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Some results on star-factor deleted graphs

Sufang Wang[a,*], Wei Zhang[b]

[a] *School of Public Management, Jiangsu University of Science and Technology, Zhenjiang, Jiangsu 212100, China*
[b] *School of Economics and Management, Wenzhou University of Technology, Wenzhou, Zhejiang 325000, China*

Abstract. Let G be a graph and let $k \geq 2$ be an integer. A $\{K_{1,j} : 1 \leq j \leq k\}$-factor of G is a spanning subgraph of G, in which every component is isomorphic to a member in $\{K_{1,j} : 1 \leq j \leq k\}$. A graph G is called a $\{K_{1,j} : 1 \leq j \leq k\}$-factor deleted graph if for any $e \in E(G)$, G has a $\{K_{1,j} : 1 \leq j \leq k\}$-factor excluding e. In this article, we first give a characterization of $\{K_{1,j} : 1 \leq j \leq k\}$-factor deleted graph. Then we show a lower bound on the binding number (resp. the size) of G to ensure that G is a $\{K_{1,j} : 1 \leq j \leq k\}$-factor deleted graph. Finally, we construct two extremal graphs to claim that the bounds derived in this article are sharp.

1. Introduction

In this article, we only deal with finite and undirected graphs which possess neither loops nor multiple edges. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The order of a graph G is the number $n = |V(G)|$ of its vertices and its size is the number $q = |E(G)|$ of its edges. Let $Iso(G)$ and $iso(G)$ denote the set of isolated vertices and the number of isolated vertices of G, respectively. A graph G is said to be trivial if $n = 1$, otherwise we call G non-trivial. For a vertex $v \in V(G)$, we denote by $E_G(v)$ the set of edges which are incident to v. The degree of v, denoted by $d_G(v)$, is $|E_G(v)|$. An isolated vertex is a vertex with degree 0, and an isolated edge is an edge which doesn't admit a common endpoint with any edge in G. A pendant edge is an edge with an endpoint of degree 1. For a vertex $v \in V(G)$, the set of neighbors of v is denoted by $N_G(v)$. For a subset $S \subseteq V(G)$, the neighborhood of S is defined by $N_G(S) = \bigcup_{v \in S} N_G(v)$. For any $S \subseteq V(G)$, we denote by $G[S]$ the subgraph of G induced by S, and by $G - S$ the subgraph formed from G by removing the vertices in S and the edges incident to vertices in S. For any $E' \subseteq E(G)$, we use $G - E'$ to denote the subgraph formed from G by removing the edges in E'. For convenience, denote $G - \{v\}$ and $G - \{uv\}$ by $G - v$ and $G - uv$, respectively. For disjoint sets $S, T \subseteq V(G)$, $E_G(S, T)$ denotes the set of edges of G joining a vertex in S and a vertex in T, and set $e_G(S, T) = |E_G(S, T)|$. For two distinct graphs G_1 and G_2, the join and the union of G_1 and G_2 are denoted by $G_1 \vee G_2$ and $G_1 \cup G_2$, respectively. For a graph G and an integer $k \geq 2$, we denote by kG the disjoint union of k copies of G. As usual, a path, a complete graph and a star of order n are denoted by P_n, K_n and $K_{1,n-1}$, respectively.

The binding number of G, denoted by $bind(G)$, was first introduced by Woodall [16] and is defined by

$$bind(G) = min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\right\}.$$

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* Corresponding author: Sufang Wang
Email addresses: wangsufangjust@163.com (Sufang Wang), zw_wzu@163.com (Wei Zhang)

A subgraph of a graph G is spanning if the subgraph covers all vertices of G. Let $\mathcal{H}$ be a set of connected graphs. Then a spanning subgraph H of a graph G is called an $\mathcal{H}$-factor if each component of H is isomorphic to a member of $\mathcal{H}$. An $\mathcal{H}$-factor is also referred as a component factor. Let $k \geq 2$ be an integer. A $\{K_{1,j} : 1 \leq j \leq k\}$-factor is also called a star-factor of G. A $P_{\geq k}$-factor of G is its spanning subgraph each of whose components is a path of order at least k.

In mathematical literature, the study on component factors attracted much attention. Kaneko [6] presented a characterization for a graph with a $P_{\geq 3}$-factor. Kano, Katona and Király [7] posed a simple proof for this characterization on a graph with a $P_{\geq 3}$-factor. Gao and Wang [4] characterized a $P_{\geq 3}$-factor with respect to binding number. Zhou, Bian and Pan [24] established a relationship between binding number and $P_{\geq 3}$-factors of graphs. Wang and Zhang [13] presented an isolated toughness condition for a graph to have a $P_{\geq 3}$-factor. Wu [17] provided two degree conditions for the existence of $P_{\geq 3}$-factors in graphs. Zhou et al [19, 20, 23, 29, 30] showed some sufficient conditions for graphs to possess $P_{\geq 3}$-factors. Amahashi and Kano [1], and Las Vergnas [10] independently derived a necessary and sufficient condition for a graph with a $\{K_{1,j} : 1 \leq j \leq k\}$-factor, where $k \geq 2$ is an integer. Kano and Saito [9] obtained a sufficient condition for a graph to possess a $\{K_{1,j} : m \leq j \leq 2m\}$-factor, where $m \geq 2$ is an integer. Zhou, Xu and Sun [31] derived some results on the existence of star-factors in graphs. Kano, Lu and Yu [8] proved that a graph G has a $\{K_{1,2}, K_{1,3}, K_5\}$-factor if $iso(G - S) \leq \frac{|S|}{2}$ for every $S \subseteq V(G)$. Wang and Zhang [14], Zhou [18, 21], Zhou, Liu and Xu [28] established the relationships between degree condition and graph factors. Wang and Zhang [15], Zhou, Bian and Sun [25] investigated the connections between binding number and graph factors. Zhou [22], Gao, Wang and Chen [5] showed some isolated toughness conditions for the existence of graph factors. Some other results on graph factors were obtained by Zhou and Liu [26, 27], Plummer [12], Bekkai and Kouider [2], Egawa, Furuya and Kano [3].

A graph G is called a $\{K_{1,j} : 1 \leq j \leq k\}$-factor (or a star-factor) deleted graph if for any $e \in E(G)$, G has a $\{K_{1,j} : 1 \leq j \leq k\}$-factor (or a star-factor) excluding e. In this article, we first establish a necessary and sufficient condition for a graph to be $\{K_{1,j} : 1 \leq j \leq k\}$-factor deleted. Based on this result, we derive two sufficient conditions, with respect to binding number and isolated toughness, to determine whether a graph G is $\{K_{1,j} : 1 \leq j \leq k\}$-factor deleted.

Theorem 1.1. Let $k \geq 2$ be an integer. Then a graph G is a $\{K_{1,j} : 1 \leq j \leq k\}$-factor deleted graph if and only if

$$iso(G - S) \leq k|S| - \varepsilon(S)$$

for any $S \subseteq V(G)$, where $\varepsilon(S)$ is defined by

$$\varepsilon(S) = \begin{cases} 2, & \text{if there exists an isolated edge in } G - S; \\ 1, & \text{if there exists a pendant edge that is not an isolated edge in } G - S; \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 1.2. Let $k \geq 2$ be an integer, and let G be a graph with $\delta(G) \geq 2$. If its binding number $\mathrm{bind}(G) > \frac{1}{k-1}$, then G is a $\{K_{1,j} : 1 \leq j \leq k\}$-factor deleted graph.

Theorem 1.3. Let $k \geq 2$ be an integer, and let G be a connected graph of order $n \geq k+2$. If $|E(G)| > \binom{n-k+1}{2} + k - 1$, then G is a $\{K_{1,j} : 1 \leq j \leq k\}$-factor deleted graph.

2. The proof of Theorem 1.1

The following characterization for a graph with a $\{K_{1,j} : 1 \leq j \leq k\}$-factor, derived by Amahashi and Kano [1] and Las Vergnas [10], will be used to verify Theorem 1.1.

Lemma 2.1 ([1, 10]). Let $k \geq 2$ be an integer. Then a graph G contains a $\{K_{1,j} : 1 \leq j \leq k\}$-factor if and only if

$$iso(G - S) \leq k|S|$$

for any $S \subseteq V(G)$.

Proof of Theorem 1.1. Necessity: For any $e \in E(G)$, let $G_e = G - e$. A graph G is a $\{K_{1,j} : 1 \leq j \leq k\}$-factor deleted graph if and only if G_e has a $\{K_{1,j} : 1 \leq j \leq k\}$-factor. According to Lemma 2.1, we possess

$$iso(G_e - S) \leq k|S| \tag{1}$$

for any $S \subseteq V(G)$. The following proof will be divided into three cases.
Case 1. There exists an isolated edge e in $G - S$.
In this case, we possess $\varepsilon(S) = 2$ and $iso(G_e - S) = iso(G - S) + 2$. Together with (1), we derive

$$iso(G - S) = iso(G_e - S) - 2 \leq k|S| - 2 = k|S| - \varepsilon(S).$$

Case 2. There exists a pendant edge e that is not an isolated edge in $G - S$.
In this case, we have $\varepsilon(S) = 1$ and $iso(G_e - S) = iso(G - S) + 1$. Combining these with (1), we get

$$iso(G - S) = iso(G_e - S) - 1 \leq k|S| - 1 = k|S| - \varepsilon(S).$$

Case 3. Neither Case 1 nor Case 2 holds.
In this case, it is obvious that $\varepsilon(S) = 0$ and $iso(G_e - S) = iso(G - S)$. In terms of (1), we infer

$$iso(G - S) = iso(G_e - S) \leq k|S| = k|S| - \varepsilon(S).$$

Sufficiency: Note that $iso(G - S) \leq k|S| - \varepsilon(S) \leq k|S|$. Combining this with Lemma 2.1, G contains a $\{K_{1,j} : 1 \leq j \leq k\}$-factor. Suppose, to the contrary, that G is not a $\{K_{1,j} : 1 \leq j \leq k\}$-factor deleted graph. Then there exists $e \in E(G)$ such that G_e doesn't possess a $\{K_{1,j} : 1 \leq j \leq k\}$-factor. By virtue of Lemma 2.1, we obtain

$$iso(G_e - S) > k|S|. \tag{2}$$

In what follows, we consider three cases by the position of e.
Case 1. $e \in E(G[S])$ or $e \in E(S, V(G) - S)$.
Obviously, $iso(G - S) = iso(G_e - S)$. In view of (2) and $\varepsilon(S) \leq 2$, we infer $iso(G - S) = iso(G_e - S) > k|S| \geq k|S| - \varepsilon(S)$, a contradiction.
Case 2. $e \in E(G - S)$.
Subcase 2.1. e is an isolated edge in $G - S$.
In this subcase, $\varepsilon(S) = 2$ and $iso(G_e - S) = iso(G - S) + 2$. Using (2), $iso(G - S) = iso(G_e - S) - 2 > k|S| - 2 = k|S| - \varepsilon(S)$, a contradiction.
Subcase 2.2. e is a pendant edge that is not an isolated edge in $G - S$.
In this subcase, $\varepsilon(S) = 1$ and $iso(G_e - S) = iso(G - S) + 1$. It follows from (2) that $iso(G - S) = iso(G_e - S) - 1 > k|S| - 1 = k|S| - \varepsilon(S)$, a contradiction.
Subcase 2.3. e is not a pendant edge in $G - S$.
In this subcase, $\varepsilon(S) = 0$ and $iso(G_e - S) = iso(G - S)$. According to (2), we infer $iso(G - S) = iso(G_e - S) > k|S| = k|S| - \varepsilon(S)$, a contradiction. This completes the proof of Theorem 1.1. □

3. The proof of Theorem 1.2

Proof of Theorem 1.2. Suppose, to the contrary, that G is not a $\{K_{1,j} : 1 \leq j \leq k\}$-factor deleted graph. Then it follows from Theorem 1.1 that

$$iso(G - S) \geq k|S| - \varepsilon(S) + 1 \tag{3}$$

for some $S \subseteq V(G)$.
Claim 1. $S \neq \emptyset$.
Proof. Assume that $S = \emptyset$. Together with $\delta(G) \geq 2$, $G - S$ has neither isolated edge nor pendant edge. In view of the definition of $\varepsilon(S)$, we deduce $\varepsilon(S) = 0$. Together with (3), we obtain

$$0 = iso(G) = iso(G - S) \geq k|S| - \varepsilon(S) + 1 = 1,$$

which is a contradiction. This completes the proof of Claim 1. □

The following proof will be divided into two cases by the value of $iso(G-S)$.

Case 1. $iso(G-S)=0$.

According to (3) and $\varepsilon(S)\le 2$, we possess

$$0=iso(G-S)\ge k|S|-\varepsilon(S)+1\ge k|S|-1,$$

which yields $|S|\le \frac{1}{k}$. According to the integrity of $|S|$, we infer $|S|=0$. Recall that $\delta(G)\ge 2$. Then G has neither isolated edge nor pendant edge. In view of $|S|=0$ and the definition of $\varepsilon(S)$, we derive $\varepsilon(S)=0$. Combining this with (3), we obtain

$$0=iso(G)=iso(G-S)\ge k|S|-\varepsilon(S)+1=1,$$

which is a contradiction.

Case 2. $iso(G-S)\ge 1$.

In this case, we have $Iso(G-S)\ne\emptyset$, $N_G(Iso(G-S))\ne V(G)$ and $|N_G(Iso(G-S))|\le|S|$. By virtue of the definition of bind(G), we get

$$\frac{1}{k-1}<\mathrm{bind}(G)\le\frac{|N_G(Iso(G-S))|}{|Iso(G-S)|}\le\frac{|S|}{iso(G-S)},$$

which leads to

$$iso(G-S)<k|S|-|S|. \tag{4}$$

It follows from (3), (4), Claim 2 and $\varepsilon(S)\le 2$ that

$$k|S|-1\ge k|S|-|S|>iso(G-S)\ge k|S|-\varepsilon(S)+1\ge k|S|-1,$$

which is a contradiction. This completes the proof of Theorem 1.2. □

4. The proof of Theorem 1.3

In this section, we first provide the following lemma, which will be used to prove Theorem 1.3.

Lemma 3.1 ([11]). Let $k\ge 2$ be an integer, and let G be a connected graph of order n.

(i) For $n\ge k+2$ and $(k,n)\notin\{(2,7),(3,9)\}$, if $|E(G)|>\binom{n-k-1}{2}+k+1$, then G has a $\{K_{1,j}:1\le j\le k\}$-factor.

(ii) For $(k,n)=(2,7)$, if $|E(G)|>11$, then G has a $\{K_{1,1},K_{1,2}\}$-factor.

(iii) For $(k,n)=(3,9)$, if $|E(G)|>15$, then G has a $\{K_{1,1},K_{1,2},K_{1,3}\}$-factor.

Proof of Theorem 1.3. Suppose, to the contrary, that G is not a $\{K_{1,j}:1\le j\le k\}$-factor deleted graph. Choose a connected graph G such that its size is as large as possible. Then we proceed with the following two cases.

Case 1. G contains no $\{K_{1,j}:1\le j\le k\}$-factor.

For $n\ge k+2$ and $(k,n)\notin\{(2,7),(3,9)\}$, using Lemma 3.1 (i), we possess $|E(G)|\le\binom{n-k-1}{2}+k+1<\binom{n-k+1}{2}+k-1$, which contradicts the hypothesis that $|E(G)|>\binom{n-k+1}{2}+k-1$.

For $(k,n)=(2,7)$, by Lemma 3.1 (ii), we possess $|E(G)|\le 11<16=\binom{7-1}{2}+1$, a contradiction.

For $(k,n)=(3,9)$, from Lemma 3.1 (iii), we have $|E(G)|\le 15<23=\binom{9-2}{2}+2$, a contradiction.

Case 2. G contains a $\{K_{1,j}:1\le j\le k\}$-factor.

In terms of Lemma 2.1, we derive

$$iso(G-S)\le k|S| \tag{5}$$

for any $S\subseteq V(G)$. By virtue of Theorem 1.1, G is not a $\{K_{1,j}:1\le j\le k\}$-factor deleted graph if and only if there exists some subset $S\subseteq V(G)$ such that at least one of the following two statements is true.

(1) $G-S$ has an isolated edge, and $iso(G-S)\ge k|S|-1$;

(2) $G-S$ has a pendant edge that is not an isolated edge, and $iso(G-S) \geq k|S|$.

Subcase 2.1. Statement (1) is true.

According to (5) and Statement (1), we derive $k|S| \geq iso(G-S) \geq k|S|-1$. In view of the choice of G, one has

• $G[S]$ is a complete graph;

• $G-S$ contains at most two non-trivial connected components, one of which is a K_2 component, the other is a complete graph, say G_1;

• G is the join of $G[S]$ and $G-S$, namely, $G = G[S] \vee (G-S)$.

For convenience, we write $|S| = s$ and $|V(G_1)| = n_1$. Obviously, $n_1 = 0$ or $n_1 \geq 2$. Then we proceed by showing the following fact.

Fact 1. Let G and S satisfy the conditions in Subcase 2.1.

(a) If $iso(G-S) = ks-1$, then $|E(G)| \leq \binom{n-k+1}{2} + k - 1$;

(b) If $iso(G-S) = ks$, then $|E(G)| < \binom{n-k+1}{2} + k - 1$.

Proof. (a) Recall that $iso(G-S) = ks-1$ ($s \geq 1$). Then we possess $n = (k+1)s + 1 + n_1$ and $|E(G)| = \binom{n-ks-1}{2} + s(ks-1) + 2s + 1 = \binom{n-ks-1}{2} + s(ks+1) + 1$. By a simple computation, we obtain

$$\begin{aligned}
\binom{n-k+1}{2} + k - 1 - |E(G)| &= \binom{n-k+1}{2} + k - 1 - \binom{n-ks-1}{2} - s(ks+1) - 1 \\
&= \frac{1}{2}((2ks-2k+4)n - (k^2+2k)s^2 - (3k+2)s + k^2 + k - 6) \\
&= \frac{1}{2}((2ks-2k+4)((k+1)s+1+n_1) - (k^2+2k)s^2 - (3k+2)s + k^2 + k - 6) \\
&= \frac{1}{2}(k^2s^2 - (2k^2-k-2)s + (2ks-2k+4)n_1 + k^2 - k - 2) \\
&\geq \frac{1}{2}(k^2s^2 - (2k^2-k-2)s + k^2 - k - 2) \\
&= \frac{1}{2}(s-1)(k^2s - k^2 + k + 2) \\
&\geq 0,
\end{aligned}$$

which leads to $|E(G)| \leq \binom{n-k+1}{2} + k - 1$.

(b) Recall that $iso(G-S) = ks$ ($s \geq 0$). Then we obtain $n = (k+1)s + 2 + n_1$ and $|E(G)| = \binom{n-ks-2}{2} + s(ks) + 2s + 1 = \binom{n-ks-2}{2} + s(ks+2) + 1$.

We easily see that $s + n_1 \neq 0$ (otherwise $s = n_1 = 0$, which yields $n = 2$, which contradicts $n \geq k+2$). Furthermore, we easily see $s \geq 1$ (otherwise $s = 0$ and $n_1 \neq 0$, which implies that G has at least two connected components G_1 and K_2, which contradicts that G is a connected graph). By a simple computation, we derive

$$\begin{aligned}
\binom{n-k+1}{2} + k - 1 - |E(G)| &= \binom{n-k+1}{2} + k - 1 - \binom{n-ks-2}{2} - s(ks+2) - 1 \\
&= \frac{1}{2}((2ks-2k+6)n - (k^2+2k)s^2 - (5k+4)s + k^2 + k - 10) \\
&= \frac{1}{2}((2ks-2k+6)((k+1)s+2+n_1) - (k^2+2k)s^2 - (5k+4)s + k^2 + k - 10) \\
&\geq \frac{1}{2}((2ks-2k+6)((k+1)s+2) - (k^2+2k)s^2 - (5k+4)s + k^2 + k - 10) \\
&= \frac{1}{2}(s-1)(k^2s - k^2 + 3k + 2) + 2 \\
&> 0,
\end{aligned}$$

which yields $|E(G)| < \binom{n-k+1}{2} + k - 1$. Fact 1 is verified. □

Based on Fact 1, we possess $|E(G)| \le \binom{n-k+1}{2} + k - 1$, a contradiction.

Subcase 2.2. Statement (2) is true.

By virtue of (5) and Statement (2), we obtain $k|S| \ge iso(G-S) \ge k|S|$, namely, $iso(G-S) = k|S|$. According to the choice of G, one has

• $G[S]$ is a complete graph;

• $G-S$ contains one non-trivial connected component, say G_1. And G_1 is a complete graph to which an edge has been attached;

• G is the join of $G[S]$ and $G-S$, that is, $G = G[S] \vee (G-S)$.

We also write $|S| = s$ and $|V(G_1)| = n_1$. Clearly, $n_1 \ge 3$. Then we proceed by showing the following fact.

Fact 2. $|E(G)| \le \binom{n-k+1}{2} + k - 1$.

Proof. Observe that $n = (k+1)s + n_1$ and $n_1 \ge 3$. Then $|E(G)| = \binom{n-ks-1}{2} + s(ks+1) + 1$. By a simple computation, we possess

$$\begin{aligned}\binom{n-k+1}{2} + k - 1 - |E(G)| = &\binom{n-k+1}{2} + k - 1 - \binom{n-ks-1}{2} - s(ks+1) - 1\\ =&\frac{1}{2}((2ks-2k+4)n - (k^2+2k)s^2 - (3k+2)s + k^2 + k - 6)\\ \ge&\frac{1}{2}((2ks-2k+4)((k+1)s+3) - (k^2+2k)s^2 - (3k+2)s + k^2 + k - 6)\\ =&\frac{1}{2}(s-1)(k^2s - k^2 + 5k + 2) + 4\\ \ge&0,\end{aligned}$$

which implies $|E(G)| \le \binom{n-k+1}{2} + k - 1$. Fact 2 is proved. □

According to Fact 2 and the hypothesis of Theorem 1.3, we have

$$\binom{n-k+1}{2} + k - 1 < |E(G)| \le \binom{n-k+1}{2} + k - 1,$$

which is a contradiction. This completes the proof of Theorem 1.3. □

5. Extremal graphs

In this section, we create two graphs to claim that the bounds established in Theorems 1.2 and 1.3 are sharp, respectively.

Theorem 5.1. Let $k \ge 2$ and $m \ge 2$ be two integers, and let $G_m = K_t \vee ((k-1)K_1 \cup K_2 \cup K_m)$, where $t = 1$. We have $\mathrm{bind}(G_m) = \frac{1}{k-1}$ and G_m is not a $\{K_{1,j} : 1 \le j \le k\}$-factor deleted graph.

Proof. Obviously, $\mathrm{bind}(G_m) = \frac{1}{k-1}$. Set $S = V(K_t)$. Then $|S| = t = 1$ and $iso(G_m - S) = k - 1$. Since $G_m - S$ contains an isolated edge, we obtain $\varepsilon(S) = 2$. Thus, we deduce

$$iso(G_m - S) = k - 1 > k - 2 = k|S| - \varepsilon(S).$$

By virtue of Theorem 1.1, G_m is not a $\{K_{1,j} : 1 \le j \le k\}$-factor deleted graph. □

Theorem 5.2. Let $k \ge 2$ and $n \ge k+2$ be two integers, and let G be a connected graph with vertex set $V(G) = V(K_{n-k+1}) \cup \{v_1, v_2, \ldots, v_{k-1}\}$ and edge set $E(G) = E(K_{n-k+1}) \cup \{uv_i : u \in V(K_{n-k+1}), i = 1, 2, \ldots, v_{k-1}\}$. Then $|E(G)| = \binom{n-k+1}{2} + k - 1$ and G is not a $\{K_{1,j} : 1 \le j \le k\}$-factor deleted graph.

Proof. It is straightforward to check that the size of G is $|E(G)| = \binom{n-k+1}{2} + k - 1$. Set $S = \emptyset$. Then $iso(G-S) = 0$ and $G-S$ contains an pendant edge that is not an isolated edge. In terms of the definition of $\varepsilon(S)$, we see $\varepsilon(S) = 1$. Thus, we infer

$$iso(G-S) = 0 > -1 = k|S| - \varepsilon(S),$$

According to Theorem 1.1, G is not a $\{K_{1,j} : 1 \le j \le k\}$-factor deleted graph. □

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