosing $\phi = u_{1} - u_{2}$ as a test function, we have
\begin{equation}
\sum_{i=1}^{N} \int_{\Omega} A_{n} \sigma_{1}(x, D_{n}u_{1}) D_{n}(u_{1} - u_{2}) \, dx = \delta \left( \int_{\Omega} f_{n}(u_{1} - u_{2}) \, dx + \int_{\Omega} B_{n} v_{n} \sum_{i=1}^{N} |v_{i}|^{p_{1}} - 2 D_{i}(u_{1} - u_{2}) \, dx \right),
\end{equation}
and
\begin{equation}
\sum_{i=1}^{N} \int_{\Omega} A_{n} \sigma_{1}(x, D_{n}u_{2}) D_{n}(u_{1} - u_{2}) \, dx = \delta \left( \int_{\Omega} f_{n}(u_{1} - u_{2}) \, dx + \int_{\Omega} B_{n} v_{n} \sum_{i=1}^{N} |v_{i}|^{p_{1}} - 2 D_{i}(u_{1} - u_{2}) \, dx \right).
\end{equation}

By subtracting (19) from (18), we get that
\begin{equation}
\sum_{i=1}^{N} \int_{\Omega} A_{n} \left( \sigma_{1}(x, D_{n}u_{1}) - \sigma_{1}(x, D_{n}u_{2}) \right) D_{n}(u_{1} - u_{2}) \, dx = 0.
\end{equation}

Putting for all $i = 1, \ldots, N$,
\begin{equation}
I_{i} = \int_{\Omega} \left( \sigma_{1}(x, D_{n}u_{1}) - \sigma_{1}(x, D_{n}u_{2}) \right) (D_{n}u_{1} - D_{n}u_{2}) \, dx.
\end{equation}

Then, By using (13), the fact that $(\sigma_{1}(x, D_{n}u_{1}) - \sigma_{1}(x, D_{n}u_{2})) (D_{n}u_{1} - D_{n}u_{2}) \geq 0$ (due (5)), and (20), we get for all $i = 1, \ldots, N$,
\begin{equation}
I_{i} = 0.
\end{equation}

Right now, we put for all $i = 1, \ldots, N$,
\begin{equation}
\Omega_{i}^{1} = \{ x \in \Omega, p_{i}(x) \geq 2 \}, \quad \text{and} \quad \Omega_{i}^{2} = \{ x \in \Omega, 1 < p_{i}(x) < 2 \}.
\end{equation}

Then, By (5) we have, for all $i = 1, \ldots, N$
\begin{equation}
I_{i} \geq c_{3} \int_{\Omega_{i}^{1}} |D_{i}(u_{1} - u_{2})|^{p_{1}} \, dx.
\end{equation}

On the other hand, by Hölder inequality, (5), Lemma 2.1, and since $u_{1}, u_{2} \in \dot{W}^{1,\pi}(\Omega)$, we have
\begin{align}
\int_{\Omega_{i}^{2}} |D_{i}(u_{1} - u_{2})|^{p_{1}} \, dx &\leq 2 \left\| \frac{|D_{i}(u_{1} - u_{2})|^{p_{1}}}{|D_{i}(u_{1}) + |D_{i}(u_{2})|^{\frac{s}{p_{1} - p_{2}}}} \right\|_{L^{\frac{2}{p_{1}}}(\Omega_{i}^{2})} \times \left( \int_{\Omega_{i}^{2}} \frac{|D_{i}(u_{1} - u_{2})|^{2}}{|D_{i}(u_{1}) + |D_{i}(u_{2})|^{\frac{2}{p_{1} - p_{2}}}} \, dx \right)^{\frac{p_{1}}{2}} \\
&\leq 2 \max \left\{ \left( \int_{\Omega_{i}^{2}} \frac{|D_{i}(u_{1} - u_{2})|^{2}}{|D_{i}(u_{1}) + |D_{i}(u_{2})|^{\frac{2}{p_{1} - p_{2}}}} \, dx \right)^{\frac{2}{p_{1}}} \right\} \\
&\quad \times \max \left\{ \left( \int_{\Omega_{i}^{2}} \frac{|D_{i}(u_{1})|^{p_{1}}}{|D_{i}(u_{1}) + |D_{i}(u_{2})|^{\frac{2}{p_{1} - p_{2}}}} \, dx \right)^{\frac{2 - p_{1}}{2}} \right\} \\
&\quad \times \max \left\{ \left( \int_{\Omega_{i}^{2}} \frac{|D_{i}(u_{2})|^{p_{1}}}{|D_{i}(u_{1}) + |D_{i}(u_{2})|^{\frac{2}{p_{1} - p_{2}}}} \, dx \right)^{\frac{2 - p_{1}}{2}} \right\} \\
&\leq 2 c \max \left\{ \left( \frac{1}{\tilde{l}_{i}^{2}} \right)^{\frac{2}{p_{1}}} \left( \tilde{l}_{i} \right)^{\frac{p_{1}}{2}} \right\} \left( 1 + \rho_{p}(|D_{i}(u_{1})| + |D_{i}(u_{2})|) \right)^{\frac{2 - p_{1}}{2}} \\
&\leq c' \max \left\{ \left( \frac{1}{\tilde{l}_{i}^{2}} \right)^{\frac{2}{p_{1}}} \left( \tilde{l}_{i} \right)^{\frac{p_{1}}{2}} \right\}.
\end{align}
By combining (22), (23), and (21), we obtain
\[\int_{\Omega} |D_i(u_1 - u_2)|^{p_i} \, dx = 0, \quad i = 1, \ldots, N.\] (24)

Then, from (24) and (iii) of Lemma 2.1 we conclude that
\[\|D_i(u_1 - u_2)\|_{p_i} = 0, \quad i = 1, \ldots, N.\] (25)

By using (7) and (25) we get
\[\|u_1 - u_2\|_{p_i} = 0, \quad i = 1, \ldots, N.\] (26)

Then, (26) implies that \(u_1 = u_2\) and so the solution of (16) is unique.

It is clear that \(\psi(v_n, 0) = 0\) for all \(v_n \in X\), because \(u_n = 0 \in L^p(\Omega)\) is the only weak solution of the problem
\[
\left\{ \begin{array}{l}
- \sum_{i=1}^{N} D_i(A_n(x)\sigma_i(x, D_iu_n)) = 0 \quad \text{in} \ \Omega, \\
u_n = 0 \quad \text{on} \ \partial \Omega.
\end{array} \right.
\]

Now we’ll give an estimate of the solution to the problem (16), for that taking \(\varphi = u_n\) as test function in (17), and using (3), (13), (8), Hölder inequality, and Young’s inequality we have
\[
\frac{c_1\alpha}{1 + \alpha} \sum_{i=1}^{N} \int_{\Omega} |D_iu_n|^{p_i} \, dx \leq c \|f_n\|_{p\xi} \|u_n\|_{p\xi} + nC' \left( C(\varepsilon) \sum_{i=1}^{N} \int_{\Omega} |v_n|^{p_i} \, dx + \varepsilon \sum_{i=1}^{N} \int_{\Omega} |D_iu_n|^{p_i} \, dx \right) \] (27)

Choosing \(\varepsilon = \frac{c_1\alpha}{2(1+\alpha)}\) in (27) and using the boundedness of \(f_n\) in \(L^{p\xi}(\Omega)\), the fact that \(\rho_{p\xi}(v_n) \leq |\Omega| + \rho_{p\xi}(v_n)\), we obtain
\[
\frac{c_1\alpha}{2(1 + \alpha)} \sum_{i=1}^{N} \int_{\Omega} |D_iu_n|^{p_i} \, dx \leq c(n)\|u_n\|_{p\xi} + c'(n)\rho_{p\xi}(v_n) + c'(n). \] (28)

Then, we have
\[
\frac{c_1\alpha}{2(1 + \alpha)} \sum_{i=1}^{N} \int_{\Omega} |D_iu_n|^{p_i} \, dx \leq C(n)\|u_n\|_{p\xi} + C'(n). \] (29)

On the other hand, we have
\[
\sum_{i=1}^{N} \int_{\Omega} |D_iu_n|^{p_i} \, dx \geq \sum_{i=1}^{N} \min\{\|D_iu_n\|^{p_i}_{p\xi}, \|D_iu_n\|^{p_i}_{p\xi'}\}.
\]

We define for all \(i = 1, \ldots, N; \quad \xi_i = \begin{cases} p_i^*, & \text{si} \|D_iu_n\|_{p\xi} < 1 \\ p_i^c, & \text{si} \|D_iu_n\|_{p\xi} \geq 1\end{cases}\), we obtain
\[
\sum_{i=1}^{N} \min\{\|D_iu_n\|^{p_i}_{p\xi}, \|D_iu_n\|^{p_i}_{p\xi'}\} \geq \sum_{i=1}^{N} \|D_iu_n\|^{\xi_i}_{p\xi} \\
\geq \sum_{i=1}^{N} \|D_iu_n\|^{p_i}_{p\xi} - \sum_{(i, \xi) \neq (i, p_i^*)} \|D_iu_n\|^{p_i}_{p\xi} - \|D_iu_n\|^{p_i}_{p\xi'} \]
\[
\geq \sum_{i=1}^{N} \|D_iu_n\|^{p_i}_{p\xi} - \sum_{i=1}^{N} \|D_iu_n\|^{p_i}_{p\xi'} \geq \left( \frac{1}{N} \sum_{i=1}^{N} \|D_iu_n\|_{p\xi} \right)^{p_i} - N.
\]
Then, we get
\[ \sum_{i=1}^{N} \int_{\Omega} |D_i u_n|^p \varphi \, dx \geq \left( \frac{1}{N} \left\| u_n \right\|_{1}^{p} \right)^{p} - N. \] (30)

From (29) and (30), we conclude
\[ \frac{c_1 \alpha}{2(1 + \alpha)N^{p\alpha}} \left\| u_n \right\|_{1}^{p\alpha} \leq C(n) \left\| u_n \right\|_{1}^{p} + C''(n). \] (31)

Si \left\| u_n \right\|_{1}^{p} \leq 1, we have
\[ \left\| u_n \right\|_{1}^{p} \leq 1. \] (32)

Si \left\| u_n \right\|_{1}^{p} > 1, from (31) we have
\[ \left\| u_n \right\|_{1}^{p-1} \leq c(n). \] (33)

Then, there exists \( c'(n) > 0 \) such that
\[ \left\| u_n \right\|_{1}^{p} \leq c'(n). \] (34)

Compactness of \( \psi \): Let \( \tilde{B} \) be a bounded of \( L^{p,1}(\Omega) \times [0, 1] \). Thus \( \tilde{B} \) is contained in a product of the type \( B \times [0, 1] \) with \( B \) a bounded of \( L^{p,1}(\Omega) \), which can be assumed to be a ball of center \( O \) and of radius \( r > 0 \). For \( u \in \psi(\tilde{B}) \), we have, thanks to (34):
\[ \left\| u \right\|_{1}^{p} \leq \rho. \]

For \( u = \psi(v, \delta) \) with \( (v, \delta) \in B \times [0, 1] \) (\( \left\| v \right\|_{p,1} \leq r \) ). This proves that \( \psi \) applies \( \tilde{B} \) in the closed ball of center \( O \) and radius \( \rho \) (\( \rho \) depend on \( n \) and \( r \) due (28)) in \( \tilde{W}^{1,p,1}(\Omega) \) and \( \tilde{W}^{1,p,1}(\Omega) \to L^{p,1}(\Omega) \) compactly due (7) and (6).

Let \( u_n \) be a sequence of elements of \( \psi'(\tilde{B}) \), therefore \( u_n = \psi(v_n, \delta_n) \) with \( (v_n, \delta_n) \in \tilde{B} \). Since \( u_n \) remains in a bounded of \( \tilde{W}^{1,p,1}(\Omega) \), it is possible to extract a sub-sequence which converges strongly to an element \( u \) of \( L^{p,1}(\Omega) \). This proves that \( \psi'(\tilde{B}) \) is compact. So \( \psi \) is compact.

Now, let’s prove that; \( \exists M > 0 \),
\[ \forall (v_n, \delta) \in X \times [0, 1] : v_n = \psi(v_n, \delta) \Rightarrow \| v_n \|_{X} \leq M. \]

For that, we give the estimate of elements of \( L^{p,1}(\Omega) \) such that \( v_n = \psi(v_n, \delta) \), then we have,
\[ \sum_{i=1}^{N} \int_{\Omega} A_{n} \sigma(x, D_{i} v_n) D_{i} \varphi \, dx = \delta \left( \int_{\Omega} f_{n} \varphi \, dx + \int_{\Omega} B_{n} \delta_{n} + \sum_{i=1}^{N} \left| v_{n,i} \right|^{p-1} D_{i} \varphi \, dx \right), \] for all \( \varphi \in \tilde{W}^{1,p,1}(\Omega). \) (35)

We use in the weak formulation (35) the test function \( \varphi = v_n \), and use (3), (13), Young’s inequality, the boundedness of \( f_{n} \) in \( L^{p,1}(\Omega) \), the fact that \( \rho_{p,1}(v_n) \leq |\Omega| + \rho_{p,1}(v_n) \), we obtain for all \( \varepsilon > 0, \varepsilon' > 0 \):
\[ \frac{c_1 \alpha}{1 + \alpha} \sum_{i=1}^{N} \int_{\Omega} |D_{i} v_{n}^{p,1}(x) \varphi \, dx \leq \int_{\Omega} f_{n} \varphi \, dx + n \int_{\Omega} B_{n} \varphi \, dx + n \sum_{i=1}^{N} \left| v_{n,i} \right|^{p-1} \int_{\Omega} |D_{i} \varphi | \, dx \]
\[ \leq C(\varepsilon') \int_{\Omega} f_{n} \varphi \, dx + \varepsilon \int_{\Omega} \left| v_{n,i} \right|^{p,1} \varphi \, dx + n \left( C(\varepsilon) \sum_{i=1}^{N} \int_{\Omega} |v_{n,i}^{p,1} \varphi \, dx + \varepsilon \sum_{i=1}^{N} \int_{\Omega} |D_{i} v_{n}^{p,1} \varphi \, dx \right) \]
\[ \leq C'(\varepsilon') + \varepsilon'(|\Omega| + \rho_{p,1}(v_n)) + n \left( NC(\varepsilon)(|\Omega| + \rho_{p,1}(v_n)) + \varepsilon \sum_{i=1}^{N} \int_{\Omega} |D_{i} v_{n}^{p,1} \varphi \, dx \right). \] (36)
Choosing $\varepsilon = \varepsilon_0 = \frac{c_1\alpha}{2(1+\alpha)}$ in (36), we get

$$
\frac{c_1\alpha}{2(1+\alpha)} \sum_{i=1}^{N} \int_{\Omega} |D\varphi\nu_{\varepsilon, \phi}^n| dx \leq \left( \varepsilon' + nNC(\varepsilon_0) \right) p_{\phi, \nu}(\nu_n) + C'(\varepsilon') + \left( \varepsilon' + nNC(\varepsilon_0) \right) |\Omega|.
$$

(37)

Then, we obtain

$$
\frac{c_1\alpha}{2(1+\alpha)} \sum_{i=1}^{N} \rho_{\phi, \nu}(D_{\nu}v_n) \leq \left( \varepsilon' + nNC(\varepsilon_0) \right) p_{\phi, \nu}(\nu_n) + C'(\varepsilon') + \left( \varepsilon' + nNC(\varepsilon_0) \right) |\Omega|.
$$

(38)

By using (iv) from Lemma 2.1, (8) (since (7)), the fact that $p_{\phi, \nu}(\nu_n) < +\infty$ (since $\nu_n \in L^{p_{\phi, \nu}}(\Omega)$), and for any fixed choice of $\varepsilon' > 0$, we obtain

$$
\|\nu_n\|_{p_{\phi, \nu}} \leq c(n).
$$

(39)

It then follows from the Leray-Schauder’s Theorem that the operator $\psi_1 : X \to X$ defined by $\psi_1(u) = \psi(u, 1)$ has a fixed point, which shows the existence of a solution of (14) in the sense of (15). □

3.1.1. A priori estimates

Lemma 3.4. Let $\{u_n\} \subset \hat{W}^{1,\overline{p}}(\Omega)$ be the sequence of approximating solutions of (15). Assume $f$, $A$, $B$ and $p_{\nu}, \sigma_i, i = 1, \ldots, N$ be restricted as in Theorem 3.2. Then

$$
u_n \text{ is bounded in } \hat{W}^{1,\overline{p}}(\Omega).
$$

(40)

Proof. After choosing $\varphi = u_n$ in the weak formulation (15), and the same technique as in the proof of (39) we can get (37) (Of course with replacement $\nu_n$ by $u_n$), and on the other hand with the use of (30), we obtain

$$
\|u_n\|_{\overline{p}_{\phi, \nu}} \leq C(n).
$$

(41)

□

Lemma 3.5. Let $\{u_n\} \subset \hat{W}^{1,\overline{p}}(\Omega)$ be the sequence of approximating solutions of (15). Assume $f$, $A$, $B$ and $p_{\nu}, \sigma_i, i = 1, \ldots, N$ be restricted as in Theorem 3.2. Then there exists a subsequence (still denoted by $(u_n)$) and $u \in \hat{W}^{1,\overline{p}}(\Omega)$, such that

$$
u_n \to u \text{ weakly in } \hat{W}^{1,\overline{p}}(\Omega) \text{ and a.e in } \Omega,
$$

(42)

Hence, up to a further subsequence, for all $i = 1, \ldots, N$,

$$
D_i u_n \to D_i u \text{ a.e in } \Omega.
$$

(43)

Proof. From (41) the sequence $(u_n)$ is bounded in $\hat{W}^{1,\overline{p}}(\Omega)$.

So, there exists a function $u \in \hat{W}^{1,\overline{p}}(\Omega)$ and a subsequence (still denoted by $(u_n)$) where for them, we find that (42) is fulfilled.

Taking $T_i(u_n)$ as test function in (15) and using (3), (13), and Young inequality it follows that for any $\varepsilon > 0$

$$
\frac{c_1\alpha}{1+\alpha} \sum_{i=1}^{N} \int_{\Omega} \left| D_i(T_i(u_n)) \right| dx \leq \left\| f \right\|_{L^1(\Omega)} + C(c)(1 + \varepsilon^2) \sum_{i=1}^{N} \int_{\Omega} \left| B_{\phi, \nu}(\phi) \right| dx + \varepsilon \sum_{i=1}^{N} \int_{\Omega} \left| D_i(T_i(u_n)) \right| dx.
$$

(44)

Thanks to (44) we deduce that for all $t > 0$ and all $i = 1, \ldots, N$,

$$
D_i(T_i(u_n)) \in L^{p_{\phi, \nu}}(\Omega) \text{ and } T_i(u_n) \to T_i(u) \text{ weakly in } \hat{W}^{1,\overline{p}}(\Omega).
$$

(45)
Now, let us define for all \( i = 1, \ldots, N \) and \( t > 0 \) fixed
\[
I_{i,t}^0(x) = \sigma_i(x, D_i(T_i(u_n))) - \sigma_i(x, D_i(T_i(u))) \left( D_i(T_i(u_n)) - D_i(T_i(u)) \right)
\]

For \( 0 < \theta < 1 \) and \( 0 < h < t \), and all \( i = 1, \ldots, N \), we can get
\[
\int_{\Omega} \left( I_{i,t}^0(x) \right)^\theta dx = \int_{\Omega} \left( I_{i,t}^0(x) \right)^\theta dx + \int_{\{|T_i(u_n) - T_i(u)| > h\}} \left( I_{i,t}^0(x) \right)^\theta dx
\]
\[
\leq \left( \int_{\Omega} \left( I_{i,t}^0(x) \right)^\theta dx \right)^\theta \left| \{|T_i(u_n) - T_i(u)| > h\} \right|^{1-\theta}
\]
\[
+ \left( \int_{\{|T_i(u_n) - T_i(u)| \leq h\}} I_{i,t}^0(x) dx \right)^\theta \left| \Omega \right|^{1-\theta}.
\]

Then, we can write
\[
\int_{\Omega} \left( I_{i,t}^0(x) \right)^\theta dx \leq J_1 + J_2.
\]

Where,
\[
J_1 = \left( \int_{\Omega} I_{i,t}^0(x) dx \right)^\theta \left| \{|T_i(u_n) - T_i(u)| > h\} \right|^{1-\theta}
\]
\[
J_2 = \left( \int_{\{|T_i(u_n) - T_i(u)| \leq h\}} I_{i,t}^0(x) dx \right)^\theta \left| \Omega \right|^{1-\theta}.
\]

For every fixed \( h \), thanks to (45), and the convergence in measure of \( T_i(u_n) \), we can get
\[
\lim_{n \to +\infty} J_1 = 0
\]

Now, choosing \( T_h(u_n - T_i(u)) \) (with \( 0 < h < t \)) as test function in (15) and using (3), (13), we obtain
\[
\sum_{i=1}^{N} \int_{\Omega} \sigma_i(x, D_i(T_i(u_n))) D_i \left( T_h(u_n) - T_i(u) \right) dx - \sum_{i=1}^{N} \int_{\{|u_n-T_i(u)|<h\}} \sigma_i(x, D_i(G_i(u_n))) D_i \left( T_i(u) \right) dx
\]
\[
\leq ch + c' \sum_{i=1}^{N} \int_{\Omega} B_n u_n |u_n|^{p-1} D_i \left( T_h(u_n) - T_i(u) \right) dx.
\]

Then, using (49) and the fact that \( I_{i,t}^0(x) \geq 0 \) (since (5)), we deduce
\[
0 \leq \sum_{i=1}^{N} \int_{\{|T_i(u_n) - T_i(u)| \leq h\}} I_{i,t}^0(x) dx = \sum_{i=1}^{N} \int_{\Omega} \left( \sigma_i(x, D_i(T_i(u_n))) - \sigma_i(x, D_i(T_i(u))) \right) D_i \left( T_h(u_n) - T_i(u) \right) dx
\]
\[
\leq ch + c' \sum_{i=1}^{N} \int_{\Omega} B_n u_n |u_n|^{p-1} D_i \left( T_h(u_n) - T_i(u) \right) dx
\]
\[
+ \sum_{i=1}^{N} \int_{\{|u_n-T_i(u)|<h\}} \sigma_i(x, D_i(u_n)) D_i \left( T_i(u) \right) dx
\]
\[
- \sum_{i=1}^{N} \int_{\Omega} \sigma_i(x, D_i(T_i(u))) D_i \left( T_h(u_n) - T_i(u) \right) dx.
\]
By noticing that $|u_n - T_i(u)| < h \subset |u_n| \leq h + t \subset |u_n| \leq 2l$, and that $(\sigma_i(x, D_i(T_2(u_n))))$ is bounded in $L^{p_1}_{\sigma}(\Omega)$, and (45), we can pass to the limit with respect to $n$ in (50) when $n \to +\infty$, we get

$$
\lim_{n \to +\infty} \sup \sum_{i=1}^{N} \int_{|T_i(u_n) - T_i(u)| < h} l_{i,n}(x) \, dx \leq c + c' \sum_{i=1}^{N} \int_{\Omega} B u|u|^{p_1} - 2 D_i(T_2(G_i(u))) \, dx + \sum_{i=1}^{N} \int_{|A_i| < t < |A_i| + h} \tau_i D_i(T_i(u)),
$$

(51)

where $\tau_i \in L^{p_1}_{\sigma}(\Omega)$ is the weak limit of $\sigma_i(x, D_i(T_2(u_n)))$.

After letting $h \to 0$ in (51), we obtain

$$
\lim_{n \to +\infty} J_2 = 0.
$$

(52)

We combine (47), (48), (52), and using (5), we get

$$
\lim_{n \to +\infty} \int_{\Omega} \left( l_{i,n}(x) \right)^{\theta} \, dx = 0.
$$

(53)

From (53) we deduce, like in [1], that: for every $t > 0$ and every $i = 1, \ldots, N$

$$
D_i(T_i(u_n)) \to D_i(T_i(u)), \text{ almost everywhere in } \overline{\Omega}.
$$

(54)

And through the results obtained in [2] we can get (43). □

**Lemma 3.6.** Let $A(\cdot), B(\cdot)$ are in $W^{1,\overline{p}_1}(\Omega)$, and $A_n(\cdot), B_n(\cdot)$ be defined in (12). Then $A_n(\cdot), B_n(\cdot)$ are bounded in $W^{1,\overline{p}_1}(\Omega)$ and

$$
A_n \to A, \text{ Strongly in } W^{1,\overline{p}_1}(\Omega)
$$

(55)

and

$$
B_n \to B, \text{ Strongly in } W^{1,\overline{p}_1}(\Omega).
$$

(56)

**Proof.** Since, for all $x \in \overline{\Omega}$

$$
D_i A_n(x) = \frac{D_i A(x)}{\left(1 + \frac{A(x)}{\pi} \right)^{\frac{\theta}{\overline{p}_1}}}, \quad i = 1, \ldots, N,
$$

we have that $|D_i A_n(x)| \leq |D_i A(x)|$, and therefore $A_n(\cdot) \in W^{1,\overline{p}_1}(\Omega)$, due to $0 < A_n(x) \leq A(x)$, we obtain that, $A_n$ is bounded in $W^{1,\overline{p}_1}(\Omega)$, and (55). In a similar way we get the boundedness of $B_n$ in $W^{1,\overline{p}_1}(\Omega)$ and (56). □

3.2. **Proof of the Theorem 3.2:**

From (4) and (40), we get for all $i = 1, \ldots, N$

$$
\int_{\Omega} |\sigma_i(x, D_i u_n)|^{p_1} \, dx \leq (1 + c_2^{p_1}) \int_{\Omega} \left( \sum_{j=1}^{N} |D_j u_n|^{p_1} + |h| \right) \, dx \leq (1 + c_2^{p_1}) \int_{\Omega} \left( \sum_{j=1}^{N} |D_j u_n|^{p_1} + |h| \right) \, dx \leq C \|u_n\|^{p_1}_{W^{1,\overline{p}_1}} + C' \leq C''.
$$

And therefore

$$
\sigma_i(x, D_i u_n) \text{ is bounded in } L^{p_1}_{\sigma}(\Omega), \quad i = 1, \ldots, N.
$$

(57)
By (43) and (57) we have, for all \(i = 1, \ldots, N\)

\[
\sigma_i(x, D_i u_n) \rightharpoonup \sigma_i(x, D_i u) \quad \text{weakly in } L^{p_i'}(\Omega), \quad p_i' = \frac{p_i}{p_i-1}.
\]

(58)

Now, we have

\[
\int_{\Omega} |u_n|^{p_i'-1} u_n' \, dx = \int_{\Omega} |u_n|^{p_i'} \, dx \leq C.
\]

Then, we obtain

\[
(u_n|u_n|^{p_i'-2}) \text{ is bounded in } L^{p_i'}(\Omega), \quad i = 1, \ldots, N.
\]

(59)

By (42) and (59) we have, for all \(i = 1, \ldots, N\)

\[
u_n|u_n|^{p_i'-2} \rightharpoonup u|u|^{p_i'-2} \quad \text{weakly in } L^{p_i'}(\Omega).
\]

(60)

Then, from (58), (60), (55), and (56), we can pass to the limit in the weak formulation (15). This proves Theorem 3.2.

References


