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A bivariate probability generator for the odd generalized exponential model: Mathematical structure and data fitting

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Abstract. The generalized exponential (GE) distribution is the well-established generalization of the exponential distribution in statistical literature. Tahir et al. (2015) proposed a flexible probability generator called the odd generalized exponential-G (OGE-G) family of distributions. In this article, we propose a bivariate extension of the OGE-G class, in the so-called the bivariate odd generalized exponential-G (BOGE-G) family of distributions, whose marginal distributions are OGE-G families. Important mathematical and statistical properties of the BOGE-G family including joint density function with its marginals, Marshall-Olkin copula, product moments, covariance, conditional densities, median correlation coefficient, joint reliability function, joint hazard rate function with its marginal functions, marginal asymptotic, and distributions for both $\max(X_1, X_2)$ and $\min(X_1, X_2)$, are derived. After the general class is introduced, a sub-model is discussed in detail. The maximum likelihood approach is utilized for estimating the bivariate family parameters. A simulation study is carried out to assess the performance of the sub-model parameters. A real-life data set is analyzed to illustrate the flexibility of the proposed bivariate class.

1. Introduction

The exponential (E) distribution is the basic and well-recognized probability model in teaching and research. For lifetime phenomenon and reliability studies, the E model is rarely used due to having constant hazard rate, but its memoryless property has utility in queuing theory. Various extensions of the E distribution have been reported in the literature and have received increasing attention, especially the exponentiated exponential (EE) distribution. In the literature, the EE distribution is also called the generalized exponential (GE) distribution (see, Gupta et al. [1] and Gupta and Kundu [2, 3]). The cumulative distribution function (CDF) of the GE distribution is given by

$$F_{GE} = (1 - e^{-\alpha x})^{\beta}; \quad x > 0, \tag{1}$$

where $\beta > 0$ is the power parameter, and $\alpha > 0$ is the scale parameter. To bring the flexibility in a probability distribution, the generalization or extension is the basic idea "parameter induction approach". Azzalini [4],

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Marshall-Olkin [5] and Gupta et al. [1] were the pioneers who proposed skew-normal, Marshall-Olkin, and exponentiated-G families of distributions. Gupta et al. [1] discussed generalized families using Lehmann alternatives 1 and 2 (LA1 and LA2) $F_{LA1}(x) = G(x)^{\beta}$ and $F_{LA2}(x) = 1 - [\bar{G}(x)]^{\beta}$, where $G(\cdot)$ and $\bar{G}(\cdot) = 1 - G(\cdot)$ are the CDF and survival function (SF) of the baseline distribution, and $\beta > 0$ is the shape parameter. For more detail on generalized families, the reader is referred to Tahir and Nadarajah [6] and Tahir and Cordeiro [7]. Alzaatreh et al. [8] introduced a general approach for the construction of generalized families by utilizing the transformed-transformer (T-X) technique. Based on Alzaatreh et al. [8] method, some odd univariate distributions have been proposed in literature such as odd log-logistic-G (OLL-G) (Gleaton and Lynch [9]), odd gamma-G (OGa-G) (Torabi [10]), odd Weibull-G (OW-G) (Bourguignon et al. [11]), odd Birnbaum-Saunders-G (OBS-G) (Ortega et al. [12]), odd Burr-G (OB-G) (Alizadeh et al. [13]), generalized odd half-Cauchy-G (GOHCa-G) (Cordeiro et al. [14]) and odd Lindley-G (OL-G) (Silva et al. [15]). Recently, Tahir et al. [16] introduced a new odd function of distributions called the odd generalized exponential-G (OGE-G) which exhibit flexible hazard rate shapes such as increasing, decreasing, bathtub or upside-down bathtub. For more details about odd univariate generator (Alzaatreh et al. [8]). The CDF and PDF of the OGE-G family are, respectively, given by

$$F_{OGE-G}(x;\alpha,\beta,\xi) = \left(1 - e^{-\alpha \frac{G(x;\xi)}{\bar{G}(x;\xi)}}\right)^{\beta}; \quad x > 0 \tag{2}$$

and

$$f_{OGE-G}(x;\alpha,\beta,\xi) = \frac{\alpha\beta g(x;\xi)}{[\bar{G}(x;\xi)]^2} e^{-\alpha \frac{G(x;\xi)}{\bar{G}(x;\xi)}} \left(1 - e^{-\alpha \frac{G(x;\xi)}{\bar{G}(x;\xi)}}\right)^{\beta-1}; \quad x > 0, \tag{3}$$

where $\alpha > 0$ and $\beta > 0$ are scale and shape parameters, respectively, and ξ is the vector of parameters. The bivariate (BV) or multivariate (MV) probability distributions have been derived and developed by many statisticians which have wide applications in various fields including drought, reliability, engineering, weather, sports, among others. More detail is given in Balakrishnan and Lai [17], and Sarabia and Gomez [18]. The construction or development of BV (MV) discrete and continuous models are mainly such as the compounding "power series class", marginals, copulas, reduction, and conditioning. Recently, the trend in introducing new BV compounded, weighted and generalized (G-) families of distributions which have received increased attention, which is briefly described below:

1. BV compounded distributions and families: Dimitrakopoulou et al. [19] obtained four BV extended exponential-geometric (BVEEG) distributions from the extended exponential-geometric (EEG) model introduced by Adamidis et al. [20]. Kundu and Gupta [21] proposed and studied BV Weibull-geometric (BVWG) distribution and discussed some of its properties and estimation methods. Kundu [22] introduced a five-parameter BV complementary GE-geometric (BVCGEG) model and investigated some of its important properties. Roozegar and Jafari [23] introduced and studied complementary BV generalized linear failure rate-power series (BVGLFRPS) family of distributions. Nadarajah and Roozegar [24] proposed BV Weibull-power series (BVWPS) family of distribution which generalizes the work of [21]. Jafari and Roozegar [25] obtained BV generalized-exponential power-series (BVGEPS) family of distributions by compounding GE and power-series distributions. Bidram [26] compounded two discrete distributions and proposed BV geometric-Poisson distribution.

2. BV weighted distributions: The Ronald Fisher's idea of weighted distribution received attention due to the work of [27, 28] who applied it in proposing sampling plans for human families and wildlife populations. Later, their applications were found useful in the fields of line transcend sampling, renewal theory, etiological studies, ecology, arial surveys, reliability modeling and bimedical sciences, among others. Mahfond and Patil [29] first proposed and studied weight function for BV distributions. Patil et al. [30] gave different weight functions, and illustrated their applications in various models. Further work on BV weighted distributions appeared in [31–34]. Al-Mutairi et al. [35] first introduced an absolute continuous BV weighted exponential (BVWtE) distribution from weighted exponential (WtE) marginal. Mahdavi et al. [36] proposed BVWtE distribution from GE marginals and studied some of its mathematical properties. Jamalizadeh and Kundu [37] introduced a different approach for the construction of BVWtE model as compared to Al-Mutairi's approach, called it weighted Marshall-Olkin BV exponential (WtMOBVE) distribution. Recently, Ghosh and Alzaatreh [38], and Arnold et al. [39] proposed and studied BVWtE and BVWtGE models via conditioning.

3. BV G-families: Gupta and Kundu [40] introduced BV proportional reversed hazard rate (BVPRHR) family, that is, using LA1 class $F_{PRHR}(x) = [G(x)]^{\beta}$ and discussed some of its properties. Sarabia et al. [41] proposed three BV beta-generated (BVBG) families and also studied properties of some specific BVBG distributions. The CDF of the well-known beta-G family (Eugene et al. [42]) of distributions can be listed as

$$F_{BG}(x) = \int_0^{G(x)} [B(a,b)]^{-1}\, t^{a-1}\, (1-t)^{b-1}\, dt = I_{G(x)}(a,b), \tag{4}$$

where $a > 0$ and $b > 0$ are shape parameters, $I_t(p,q) = B(p,q)/B_t(p,q)$, $B(p,q)$ and $B_t(p,q)$ are incomplete beta function ratio, beta function and incomplete beta functions, respectively. Balakrishnan and Ristić [43] introduced BV Zografos-Balaktishnan gamma-G (BVZBGaG) family from ZBGaG family proposed in Zografos and Balakrishnan [44]. The CDF of ZBGaG family can be formulated as

$$F_{ZBGaG}(x) = \int_0^{-\log[1-G(x)]} [\Gamma(a)]^{-1}\, t^{a-1}\, \exp(-t)\, dt = [\Gamma(a)]^{-1}\, \gamma\left(a, -\log[1-G(x)]\right), \tag{5}$$

where $\Gamma(\cdot)$ and $\gamma(\cdot)$ are complete gamma and upper incomplete gamma functions, respectively. Ghosh and Hamedani [45] proposed BV Ristić-Balaktishnan gamma-G (BVRBGaG) family from RBGaG family proposed by Ristić and Balaktishnan [46]. The CDF of RBGaG family can be expressed as

$$F_{RBGaG}(x) = 1 - \int_0^{-\log\, G(x)} [\Gamma(a)]^{-1}\, t^{a-1}\, \exp(-t)\, dt = 1 - [\Gamma(a)]^{-1}\, \gamma\left(a, -\log\, G(x)\right). \tag{6}$$

Roozegar and Jafari [47] proposed Marshall-Olkin type BV exponentiated extended Weibull (BVEeW) family of distributions using LA1 approach to Gurvich et al. [48] extended-Weibull (eW) family. The CDF of exponentiated-eW (EeW) family of distributions is given by

$$F_{EeW}(x) = \left(1 - e^{-\lambda\, H(x;\xi)}\right)^{\alpha};\ x > 0, \tag{7}$$

where $\lambda > 0$ and $\alpha > 0$ are the scale and power parameters, respectively, and $H(x;\xi)$ is a non-negative monotonically increasing function which depends on the parameter vector ξ.

The aim of our paper is to introduce a new bivariate family, the bivariate odd generalized exponential-G (BOGE-G) family based on Marshall-Olkin shock model [49], whose marginal distributions are OGE-G families. A random vector $\mathbf{X} = (X_1, X_2)$ follows the BV Marshall-Olkin model if and only if there exist three independent random variables (RVs) U_1, U_2 and U_3 such that "$X_1 = \max\{U_1, U_3\}$ and $X_2 = \max\{U_2, U_3\}$" or "$X_1 = \min\{U_1, U_3\}$ and $X_2 = \min\{U_2, U_3\}$". The proposed BOGE-G class was generated from three independent OGE-G families for distributions that use a maximization process. Some features of the BOGE-G family can be listed as follows:

1. The joint CDF can be formulated as a mixture of an absolute continuous and a singular functions;
2. The joint PDF, joint CDF and joint survival function can be proposed in explicit forms;
3. The joint hazard rate function (HRF) can take various shapes depending on the BOGE-G parameters;
4. The marginals of the BOGE-G class can be utilized to analyze various kindes of failure rates;
5. The stress-strength model is not based on the baseline function, but only on the parameters of the bivariate class;
6. It can be applied to a maintenance model or a stress model;
7. This class contains several bivariate special distributions depending on the baseline function (BF);
8. This family can be utilized to model asymmetric data;

9. It provide consistently better fits than other generated distributions under the same BF.

The paper is unfolded as follows. In Section 2, the BOGE-G family and its margins are defined. Some mathematical properties of the BOGE-G class are obtained in Section 3. A sub-model of the BOGE-G family called BOGE-Gompertz distributionis discussed in Section 4. In Section 5, the family parameters are estimated via the maximum likelihood technique. A simulation study is performed in Section 6. In Section 7, the usefulness of the BOGE-G class is illustrated by means of a real data set. Finally, Section 8 offers some concluding remarks and future work.

2. The BOGE-G Family: Methodology and Structure

Assume $U_i \sim OGE-G(\alpha,\xi,\beta_i)$; $i = 1,2,3$ are three independent RVs. Define $X_m = \max\{U_m, U_3\}$; $m = 1,2$. Thus, the bivariate vector (BVr) $\mathbf{X} = (X_1, X_2)$ has the BOGE-G class with parameters vector $\mathbf{\Omega} = (\alpha,\xi,\beta_1,\beta_2,\beta_3)$. The joint CDF of the BVr $\mathbf{X}$ can be formulated as

$$F_{X_1,X_2}(x_1,x_2) = F_{OGE-G}(z;\alpha,\xi,\beta_3)\prod_{i=1}^{2}F_{OGE-G}(x_i;\alpha,\xi,\beta_i), \tag{8}$$

where $z = \min(x_1, x_2)$. Equation (8) can be written as follows

$$F_{X_1,X_2}(x_1,x_2) = \begin{cases} F_{OGE-G}(x_1;\alpha,\xi,\beta_1+\beta_3) \times F_{OGE-G}(x_2;\alpha,\xi,\beta_2) \text{ if } x_1 \leq x_2 \\ F_{OGE-G}(x_1;\alpha,\xi,\beta_1) \times F_{OGE-G}(x_2;\alpha,\xi,\beta_2+\beta_3) \text{ if } x_1 > x_2. \end{cases} \tag{9}$$

The corresponding joint PDF of Equation (9) can be expressed as

$$f_{X_1,X_2}(x_1,x_2) = \begin{cases} f_1(x_1,x_2) & \text{if } 0 < x_1 < x_2 < \infty \\ f_2(x_1,x_2) & \text{if } 0 < x_2 < x_1 < \infty \\ f_3(x,x) & \text{if } 0 < x_1 = x_2 = x < \infty, \end{cases} \tag{10}$$

where

$$f_1(x_1,x_2) = f_{OGE-G}(x_2;\alpha,\xi,\beta_2) \times f_{OGE-G}(x_1;\alpha,\xi,\beta_1+\beta_3),$$

$$f_2(x_1,x_2) = f_{OGE-G}(x_1;\alpha,\xi,\beta_1) \times f_{OGE-G}(x_2;\alpha,\xi,\beta_2+\beta_3),$$

and

$$f_3(x,x) = \frac{\beta_3}{\beta_1+\beta_2+\beta_3} f_{OGE-G}(x;\alpha,\xi,\beta_1+\beta_2+\beta_3).$$

The expressions $f_1(x_1,x_2)$ and $f_2(x_1,x_2)$ can be derived by differentiating Equation (8) with respect to x_1 and x_2 in case of $x_1 < (>)\, x_2$, respectively. Whereas $f_3(x,x)$ cannot be derived in the same approach. Thus, the following fact can be utilized to get $f_3(x,x)$

$$\int_0^\infty f_3(x,x)\,dx = 1 - \int_0^\infty\int_0^{x_2} f_1(x_1,x_2)dx_1dx_2 - \int_0^\infty\int_0^{x_1} f_2(x_1,x_2)dx_2dx_1. \tag{11}$$

The marginal CDFs of the BOGE-G class can be represented as

$$F_{X_i}(x_i) = F_{OGE-G}(x_i;\alpha,\xi,\beta_i+\beta_3);\ \ i = 1,2. \tag{12}$$

Therefore, the marginal PDFs to Equation (12) can be formulated as

$$f_{X_i}(x_i) = f_{OGE-G}(x_i;\alpha,\xi,\beta_i+\beta_3);\ \ i = 1,2, \tag{13}$$

The asymptotics of Equations (12) and (13) as $G(x_i;\xi) \to 0$ are given by

$$F_{X_i}(x_i) \sim [\alpha G(x_i;\xi)]^{\beta_i+\beta_3}, \tag{14}$$

and

$$f_{X_i}(x_i) \sim (\beta_i+\beta_3)\,\alpha^{\beta_i+\beta_3} g(x_i;\xi)\,[G(x_i;\xi)]^{\beta_i+\beta_3-1}, \tag{15}$$

respectively. Further, when $x_i \to \infty$, the asymptotics of Equations (12) and (13) are given by

$$1 - F_{X_i}(x_i) \sim (\beta_i+\beta_3)\, e^{-\frac{\alpha}{\overline{G}(x_i;\xi)}}, \tag{16}$$

and

$$f_{X_i}(x_i) \sim \frac{\alpha\,(\beta_i+\beta_3)}{\left[\overline{G}(x_i;\xi)\right]^2} g(x_i;\xi) e^{-\frac{\alpha}{\overline{G}(x_i;\xi)}}, \tag{17}$$

respectively. A useful linear representation can be provided for the marginal PDFs of the BOGE-G class. Using the generalized binomial expansion, expanding the exponential function in power series, and a result of Gradshteyn and Ryzhik (2000, Section 0.314) for a power series raised to a positive integer, we get

$$f_{X_i}(x_i) = \sum_{k=0}^{\infty} \varpi_{k,(\beta_i+\beta_3)} h_{\beta_i+\beta_3+k}(x_i);\ i = 1,2, \tag{18}$$

where $h_{\beta_i+\beta_3+k}(x_i) = (\beta_i+\beta_3+k)\, g(x_i,\xi)\,[G(x_i,\xi)]^{\beta_i+\beta_3+k-1}$ is a RV having the exponential-G PDF with power parameter $(\beta_i+\beta_3+k)$,

$$\varpi_{k,(\beta_i+\beta_3)} = \sum_{m=0,l=m}^{\infty} \frac{(-1)^{l-m}}{l!}\,(\beta_i+\beta_3)_l \binom{l}{m} L_{m,k} \text{ with } (\beta_i+\beta_3)_0 = 1,$$

$$(\beta_i+\beta_3)_l = (\beta_i+\beta_3)\,(\beta_i+\beta_3-1)\ldots(\beta_i+\beta_3-l+1),$$

$$L_{m,k} = \frac{1}{kb_0}\sum_{n=1}^{s}(n(m+1)-k)\,b_n L_{m,k-n} \text{ with } L_{m,0} = b_0^m, m = 1,2,3,\ldots,$$

$$b_n = a_{n+1} \text{ with } a_n = \sum_{(s,j)\in I_n} \frac{(-1)^{s+j+1}}{s!}\alpha^i \binom{-s}{j},$$

and

$$I_n = \{(s,j) : s+j = n, s = 1,2,3,\ldots; j = 0,1,2,\ldots\}.$$

3. Statistical Generator Properties

3.1. Absolute continuous and singular parts with copula

The BOGE generator has both an absolute continuous and a singular parts similar to Marshall and Olkin's bivariate exponential model. The joint distribution function of X_1 and X_2 has a singular part along the line $x_1 = x_2$ with weight $\frac{\beta_3}{\beta_1+\beta_2+\beta_3}$, and has an absolute continuous part on $0 < x_1 \neq x_2 < \infty$ with weight $\frac{\beta_1+\beta_2}{\beta_1+\beta_2+\beta_3}$. Interestingly, the BOGE-G family can be derived via the Marshall-Olkin copula with the marginals as the OGE-G family of distributions. To every bivariate distribution function $F_{X_1,X_2}(x_1,x_2)$ with continuous marginals $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$ corresponds a unique bivariate distribution function with uniform margins $C : [0,1]^2 \to [0,1]$ called a copula, such that

$$F_{X_1,X_2}(x_1,x_2) = C\left(F_{X_1}(x_1), F_{X_2}(x_2)\right); \text{ for all } (x_1,x_2) \in R^2, \tag{19}$$

(see, Nelsen [51]). The Marshall-Olkin copula can be expressed as

$$C_{\delta_1,\delta_2}(v_1,v_2) = v_1^{1-\delta_1} v_2^{1-\delta_2} \min\left(v_1^{\delta_1}, v_2^{\delta_2}\right); \text{ for } 0 < \delta_1, \delta_2 < 1. \tag{20}$$

Utilizing $v_i = F_{X_i}(x_i)$ where $X_i \sim OGE-G(\alpha,\xi,\beta_i+\beta_3)$, and $\delta_i = \frac{\beta_3}{\beta_i+\beta_3}$; $i = 1,2$ gives the same CDF as Equation (9). A copula is a function that links the marginal distributions of random variables to their joint distribution. A copula can have both continuous and singular parts, where the continuous part describes the dependence between random variables and the singular part describes the probability of perfect dependence or independence. To manipulate a copula analytically, one needs to use derivatives and integrals, which can be done using generalized functions such as the Heaviside and Dirac functions. A copula with a singular part can be used to model phenomena such as perfect correlation, extreme events, or discrete data.

3.2. *Distributions of $T = \max(X_1, X_2)$ and $S = \min(X_1, X_2)$*

In applied fields, particularly in the industrial, medical, insurance and military areas, it is significant to derive the distributions of the RVs T and S, because the RVs X_1 and X_2 could be exchange rates in two time periods, or remission times two chemicals when administered in two types of mechanical systems, or two types of ammunition that will penetrate their target in military warfare, or future observations about the stability of an engineering design. Assume the BVr $\mathbf{X}$ has the BOGE-G class, then the distributions of the RVs T and S can be expressed, respectively, as

$$F_T(t) = F_{OGE-G}(t;\alpha,\xi,\beta_1+\beta_2+\beta_3), \tag{21}$$

and

$$F_S(t) = F_{OGE-G}(t;\alpha,\xi,\beta_1+\beta_3) + F_{OGE-G}(t;\alpha,\xi,\beta_2+\beta_3) - F_{OGE-G}(t;\alpha,\xi,\beta_1+\beta_2+\beta_3). \tag{22}$$

3.3. *Moments of the margins and RV $Z = X_1^r X_2^r$*

Descriptive statistics is an invaluable tool used in the analysis of data. It allows us to summaries and interpret large datasets, providing a concise overview of the data's key points. Descriptive statistics provide quantifiable information about the data such as the mean, median, mode, variance, standard deviation and range, which can be presented in graphical form. This enables us to quickly identify patterns and trends within the data, allowing for more accurate conclusions to be drawn. Additionally, descriptive statistics can be used to identify outliers in the data, thus providing an even more comprehensive understanding of the dataset. In this segment, we derive the *rth* moment and *mth* incomplete moment of X_i when $X_i \sim OGE-G(\alpha,\xi,\beta_i)$, such that $i = 1,2$. Further, the product moment, say $E(X_1^r X_2^r)$, covariance, and coefficient of correlation of the BVr $\mathbf{X}$ are derived. The *rth* moment of X_i, say $E(X_i^r)$, can be expressed as

$$E(X_i^r) = \sum_{k=0}^{\infty} \varpi_{k,(\beta_i+\beta_3)} E(Y_{i,k}^r); i = 1,2, \tag{23}$$

where $Y_{i,k}; i = 1,2$ are the RVrs having the exponential-G PDF $h_{\beta_i+\beta_3+k}(x_i)$ with power parameter $(\beta_i+\beta_3+k)$. The moments of the exponential-G distributions were discussed by Nadarajah and Kotz [53]. Setting $r = 1$ in Equation (23), we obtain the mean of $X_i; i = 1,2$. The variance of X_i, say $Var(X_i)$, can be expressed as

$$Var(\mathrm{X}_i) = \sum_{k=0}^{\infty} \varpi_{k,(\beta_i+\beta_3)} E(Y_{i,k}^2) - \left[E(X_i^1)\right]^2; i = 1,2. \tag{24}$$

For empirical purposes, the shapes of many distributions can be usefully described by what we call the first incomplete moment which plays an important role for measuring inequality, for example, income quantiles, and Lorenz and Bonferroni curves. The *mth* incomplete moment of X_1 and X_2 can be expressed as

$$M_{m_i}(.) = (\beta_i + \beta_3) \sum_{k=0}^{\infty} d_k \int_0^{G(.)} Q_G(w) w^{\beta_i+\beta_3+k-1} dw; \; i = 1, 2. \tag{25}$$

The last integral can be calculated for most G distributions. Utilizing Equations (10) and (18), the product moments can be formulated as

$$E(X_1^r X_2^r) = \int_0^{\infty} \int_0^{x_2} x_1^r x_2^r f_1(x_1, x_2) dx_1 dx_2 + \int_0^{\infty} \int_0^{x_1} x_1^r x_2^r f_2(x_1, x_2) dx_2 dx_1 + \int_0^{\infty} x^{2r} f_3(x, x)\, dx$$

$$\begin{aligned} &= \sum_{k=0}^{\infty} \sum_{k^*=0}^{\infty} \left[\varpi_{k,(\beta_1+\beta_3)} \varpi_{k^*,(\beta_2)} \Psi_2^{(r)}(k, \omega) + \varpi_{k,(\beta_2+\beta_3)} \varpi_{k^*,(\beta_1)} \Psi_1^{(r)}(k, \omega) \right] \\ &+ \frac{\beta_3}{\beta_1 + \beta_2 + \beta_3} \sum_{k^*=0}^{\infty} \varpi_{k^*,(\beta_1+\beta_2+\beta_3)} \Psi_*^{(r)}(k, \omega), \end{aligned} \tag{26}$$

where

$$\begin{aligned} \Psi_i^{(r)}(k, \omega) &= \int_0^{\infty} x_i^r \, \Delta^{(r)}(x_i) h_{\beta_i+\beta_3+k^*}(x_i)\, dx_i, \\ \Delta^{(r)}(x_i) &= \int_0^{x_i} x_{3-i}^r \, h_{\beta_{3-i}+\beta_3+k}(x_{3-i}) dx_{3-i}, \end{aligned}$$

and

$$\Psi_*^{(r)}(k, \omega) = \int_0^{\infty} x^{2r} \, h_{\beta_i+\beta_3+k^*}(x)\, dx \;\; ; i = 1, 2.$$

Based on Equations (23) and (26) when $r = 1$, the covariance, say $Cov(X_1, X_2)$, of the RVr $\mathbf{X}$ can be expressed as

$$\begin{aligned} Cov(X_1, X_2) &= \sum_{k=0}^{\infty} \sum_{k^*=0}^{\infty} \left[\varpi_{k,(\beta_1+\beta_3)} \varpi_{k^*,(\beta_2)} \Psi_2^{(1)}(k, \omega) + \varpi_{k,(\beta_2+\beta_3)} \varpi_{k^*,(\beta_1)} \Psi_1^{(1)}(k, \omega) \right] \\ &+ \frac{\beta_3}{\beta_1 + \beta_2 + \beta_3} \sum_{k^*=0}^{\infty} \varpi_{k^*,(\beta_1+\beta_2+\beta_3)} \Psi_*^{(1)}(k, \omega) - \sum_{k=0}^{\infty} \varpi_{k,(\beta_1+\beta_3)} E(Y_{1,k}^1) \\ &\times \sum_{k=0}^{\infty} \varpi_{k,(\beta_2+\beta_3)} E(Y_{2,k}^1). \end{aligned} \tag{27}$$

According to Equations (23) and (27), the coefficient of correlation, say $\vartheta(X_1, X_2)$, can be represented as

$$\begin{aligned} \vartheta(X_1, X_2) &= \left[\left(\sum_{k=0}^{\infty} \varpi_{k,(\beta_1+\beta_3)} E(Y_{1,k}^2) - \left[E(X_1^1)\right]^2\right) \left(\sum_{k=0}^{\infty} \varpi_{k,(\beta_2+\beta_3)} E(Y_{2,k}^2) - \left[E(X_2^1)\right]^2\right) \right]^{\frac{-1}{2}} \\ &\times \Big[\sum_{k=0}^{\infty} \sum_{k^*=0}^{\infty} \left[\varpi_{k,(\beta_1+\beta_3)} \varpi_{k^*,(\beta_2)} \Psi_2^{(1)}(k, \omega) + \varpi_{k,(\beta_2+\beta_3)} \varpi_{k^*,(\beta_1)} \Psi_1^{(1)}(k, \omega) \right] \\ &+ \frac{\beta_3}{\beta_1 + \beta_2 + \beta_3} \sum_{k^*=0}^{\infty} \varpi_{k^*,(\beta_1+\beta_2+\beta_3)} \Psi_*^{(1)}(k, \omega) - \sum_{k=0}^{\infty} \varpi_{k,(\beta_1+\beta_3)} E(Y_{1,k}^1) \end{aligned}$$

$$\times \sum_{k=0}^{\infty} \varpi_{k,(\beta_2+\beta_3)} E(Y_{2,k}^{1})], \tag{28}$$

where $\vartheta(X_1, X_2) = \frac{Cov(X_1,X_2)}{\sigma_{X_1}\sigma_{X_2}}$, $\sigma_{X_i} = +\sqrt{Var(X_i)}$ for $i = 1, 2$.

3.4. Conditional densities

A conditional distribution is a statistical concept that describes the probability of an event occurring, given that another event has already occurred. This type of distribution is useful for examining the probability of outcomes that are dependent on a specific event. It enables us to predict the probability of an event based on the probability of a different event. The conditional distribution can also be used to analyze the relationship between two variables. This type of analysis can help us understand how a change. Assume the RVr $\mathbf{X}$ has the BOGE-G class, the conditional PDF of X_i given $X_j = x_j$, $(i,\ j = 1, 2, i \neq j)$, can be expressed as

$$f_{X_i|X_j}(x_i \mid x_j) = \begin{cases} f^{(1)}_{X_i|X_j}(x_i \mid x_j) & \text{if } 0 < x_i < x_j < \infty \\ f^{(2)}_{X_i|X_j}(x_i \mid x_j) & \text{if } 0 < x_j < x_i < \infty \\ f^{(3)}_{X_i|X_j}(x_i \mid x_j) & \text{if } 0 < x_i = x_j < \infty, \end{cases} \tag{29}$$

where

$$f^{(1)}_{X_i|X_j}(x_i \mid x_j) = \frac{\alpha\beta_j\left(\beta_i+\beta_3\right)g(x_i;\xi)e^{-\alpha\frac{G(x_i;\xi)}{\overline{G}(x_i;\xi)}}\left(1-e^{-\alpha\frac{G(x_i;\xi)}{\overline{G}(x_i;\xi)}}\right)^{\beta_i+\beta_3-1}}{\left(\beta_j+\beta_3\right)\left(\overline{G}(x_i;\xi)\right)^2\left(1-e^{-\alpha\frac{G(x_j;\xi)}{\overline{G}(x_j;\xi)}}\right)^{\beta_3}},$$

$$f^{(2)}_{X_i|X_j}(x_i \mid x_j) = \frac{\alpha\beta_i}{\left(\overline{G}(x_i;\xi)\right)^2} g(x_i;\xi)e^{-\alpha\frac{G(x_i;\xi)}{\overline{G}(x_i;\xi)}}\left(1-e^{-\alpha\frac{G(x_i;\xi)}{\overline{G}(x_i;\xi)}}\right)^{\beta_i-1},$$

and

$$f^{(3)}_{X_i|X_j}(x_i \mid x_j) = \frac{\beta_3}{\beta_j+\beta_3}\left(1-e^{-\alpha\frac{G(x_i;\xi)}{\overline{G}(x_i;\xi)}}\right)^{\beta_i}.$$

Equation (29) can be derived by substituting from Equations (10) and (18) in the following relation

$$f_{X_i|X_j}(x_i \mid x_j) = \frac{\text{joint}(X_i, X_j) \text{ density at } (x_i, x_j)}{\text{marginal } X_j \text{ density at } x_j};\ \ (i \neq j = 1, 2). \tag{30}$$

3.5. Median correlation coefficient (MCC)

Domma [52] presented the MCC, say M_{X_1,X_2}, as a form

$$M_{X_1,X_2} = 4F_{X_1,X_2}(M_{X_1}, M_{X_2}) - 1, \tag{31}$$

where M_{X_1} and M_{X_2} denote the median of X_1 and X_2, respectively. If $X_1 \sim OGE-G(\alpha, \xi, \beta_1+\beta_3)$ and $X_2 \sim OGE-G(\alpha, \xi, \beta_2+\beta_3)$, then

$$M_{X_i} = Q_G\left(\frac{-\log\left[1-U^{\frac{1}{\beta_i+\beta_3}}\right]}{\alpha-\log\left[1-U^{\frac{1}{\beta_i+\beta_3}}\right]}\right);\quad i = 1, 2, \tag{32}$$

where U has a uniform $U(0,1)$ distribution, and $Q_G(.) = G^{-1}(.)$ the baseline quantile function. So, the MCC between X_1 and X_2 at $U = 0.5$ can be formulated as

$$M_{X_1,X_2} = \begin{cases} 4F_{OGE-G}\left(M_{X_2};\alpha,\xi,\beta_2\right)\times F_{OGE-G}\left(M_{X_1};\alpha,\xi,\beta_1+\beta_3\right)-1 & \text{if } x_1 < x_2 \\ 4F_{OGE-G}\left(M_{X_1};\alpha,\xi,\beta_1\right)\times F_{OGE-G}\left(M_{X_2};\alpha,\xi,\beta_2+\beta_3\right)-1 & \text{if } x_1 > x_2. \end{cases} \tag{33}$$

3.6. *Joint reliability function*

Survival functions are an important tool in statistics, with many applications in understanding the behavior of living organisms. In essence, a survival function is a mathematical expression of the probability of an organism surviving until a certain time, given a certain set of conditions. The form of the survival function can vary greatly, depending on the type of organism being studied. In contrast, a survival function for a species of insect may only consider the rate of reproduction. For any organism, the survival function is a reflection of the environment in which it exists. Factors such as climate, competition, and predation all play a role in determining the survival rate. As a result, survival functions can provide valuable insight into the ecology of a species. In addition to its use in ecology, the survival function is also used in economics and other fields. Assume (X_1, X_2) be two dimensional RVr with CDF $F_{X_1,X_2}(x_1, x_2)$, and the marginal functions are $F_{X_1}(x_1)$ and $F_{X_2}(x_2)$. Then, the joint RF can be expressed as

$$R_{X_1,X_2}(x_1,x_2) = 1 - F_{X_1}(x_1) - F_{X_2}(x_2) + F_{X_1,X_2}(x_1,x_2). \tag{34}$$

Assume the RVr $\mathbf{X}$ has the BOGE-G class. Then, the joint RF can be formulated as

$$R_{X_1,X_2}(x_1,x_2) = \begin{cases} R_1(x_1,x_2) & \text{if } 0 < x_1 < x_2 < \infty \\ R_2(x_1,x_2) & \text{if } 0 < x_2 < x_1 < \infty \\ R_3(x,x) & \text{if } 0 < x_1 = x_2 = x < \infty, \end{cases} \tag{35}$$

where

$$R_1(x_1,x_2) = 1 - F_{OGE-G}(x_1;\alpha,\xi,\beta_1+\beta_3) - F_{OGE-G}(x_2;\alpha,\xi,\beta_2+\beta_3) + \\ F_{OGE-G}(x_2;\alpha,\xi,\beta_2)\times F_{OGE-G}(x_1;\alpha,\xi,\beta_1+\beta_3),$$

$$R_2(x_1,x_2) = 1 - F_{OGE-G}(x_1;\alpha,\xi,\beta_1+\beta_3) - F_{OGE-G}(x_2;\alpha,\xi,\beta_2+\beta_3) + \\ F_{OGE-G}(x_1;\alpha,\xi,\beta_1)\times F_{OGE-G}(x_2;\alpha,\xi,\beta_2+\beta_3),$$

and

$$R_3(x,x) = 1 - F_{OGE-GF}(x;\alpha,\xi,\beta_1+\beta_3) - F_{OGE-GF}(x;\alpha,\xi,\beta_2+\beta_3) + \\ F_{OGE-GF}(x;\alpha,\xi,\beta_1+\beta_2+\beta_3).$$

3.7. *Joint hazard rate function: Marginal distributions with asymptotic*

Assume (X_1, X_2) be two dimensional RV with joint PDF $f_{X_1,X_2}(x_1, x_2)$, and joint RF $R_{X_1,X_2}(x_1, x_2)$. Basu [54] defined the joint HRF as follows

$$h_{X_1,X_2}(x_1,x_2) = \frac{f_{X_1,X_2}(x_1,x_2)}{R_{X_1,X_2}(x_1,x_2)}. \tag{36}$$

Based on Equation (36), the joint HRF of the BOGE-G class can be listed as

$$h_{X_1,X_2}(x_1,x_2) = \begin{cases} h_1(x_1,x_2) & \text{if } 0 < x_1 < x_2 < \infty \\ h_2(x_1,x_2) & \text{if } 0 < x_2 < x_1 < \infty \\ h_3(x,x) & \text{if } 0 < x_1 = x_2 = x < \infty, \end{cases} \tag{37}$$

where

$$h_1(x_1,x_2) = f_{OGE-G}\,(x_2;\alpha,\xi,\beta_2) \times f_{OGE-G}\,(x_1;\alpha,\xi,\beta_1+\beta_3) \times \\ [1 - F_{OGE-G}(x_1;\alpha,\xi,\beta_1+\beta_3) - F_{OGE-G}(x_2;\alpha,\xi,\beta_2+\beta_3) + \\ F_{OGE-G}(x_2;\alpha,\xi,\beta_2) \times F_{OGE-G}(x_1;\alpha,\xi,\beta_1+\beta_3)]^{-1},$$

$$h_2(x_1,x_2) = f_{OGE-G}\,(x_1;\alpha,\xi,\beta_1) \times f_{OGE-G}\,(x_2;\alpha,\xi,\beta_2+\beta_3) \times \\ [1 - F_{OGE-G}(x_1;\alpha,\xi,\beta_1+\beta_3) - F_{OGE-G}(x_2;\alpha,\xi,\beta_2+\beta_3) + \\ F_{OGE-G}(x_1;\alpha,\xi,\beta_1) \times F_{OGE-G}(x_2;\alpha,\xi,\beta_2+\beta_3)]^{-1},$$

and

$$h_3(x,x) = \frac{\beta_3}{\beta_1+\beta_2+\beta_3} f_{OGE-G}\,(x;\alpha,\xi,\beta_1+\beta_2+\beta_3) \times \\ [1 - F_{OGE-G}(x;\alpha,\xi,\beta_1+\beta_3) - F_{OGE-G}(x;\alpha,\xi,\beta_2+\beta_3) + \\ F_{OGE-G}(x;\alpha,\xi,\beta_1+\beta_2+\beta_3)]^{-1}.$$

Further, the marginal functions of the joint HRF, say $h_i(x_i); i = 1,2$, of the BOGE-G class can be reportd as

$$h_i(x_i) = \frac{f_{OGE-GF}\,(x_i;\alpha,\xi,\beta_i+\beta_3)}{1 - F_{OGE-GF}\,(x_i;\alpha,\xi,\beta_i+\beta_3)};\ i = 1,2. \tag{38}$$

The asymptotic of Equation (38) as $G(x_i;\xi) \to 0$ can be listed as

$$h_i(x_i) \sim (\beta_i+\beta_3)\,\alpha^{\beta_i+\beta_3} g(x_i;\xi)\,[G(x_i;\xi)]^{\beta_i+\beta_3-1}. \tag{39}$$

Also, the asymptotic of Equation (38) as $x_i \to \infty$ is given by

$$h_i(x_i) \sim \frac{\alpha}{\left(\overline{G}(x_i;\xi)\right)^2} g(x_i;\xi). \tag{40}$$

4. Bivariate Odd Generalized Exponential-Gompertz Distribution

The Gompertz distribution is a continuous probability distribution that is often used to model the distribution of time until an event occurs, particularly in survival analysis and reliability engineering. It is named after the British mathematician Benjamin Gompertz, who introduced it in the early 19th century. The Gompertz distribution is characterized by its shape, which features exponential growth in hazard rate as time progresses. Assume

$$G(x;a,b) = 1 - e^{-\frac{a}{b}(e^{bx}-1)};\ a > 0, b > 0, \tag{41}$$

is the CDF of the Gompertz RV. The joint CDF of the bivariate generalized exponential-Gompertz (BOGEGz) is formulated using Equations (2), (8) and (41). The joint PDF can be extracted using Equations (3), (10) and (41). The joint RF can be proposed using Equations (2), (35) and (41). Similarly, the joint HRF can be derived using Equations (3), (37) and (41). As an example, the joint CDF of the BOGEGz distribution can be formulated as

$$F_{X_1,X_2}(x_1,x_2) = \left(1 - e^{-\alpha\left[e^{\frac{a}{b}\left(e^{bz}-1\right)}-1\right]}\right)^{\beta_3} \prod_{i=1}^{2} \left(1 - e^{-\alpha\left[e^{\frac{a}{b}\left(e^{bx_i}-1\right)}-1\right]}\right)^{\beta_i}. \tag{42}$$

Figures 1, 2, and 3 show some plots of the joint PDF, joint HRF, and joint RF of the BOGEGz model based on various values of the model parameters.

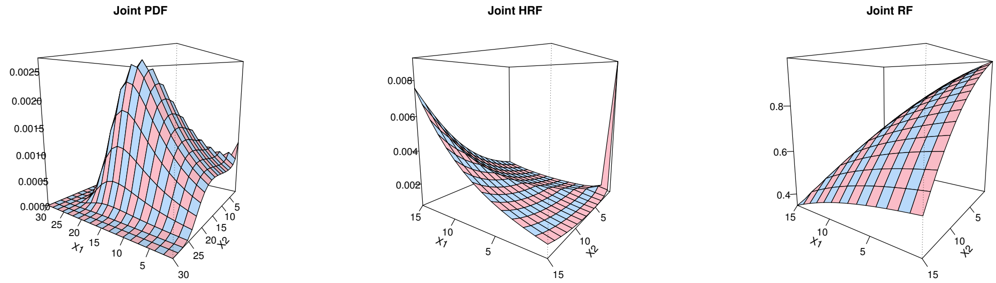

Figure 1. Some distribution characterization functions for specific parameters
$\beta_1 = 0.5,\ \beta_2 = 0.5,\ \beta_3 = 0.5,\ \alpha = 0.3,\ a = 0.05$, and $b = 0.06$.

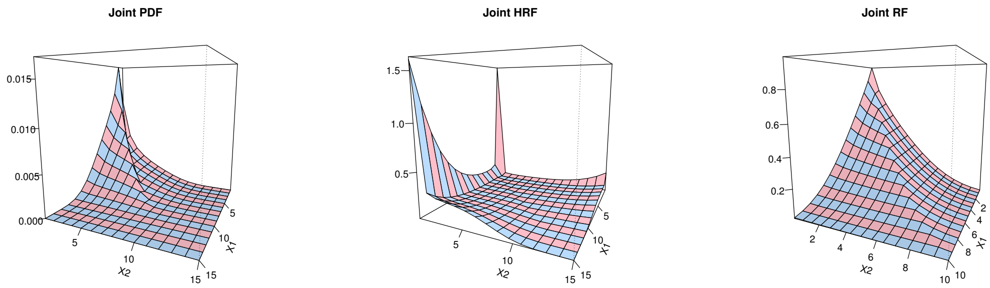

Figure 2. Some distribution characterization functions for specific parameters
$\beta_1 = 0.1,\ \beta_2 = 0.3,\ \beta_3 = 0.5,\ \alpha = 0.5,\ a = 0.2$, and $b = 0.02$.

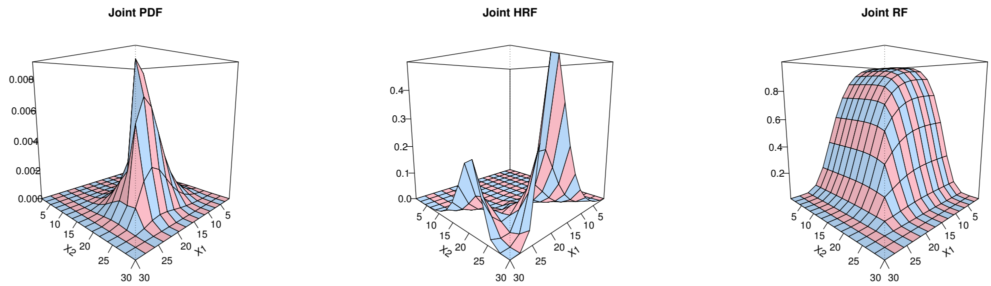

Figure 3. Some distribution characterization functions for specific parameters
$\beta_1 = 1.8,\ \beta_2 = 0.5,\ \beta_3 = 1.5,\ \alpha = 0.3,\ a = 0.05$, and $b = 0.06$.

As can be noted, the joint PDF of the BOGEGz model can be utilized to model asymmetric data. Further, it can be either decreasing or unimodal-shaped. The joint HRF can take various shapes. Thus, the BOGEGz distribution can be utilized as a probabilistic tool to model and discuss different types of failure rates.

5. A General Mathematical Formula for Parameters Estimation

In this segment, the unknown parameters of the BOGE-G class are estimated via the maximum likelihood (ML) approach. The ML estimation is a statistical technique used to estimate parameters in a model. It is widely employed in the fields of engineering and science, as it provides an efficient way of estimating parameters based on observed data. In essence, the technique works by calculating a probability distribution that best fits the observed data. Once this distribution has been determined, the ML estimates of the parameters are calculated by maximizing the likelihood function. This ML estimate is then used to make predictions. The technique is advantageous because it is relatively easy to implement and can yield accurate results, even when the data is heterogeneous and noisy. Furthermore, it is a powerful tool for testing hypotheses as it can be used to determine the significance of certain parameters and provide insight into the underlying structure of the data. Overall, ML estimation is a useful tool for making sense of complex data. Its implementation is efficient and can yield accurate results. It is thus a valuable tool for any researcher seeking to gain insight into the structure. Consider (x_{11}, x_{21}), (x_{12}, x_{22}),..., (x_{1n}, x_{2n}) is a random sample of size n from the BOGE-G family. The following notation can be utilized: $I_1 = \{x_{1i} < x_{2i}\}$, $I_2 = \{x_{1i} > x_{2i}\}$, $I_3 = \{x_{1i} = x_{2i} = x_i\}$, $I = I_1 \cup I_2 \cup I_3$, $|I_1| = n_1$, $|I_2| = n_2$, $|I_3| = n_3$, and $|I| = n_1 + n_2 + n_3 = n$. Based on the observations, the likelihood function, say $l(\mathbf{\Omega})$, of this sample can be proposed as

$$l(\mathbf{\Omega}) = \prod_{i=1}^{n_1} f_1(x_{1i}, x_{2i}) \prod_{i=1}^{n_2} f_2(x_{1i}, x_{2i}) \prod_{i=1}^{n_3} f_3(x_i, x_i). \tag{43}$$

Substituting from Equation (10) into Equation (43), the log-likelihood function, say $L(\mathbf{\Omega})$, can be written as

$$\begin{aligned}
L(\mathbf{\Omega}) = {} & n_1 \ln\left[\alpha^2 \beta_2 (\beta_1 + \beta_3)\right] - \alpha \sum_{i=1}^{n_1} \frac{G(x_{1i};\xi)}{\overline{G}(x_{1i};\xi)} + (\beta_1 + \beta_3 - 1) \sum_{i=1}^{n_1} \ln\left[1 - e^{-\alpha \frac{G(x_{1i};\xi)}{\overline{G}(x_{1i};\xi)}}\right] \\
& + \sum_{i=1}^{n_1} \ln\left[g(x_{1i};\xi)\right] - 2\sum_{i=1}^{n_1} \ln\left[\overline{G}(x_{1i};\xi)\right] + \sum_{i=1}^{n_1} \ln\left[g(x_{2i};\xi)\right] - 2\sum_{i=1}^{n_1} \ln\left[\overline{G}(x_{2i};\xi)\right] \\
& - \alpha \sum_{i=1}^{n_1} \frac{G(x_{2i};\xi)}{\overline{G}(x_{2i};\xi)} + (\beta_2 - 1) \sum_{i=1}^{n_1} \ln\left[1 - e^{-\alpha \frac{G(x_{2i};\xi)}{\overline{G}(x_{2i};\xi)}}\right] + n_2 \ln\left[\alpha^2 \beta_1 (\beta_2 + \beta_3)\right] \\
& - \alpha \sum_{i=1}^{n_2} \frac{G(x_{1i};\xi)}{\overline{G}(x_{1i};\xi)} + (\beta_1 - 1) \sum_{i=1}^{n_2} \ln\left[1 - e^{-\alpha \frac{G(x_{1i};\xi)}{\overline{G}(x_{1i};\xi)}}\right] + \sum_{i=1}^{n_2} \ln\left[g(x_{1i};\xi)\right] \\
& - 2\sum_{i=1}^{n_2} \ln\left[\overline{G}(x_{1i};\xi)\right] + \sum_{i=1}^{n_2} \ln\left[g(x_{2i};\xi)\right] - 2\sum_{i=1}^{n_2} \ln\left[\overline{G}(x_{2i};\xi)\right] - \alpha \sum_{i=1}^{n_2} \frac{G(x_{2i};\xi)}{\overline{G}(x_{2i};\xi)} \\
& + (\beta_2 + \beta_3 - 1) \sum_{i=1}^{n_2} \ln\left[1 - e^{-\alpha \frac{G(x_{2i};\xi)}{\overline{G}(x_{2i};\xi)}}\right] + \sum_{i=1}^{n_3} \ln\left[g(x_i;\xi)\right] - 2\sum_{i=1}^{n_3} \ln\left[\overline{G}(x_i;\xi)\right] \\
& + n_3 \ln\left[\alpha \beta_3\right] - \alpha \sum_{i=1}^{n_3} \frac{G(x_i;\xi)}{\overline{G}(x_i;\xi)} + (\beta_1 + \beta_2 + \beta_3 - 1) \sum_{i=1}^{n_3} \ln\left[1 - e^{-\alpha \frac{G(x_i;\xi)}{\overline{G}(x_i;\xi)}}\right].
\end{aligned} \tag{44}$$

The first partial derivatives of Equation (44) with respect to β_1, β_2, β_3, α and ξ_k $(k = 1, 2, 3, ...)$ can be expressed as

$$\begin{aligned}
\frac{\partial L(\mathbf{\Omega})}{\partial \beta_1} = {} & \frac{n_1}{\beta_1 + \beta_3} + \sum_{i=1}^{n_1} \ln\left[1 - e^{-\alpha \frac{G(x_{1i};\xi)}{\overline{G}(x_{1i};\xi)}}\right] + \frac{n_2}{\beta_1} + \sum_{i=1}^{n_2} \ln\left[1 - e^{-\alpha \frac{G(x_{1i};\xi)}{\overline{G}(x_{1i};\xi)}}\right] \\
& + \sum_{i=1}^{n_3} \ln\left[1 - e^{-\alpha \frac{G(x_i;\xi)}{\overline{G}(x_i;\xi)}}\right],
\end{aligned} \tag{45}$$

$$\begin{aligned}
\frac{\partial L(\mathbf{\Omega})}{\partial \beta_2} = {} & \frac{n_1}{\beta_2} + \sum_{i=1}^{n_1} \ln\left[1 - e^{-\alpha \frac{G(x_{2i};\xi)}{\overline{G}(x_{2i};\xi)}}\right] + \frac{n_2}{\beta_2 + \beta_3} + \sum_{i=1}^{n_2} \ln\left[1 - e^{-\alpha \frac{G(x_{2i};\xi)}{\overline{G}(x_{2i};\xi)}}\right] \\
& + \sum_{i=1}^{n_3} \ln\left[1 - e^{-\alpha \frac{G(x_i;\xi)}{\overline{G}(x_i;\xi)}}\right],
\end{aligned} \tag{46}$$

$$\frac{\partial L(\mathbf{\Omega})}{\partial \beta_3} = \frac{n_1}{\beta_1+\beta_3} + \sum_{i=1}^{n_1} \ln\left[1 - e^{-\alpha \frac{G(x_{1i};\xi)}{\overline{G}(x_{1i};\xi)}}\right] + \frac{n_2}{\beta_2+\beta_3} + \sum_{i=1}^{n_2} \ln\left[1 - e^{-\alpha \frac{G(x_{2i};\xi)}{\overline{G}(x_{2i};\xi)}}\right]$$
$$+ \frac{n_3}{\beta_3} + \sum_{i=1}^{n_3} \ln\left[1 - e^{-\alpha \frac{G(x_i;\xi)}{\overline{G}(x_i;\xi)}}\right], \tag{47}$$

$$\frac{\partial L(\mathbf{\Omega})}{\partial \alpha} = \frac{2n_1}{\alpha} - \sum_{i=1}^{n_1} \frac{G(x_{1i};\xi)}{\overline{G}(x_{1i};\xi)} + (\beta_1+\beta_3-1)\sum_{i=1}^{n_1} \frac{G(x_{1i};\xi)}{\overline{G}(x_{1i};\xi)A(x_{1i};\xi)} - \sum_{i=1}^{n_1} \frac{G(x_{2i};\xi)}{\overline{G}(x_{2i};\xi)}$$
$$+ (\beta_2-1)\sum_{i=1}^{n_1} \frac{G(x_{2i};\xi)}{\overline{G}(x_{2i};\xi)A(x_{2i};\xi)} + \frac{2n_2}{\alpha} - \sum_{i=1}^{n_2} \frac{G(x_{1i};\xi)}{\overline{G}(x_{1i};\xi)} - \sum_{i=1}^{n_2} \frac{G(x_{2i};\xi)}{\overline{G}(x_{2i};\xi)}$$
$$+ (\beta_1-1)\sum_{i=1}^{n_2} \frac{G(x_{1i};\xi)}{\overline{G}(x_{1i};\xi)A(x_{1i};\xi)} + (\beta_2+\beta_3-1)\sum_{i=1}^{n_2} \frac{G(x_{2i};\xi)}{\overline{G}(x_{2i};\xi)A(x_{2i};\xi)}$$
$$+ \frac{n_3}{\alpha} - \sum_{i=1}^{n_3} \frac{G(x_i;\xi)}{\overline{G}(x_i;\xi)} + (\beta_1+\beta_2+\beta_3-1)\sum_{i=1}^{n_3} \frac{G(x_i;\xi)}{\overline{G}(x_i;\xi)A(x_i;\xi)}, \tag{48}$$

and

$$\frac{\partial L}{\partial \xi_k} = -\alpha\sum_{i=1}^{n_1} U(x_{1i};\xi) + (\beta_1+\beta_3-1)\sum_{i=1}^{n_1} \frac{\alpha U(x_{1i};\xi)}{A(x_{1i};\xi)} + 2\sum_{i=1}^{n_1} \overline{G}(x_{1i};\xi)U(x_{1i};\xi)$$
$$+ \sum_{i=1}^{n_1} V(x_{1i};\xi) + \sum_{i=1}^{n_1} V(x_{2i};\xi) + 2\sum_{i=1}^{n_1} \overline{G}(x_{2i};\xi)U(x_{2i};\xi) + (\beta_2-1)\sum_{i=1}^{n_1} \frac{\alpha U(x_{2i};\xi)}{A(x_{2i};\xi)}$$
$$-\alpha\sum_{i=1}^{n_1} U(x_{2i};\xi) + \sum_{i=1}^{n_2} V(x_{1i};\xi) + 2\sum_{i=1}^{n_2} \overline{G}(x_{1i};\xi)U(x_{1i};\xi) + (\beta_1-1)\sum_{i=1}^{n_2} \frac{\alpha U(x_{1i};\xi)}{A(x_{1i};\xi)}$$
$$-\alpha\sum_{i=1}^{n_2} U(x_{1i};\xi) + \sum_{i=1}^{n_2} V(x_{2i};\xi) + 2\sum_{i=1}^{n_2} \overline{G}(x_{2i};\xi)U(x_{2i};\xi) - \alpha\sum_{i=1}^{n_2} U(x_{2i};\xi)$$
$$+ (\beta_2+\beta_3-1)\sum_{i=1}^{n_2} \frac{\alpha U(x_{2i};\xi)}{A(x_{2i};\xi)} + \sum_{i=1}^{n_3} V(x_i;\xi) + 2\sum_{i=1}^{n_3} \overline{G}(x_i;\xi)U(x_i;\xi)$$
$$+ (\beta_1+\beta_2+\beta_3-1)\sum_{i=1}^{n_3} \frac{\alpha U(x_i;\xi)}{A(x_i;\xi)} - \alpha\sum_{i=1}^{n_3} U(x_i;\xi), \tag{49}$$

where

$$U(x;\xi) = \frac{[G'(x;\xi)]_{\xi_k}}{\left[\overline{G}(x;\xi)\right]^2}, \quad V(x;\xi) = \frac{[g'(x;\xi)]_{\xi_k}}{g(x;\xi)},$$

and

$$[\Theta'(x;\xi)]_{\xi_k} = \partial\Theta(x;\xi)/\partial\xi_k.$$

By equating the Equations (45-49) with zeros, we get the non-linear regular equations. Therefore, the solution must be obtained numerically.

6. Behavior of Estimator: A Simulation Study

A simulation study for parameter estimation is a method of finding the optimal values of model inputs that match a set of reference data. This can be done by formulating the problem as a maximum likelihood optimization problem and using a parameter estimation study step in a software tool such as R. A simulation study can help to gain insight into the effects of model inputs on the objective function, as well as to perform curve fitting or data fitting applications. A simulation study for parameter estimation may depend on factors such as the experimental design, the error structure, and the sample size.

The behaviour of the probability model estimators refers to how the estimators perform in terms of their properties and characteristics. An estimator is a rule or a method for obtaining an estimate of an unknown parameter based on observed data. An estimate is a specific value or a range of values obtained by applying the estimator to a sample of data. Some of the properties that are used to evaluate and compare estimators are: Bias, variance, mean squared error, mean relative errors, consistency, efficiency, and sufficiency. Different estimators may have different behaviours depending on the model, the data, and the sample size. For example, some estimators may be biased but consistent, while others may be unbiased but inconsistent. Some estimators may be more efficient or more robust than others. Some estimators may be derived from graphical methods such as probability plots, while others may be derived from analytical methods such as maximum likelihood or method of moments. In this section, we assess the performance of the maximum likelihood estimation (MLE) technique with respect to sample size n using R software. Simulations are discussed based on different schemes as follows:

1. Generate $N = 1000$ samples of various sample sizes $n_i; i = 1, 2, 3, 4, 5$ from the BOGEGz distribution as follows

- Scheme I: $\beta_1 = 0.3$, $\beta_2 = 0.8$, $\beta_3 = 0.1$, $a = 0.3$, $b = 0.2$, $\alpha = 0.3| n_1 = 20$, $n_2 = 50$, $n_3 = 150$, $n_4 = 300$, $n_5 = 500$.
- Scheme II: $\beta_1 = 0.4$, $\beta_2 = 0.7$, $\beta_3 = 0.2$, $a = 0.5$, $b = 0.4$, $\alpha = 0.4| n_1 = 20$, $n_2 = 50$, $n_3 = 150$, $n_4 = 300$, $n_5 = 500$.
- Scheme III: $\beta_1 = 0.5$, $\beta_2 = 0.6$, $\beta_3 = 0.3$, $a = 0.7$, $b = 0.6$, $\alpha = 0.2| n_1 = 20$, $n_2 = 50$, $n_3 = 150$, $n_4 = 300$, $n_5 = 500$.
- Scheme IV: $\beta_1 = 0.6$, $\beta_2 = 0.5$, $\beta_3 = 0.4$, $a = 0.9$, $b = 0.8$, $\alpha = 0.5| n_1 = 20$, $n_2 = 50$, $n_3 = 150$, $n_4 = 300$, $n_5 = 500$.
- Scheme V: $\beta_1 = 0.7$, $\beta_2 = 0.4$, $\beta_3 = 0.5$, $a = 1.5$, $b = 1.4$, $\alpha = 0.6| n_1 = 20$, $n_2 = 50$, $n_3 = 150$, $n_4 = 300$, $n_5 = 500$.
- Scheme VI: $\beta_1 = 0.8$, $\beta_2 = 0.3$, $\beta_3 = 0.6$, $a = 2.0$, $b = 1.8$, $\alpha = 0.8| n_1 = 20$, $n_2 = 50$, $n_3 = 150$, $n_4 = 300$, $n_5 = 500$.

2. Compute the MLE for the 1000 samples, say $\widehat{\boldsymbol{\Omega}}_j$ for $j = 1, 2, ..., 1000$.

3. Calculate the bias, mean squared errors (MSE), mean relative errors (MRE) for $N = 1000$ samples.

4. Simulation results are reported in Tables 1, 2, and 3 and provided via Figures 4 and 5 for schemes I and II as an example. As can be noted, the bias approaches to zero when n increases. Similarly, both MSE and MRE approach to zero as n grows. These results reveal the unbiasedness, consistency, and efficiency properties of the MLE technique as n grows. Thus, we can conclude that the MLE technique works quite well under different sizes of samples.

Table 1. The empirical simulation results for schemes I and II.

		Scheme I			Scheme II		
Parameter	n	Bias	MSE	MRE	Bias	MSE	MRE
a	20	0.13628963	0.01952369	0.27223691	0.87136982	0.76096852	0.29098552
	50	0.08332697	0.00702369	0.16102369	0.54112560	0.29185002	0.17912202
	150	0.05120258	0.00224102	0.09896369	0.31201473	0.09737463	0.10408996
	300	0.03339722	0.00122147	0.06698555	0.22903692	0.05189303	0.07710369
	500	0.01903147	0.00061236	0.02901224	0.01301255	0.00086636	0.03263632
b	20	0.27663280	0.07502186	0.55410366	0.42336961	0.17489632	0.20401478
	50	0.15636996	0.02441559	0.31428859	0.25258893	0.06601255	0.17198203
	150	0.09641147	0.00923036	0.19936966	0.14619855	0.02017746	0.09811258
	300	0.06422395	0.00423974	0.12730478	0.10701236	0.01202982	0.07102585
	500	0.02498200	0.00062258	0.06602285	0.02303736	0.00097123	0.00552025
α	20	0.13930274	0.02112994	0.28130178	0.92412239	0.92410026	0.29128669
	50	0.09289630	0.00824139	0.18212703	0.59309980	0.59396691	0.19801588
	150	0.05329823	0.00396664	0.10622369	0.32714469	0.32703268	0.11102366
	300	0.03500274	0.00083255	0.07396691	0.02502369	0.02303369	0.05022558
	500	0.00666369	0.00003896	0.00092369	0.00336941	0.00410369	0.00603258
β_1	20	0.32412024	0.10423697	0.28442555	0.86032684	0.73103269	0.29032185
	50	0.20310369	0.04369951	0.13968222	0.50402369	0.25630361	0.19012553
	150	0.11299822	0.01402255	0.07774960	0.38442023	0.15203368	0.13022699
	300	0.08752332	0.00766369	0.02903259	0.20012025	0.09882690	0.08401225
	500	0.00977475	0.00141225	0.00422701	0.08410236	0.00403255	0.00640236
β_2	20	0.09901225	0.00963369	0.15831452	0.78013698	0.31336025	0.26302254
	50	0.05933674	0.00352556	0.11801255	0.47800236	0.22801259	0.16033662
	150	0.02630364	0.00067022	0.05133690	0.29702366	0.09112582	0.09302668
	300	0.01041147	0.00013625	0.02035594	0.19023664	0.04289033	0.04369956
	500	0.00536951	0.00003366	0.00612369	0.03482255	0.00141115	0.00710247
β_3	20	0.05410366	0.00270236	0.10856921	0.36266991	0.14096337	0.25044148
	50	0.03441259	0.00133694	0.07236695	0.23202238	0.05303668	0.15320369
	150	0.02110396	0.00044525	0.04201885	0.12710395	0.01703695	0.08574118
	300	0.01022369	0.00012036	0.03103368	0.09203699	0.00860125	0.06303699
	500	0.00210025	0.00007126	0.02088589	0.04612368	0.00101778	0.01385558

Table 2. The empirical simulation results for schemes III and IV.

		Scheme III			Scheme IV		
Parameter	n	Bias	MSE	MRE	Bias	MSE	MRE
a	20	0.24536629	0.12377416	0.38765541	0.76355409	0.53325985	0.37456721
	50	0.17255304	0.08402554	0.24353684	0.48397461	0.30755489	0.19554772
	150	0.11635899	0.02047651	0.19465469	0.19664829	0.16455793	0.10537780
	300	0.06654024	0.00746555	0.11984655	0.09387642	0.08845012	0.05564672
	500	0.02847476	0.00094544	0.06786541	0.02301255	0.00227455	0.01664820
b	20	0.24634654	0.06645791	0.34678224	0.55362378	0.33945672	0.36497671
	50	0.13982654	0.03456782	0.28465910	0.35031634	0.21095874	0.28466554
	150	0.08465543	0.00773547	0.17745441	0.20477618	0.11854665	0.18308574
	300	0.05344674	0.00284554	0.10663598	0.12094751	0.04578097	0.10446773
	500	0.01284650	0.00043708	0.04937510	0.03987645	0.00772965	0.03947655
α	20	0.16343680	0.04994613	0.32574781	0.88108774	0.72543404	0.55467194
	50	0.12464785	0.01076561	0.25346794	0.58355110	0.53409313	0.38947109
	150	0.08864509	0.00835461	0.18154908	0.37893744	0.34098412	0.24691049
	300	0.05491094	0.00109476	0.09814482	0.13540462	0.16534840	0.12004765
	500	0.00966344	0.00077846	0.00126491	0.07454002	0.08466551	0.00957623
β_1	20	0.77839746	0.32497404	0.41297763	0.63454709	0.49498751	0.58766294
	50	0.49234878	0.22476599	0.32773981	0.44109467	0.28499255	0.38756621
	150	0.23574976	0.13978563	0.20344621	0.27450941	0.18766094	0.23564095
	300	0.13560938	0.08946641	0.11376530	0.16554380	0.10847665	0.11645445
	500	0.05498313	0.02294865	0.07208731	0.08220947	0.02745652	0.03765001
β_2	20	0.33049875	0.12965864	0.16638912	0.38467659	0.22946641	0.32885765
	50	0.27455479	0.08834654	0.12994765	0.28476554	0.12044357	0.24634644
	150	0.21958762	0.02094764	0.08765093	0.20664862	0.08045567	0.19094862
	300	0.13694723	0.00834562	0.03389087	0.12059765	0.00947652	0.11043765
	500	0.08856553	0.00074513	0.00822544	0.04567773	0.00265455	0.05772545
β_3	20	0.23536841	0.13276734	0.24646904	0.45475513	0.29576105	0.37746515
	50	0.18641044	0.10455779	0.19277466	0.29763477	0.18565520	0.24468095
	150	0.13845652	0.06486539	0.14274550	0.18655402	0.10846524	0.15448824
	300	0.08846521	0.02094761	0.09451347	0.09330756	0.08456724	0.10025445
	500	0.00946513	0.00436650	0.02194620	0.00856652	0.00305976	0.01335985

Table 3. The empirical simulation results for schemes V and VI.

		Scheme V			Scheme VI		
Parameter	n	Bias	MSE	MRE	Bias	MSE	MRE
a	20	0.65374733	0.29837764	0.53883098	0.32945834	0.19464855	0.26029546
	50	0.32878624	0.13563804	0.28455792	0.26443793	0.10454385	0.20475194
	150	0.19436517	0.04562782	0.17355482	0.21494654	0.05472930	0.15026825
	300	0.10394864	0.00865301	0.08445378	0.12951524	0.00923645	0.09450345
	500	0.01046539	0.00184350	0.00845443	0.08454384	0.00104655	0.02045147
b	20	0.44236725	0.23703975	0.37455092	0.32749642	0.21955493	0.28496548
	50	0.36845609	0.16367840	0.31546802	0.27745839	0.17468365	0.23547819
	150	0.28456714	0.11038629	0.22648941	0.18458390	0.10458583	0.12045853
	300	0.17454472	0.05475482	0.13204642	0.10465373	0.02437483	0.08443703
	500	0.09465428	0.00309465	0.06465789	0.03964418	0.00104668	0.00745849
α	20	0.54025823	0.02112994	0.48376463	0.84648936	0.53829640	0.78294734
	50	0.41047654	0.00824139	0.37501864	0.68357490	0.38569365	0.61402695
	150	0.28288460	0.00396664	0.24093764	0.34253894	0.14036548	0.31045634
	300	0.13846521	0.00083255	0.11947641	0.19465843	0.08455638	0.15493549
	500	0.05341072	0.00003896	0.02094557	0.10465543	0.01046543	0.08586049
β_1	20	0.28665895	0.16389045	0.23309741	0.27468934	0.12538495	0.25496395
	50	0.20464198	0.12049652	0.16638901	0.21045578	0.08457852	0.19713054
	150	0.12792554	0.09443682	0.11947603	0.15384742	0.02264895	0.12053385
	300	0.08049862	0.01194354	0.04095756	0.08684965	0.00468245	0.05443840
	500	0.00443789	0.00345748	0.00395665	0.00248467	0.00064802	0.00084568
β_2	20	0.27640936	0.12948641	0.21946546	0.19846493	0.12057651	0.17434942
	50	0.18464508	0.07486554	0.13647848	0.14384094	0.08465925	0.13074657
	150	0.10386472	0.00836457	0.08465400	0.11094776	0.02058756	0.09469033
	300	0.04093691	0.00094758	0.03324573	0.08464862	0.00385408	0.04369956
	500	0.00773541	0.00004124	0.00553784	0.00438794	0.00029475	0.00285025
β_3	20	0.19476593	0.10465584	0.17483095	0.29478504	0.16459354	0.28450385
	50	0.12945730	0.01346485	0.10457842	0.20465785	0.11465833	0.19545845
	150	0.06458296	0.00448543	0.04356473	0.13056715	0.07484390	0.11056844
	300	0.00745682	0.00064244	0.00237554	0.07445930	0.00944683	0.05048664
	500	0.00045834	0.00008405	0.00013368	0.02045655	0.00074431	0.01758430

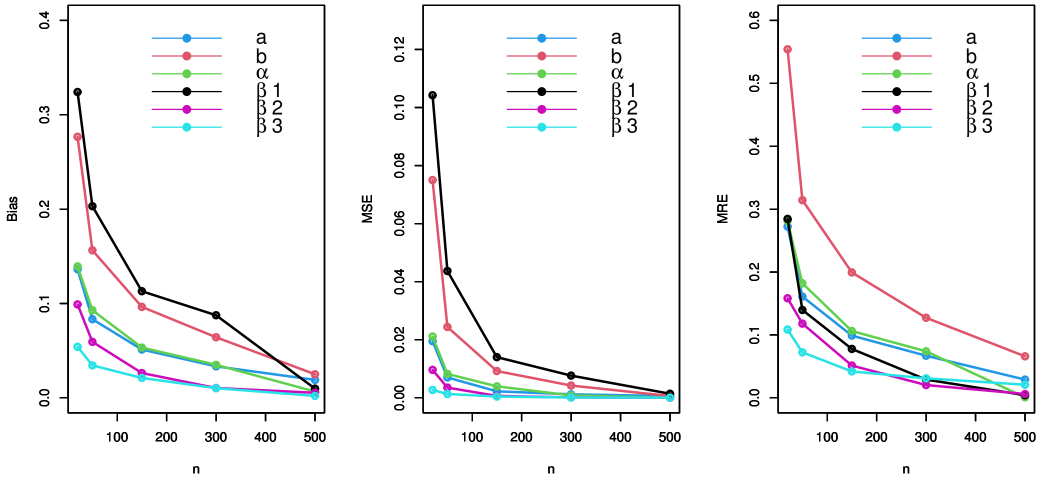

Figure 4. Visually display of the results reported in Table 1 for scheme I.

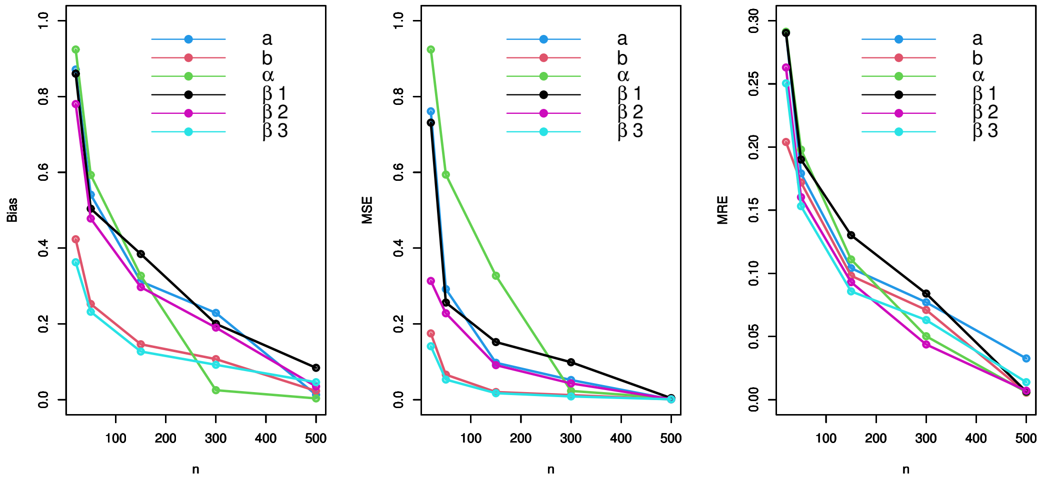

Figure 5. Visually display of the results listed in Table 1 for scheme II.

7. Data Fitting: A Comparative Study and Statistical Criteria

This data represents football (soccer) data of the UEFA Champion's League data for the year 2004-2005, and 2005-2006 (see, Meintanis [55]). The BOGEGz model and some well-known competitive distributions are considered to analyze and discuss this data. The competitive models are bivariate generalized exponential (BGE), bivariate Weibull exponential (BWE), bivariate Gumbel exponential (BGuE), bivariate Burr X exponential (BBUXE), Marshall-Olkin bivariate exponential (MOBE), bivariate generalized linear failure rate (BGLFR), bivariate exponentiated modified weibull extension (BEMWEx), bivariate generalized power Weibull (BGPW), bivariate exponentiated Weibull (BEW), bivariate Weibull (BW), bivariate exponetiated Weibull Gomperz (BEWGz), bivariate Burr X Gompertz (BBUXGz), bivariate Gompertz (BGz), and bivariate generalized Gompertez (BGGz), bivariate Gumbel Gompertz (BGuGz) models. The comparison is based on some statistical criteria, namely, negative L, Akaike information criterion (AIC), correct AIC (CAIC), Bayesian IC (BIC), and Hannan-Quinn information criterion (HQIC). Further, the Kolmogorov-Smirnov (K-S) distances and its corresponding p-values are calculated for the marginals. Figure 6 shows the bivariate data spread via scatter and box plots. It was found some extreme observations.

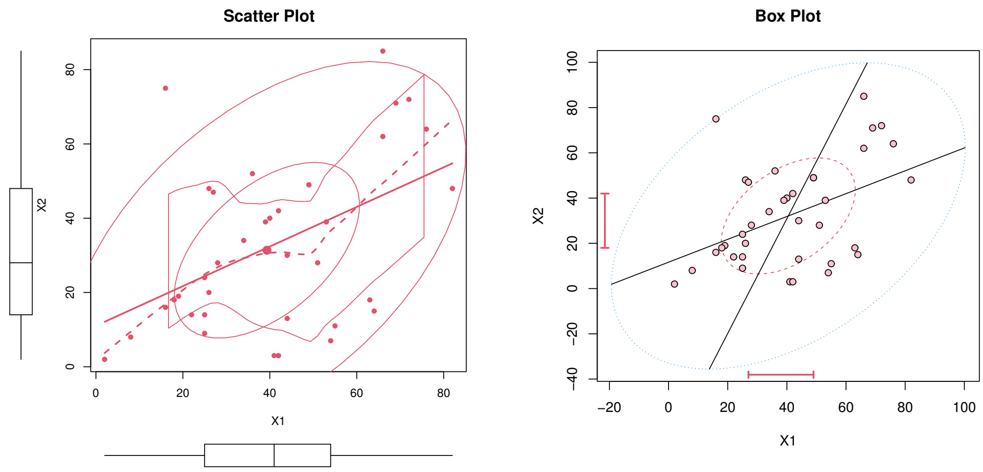

Figure 6. Diagram for real data.

For the marginals of the BOGEGz model, some non-parametric plots are sketched, namely, kernel density, quantile-quantile (QQ), box, and TTT plots. A kernel density plot, often referred to as a kernel density estimation (KDE) plot, is a non-parametric way to estimate the probability density function of a continuous random variable. It is used to visualize the distribution of data points in a dataset and provides a smoother representation compared to a histogram. Kernel density plots are useful for exploring the distribution of data, identifying modes or peaks, and getting a sense of the data's central tendency and spread. They are particularly valuable when dealing with continuous data or when you want to visualize the distribution of a single variable. For the QQ plot, it is a graphical tool used in statistics to assess whether a dataset follows a particular theoretical distribution, such as the normal distribution. It helps you visually compare the quantiles (percentiles) of your dataset against the quantiles of the chosen theoretical distribution. If the points in the QQ plot lie approximately along a straight line, it suggests that your data follows the chosen theoretical distribution. A box plot, also known as a box-and-whisker plot, is a graphical representation of the distribution of a dataset. It provides a summary of the data's central tendency, spread, and identifies potential outliers. Box plots are particularly useful for comparing the distributions of multiple datasets or for visualizing the distribution of a single dataset. In the context of testing or quality control, "Total Time in Test" (TTT) typically refers to the cumulative amount of time that an item or component is subjected to a testing process. TTT is often used in reliability testing, where the goal is to assess the durability or longevity of a product or system under various conditions. It helps in estimating the expected lifetime or failure rate of the item being tested. The results can be displayed in Figures 7, 8 and 9, and it turns out that the data is asymmetric and does not contain any outliers. Furthermore, margins have an increased shape failure rate.

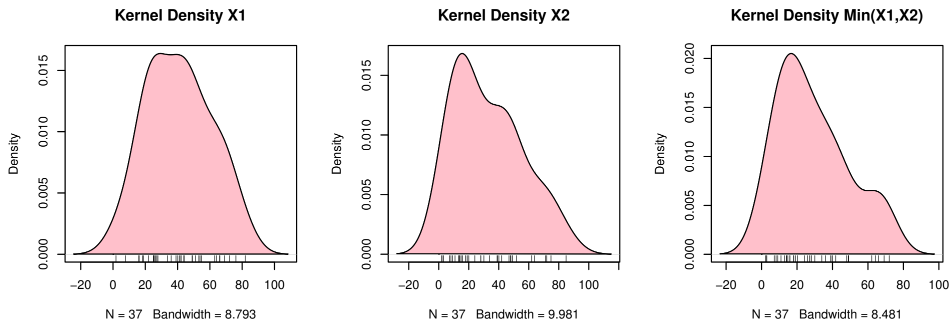

Figure 7. The kernel densities for the marginals.

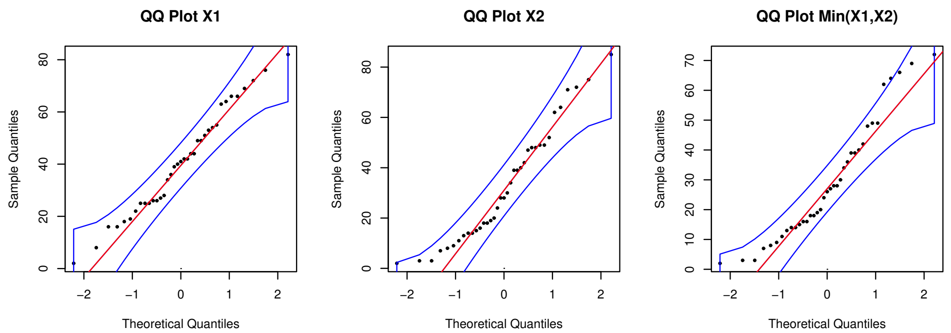

Figure 8. The QQ plots for the marginals.

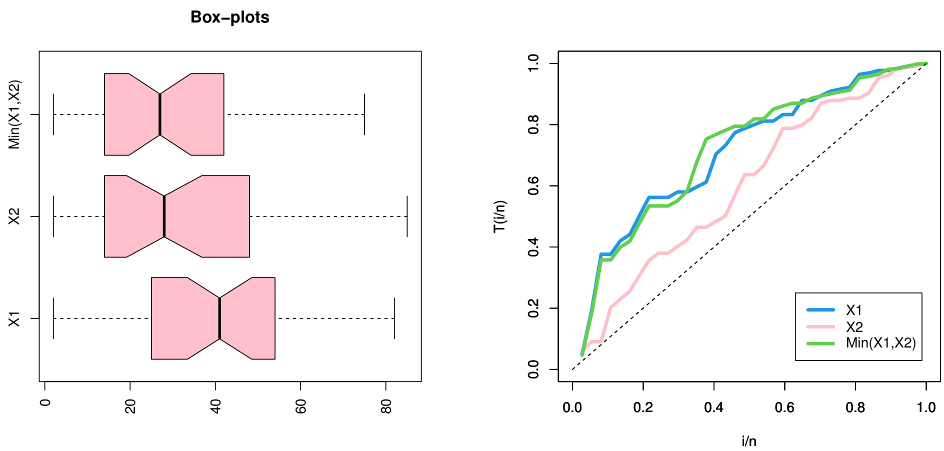

Figure 9. The box and TTT plots for the marginals.

Before discussing and analyzing bivariate data across the presented model, the margins of the proposed model should be empirically tested. Table 4 shows the goodness-of-fit (GOF) measures for the marginals.

Table 4. The GOF measures for the marginals.

RV	α	β	a	b	$-L$	KS	P-value
X_1	0.7515	1.3710	0.0163	0.0087	161.8372	0.0947	0.8944
X_2	2.7147	0.5461	0.0022	0.0311	164.4902	0.1066	0.7947
$\min(X_1, X_2)$	0.3554	0.6634	0.2831	0.0043	159.4071	0.0998	0.8545

As can be seen, the margin provides more suitable for real data (P-value > 0.05). The empirical results can be proved via Figures 10, 11, and 12.

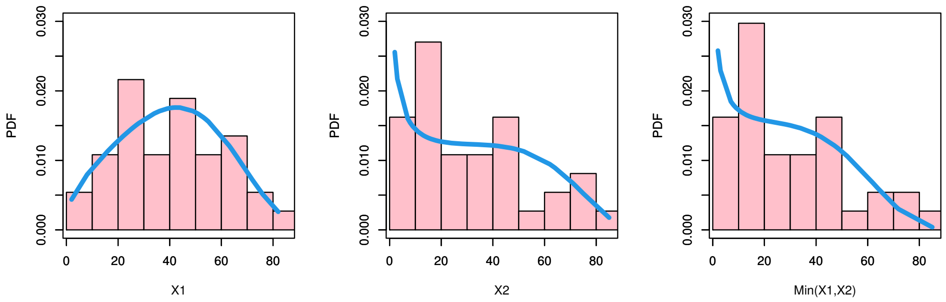

Figure 10. The estimated PDFs for the marginals.

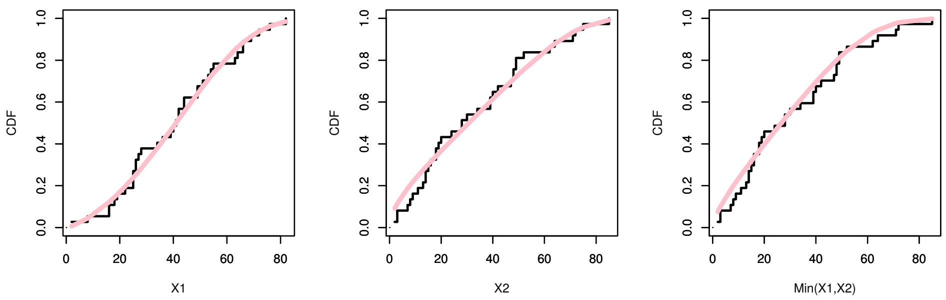

Figure 11. The empirical CDFs for the marginals.

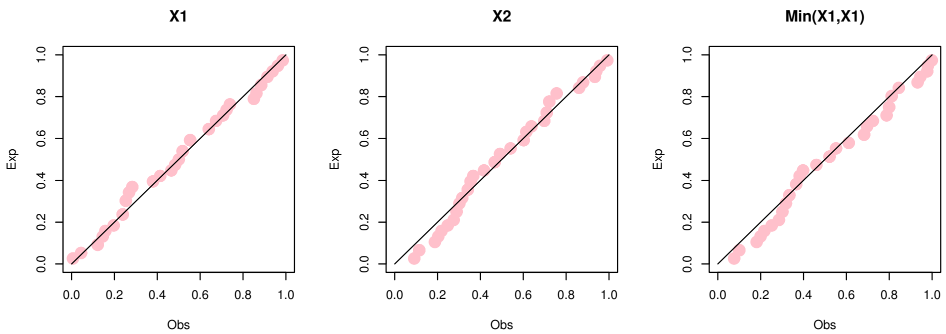

Figure 12. The probability-probability plots for the marginals.

Having established with evidence that the marginal distributions are capable of discussing the data, we can now test the BOGEGz distribution on that data. Tables 5 and 6 list the MLE and GOF measures for the BOGEGz distribution and some competitive distributions.

Table 5. The MLE for the tested distributions based on real data.

Model	$\widehat{\beta_1}$	$\widehat{\beta_2}$	$\widehat{\beta_3}$	$\widehat{a}$	$\widehat{b}$	$\widehat{\alpha}$	$\widehat{\gamma}$
BGE	1.5532	0.4993	1.1563	0.0393	−	−	−
BWE	0.1351	0.3024	0.2650	0.0251	−	−	−
BGuE	2.6784	0.9621	2.0653	5.0111	4.0814	−	−
BBUXE	0.3855	0.1362	0.3101	0.0122	−	−	−
MOBE	0.0121	0.0141	0.0221	−	−	−	−
BGLFR	0.4520	0.1567	0.3604	0.0002	0.0008	−	−
BEMWEx	0.1673	0.0613	0.1391	85.9183	4.5057	0.0254	−
BGPW	3.2294	1.9831	4.0840	0.0377	−	−	−
BEW	1.2269	0.3820	0.6611	0.0123	1.2683	−	−
BW	0.3974	0.2738	0.3389	0.0837	−	−	−
BEWGz	0.5477	0.1917	0.4446	0.4117	0.0795	0.0050	1.3587
BBUXGz	0.1320	0.1873	0.2014	0.0063	0.0154	−	−
BGz	0.0036	0.0023	0.0213	0.0406	−	−	−
BGGz	0.7428	0.2621	0.5984	0.0117	0.0294	−	−
BGuGz	0.5784	0.2044	0.4756	0.0092	0.0473	2.2784	−
BOGEGz	0.5189	0.5837	1.3435	0.0065	0.0126	2.6815	−

Table 6. The GOF measures for the tested models based on real data.

Model	$-L$	AIC	CAIC	BIC	HQIC
BGE	299.9142	607.7419	608.8894	614.2301	609.9163
BWE	291.1437	592.3103	594.2147	600.3223	595.1196
BGuE	297.8028	605.5696	607.5102	613.6426	608.4036
BBUXE	294.8127	597.6223	598.9336	604.0427	599.9744
MOBE	298.9362	607.9303	609.8102	615.9102	610.7330
BGLFR	296.8389	603.7339	605.6396	611.6896	606.5012
BEMWEx	294.0745	600.3396	603.1032	609.9325	603.7703
BGPW	344.8012	697.5412	698.8110	703.9036	699.8124
BEW	298.9336	607.9396	609.8396	615.8793	610.7399
BW	346.0174	700.0102	701.3145	706.4336	702.2892
BEWGz	294.6036	603.2112	607.1745	614.5107	607.2338
BBUXGz	301.1889	612.3892	614.3302	620.5289	615.2447
BGz	303.4996	614.9220	616.2036	621.4336	617.2302
BGGz	294.9170	599.8145	601.7163	607.9017	602.7147
BGuGz	294.2397	600.5336	603.3202	610.1230	603.9336
BOGEGz	283.7412	579.5336	582.3398	589.2112	582.9337

As can be noted, the BOGEGz model provides a better fit than the other tested models, because it has the smallest value among $-L$, AIC, CAIC, BIC, and HQIC. Based on the maximum likelihood estimators

(MLEs) of the BOGEGz parameres, the joint PDF, joint HRF, and joint RF are displayed in Figure 13.

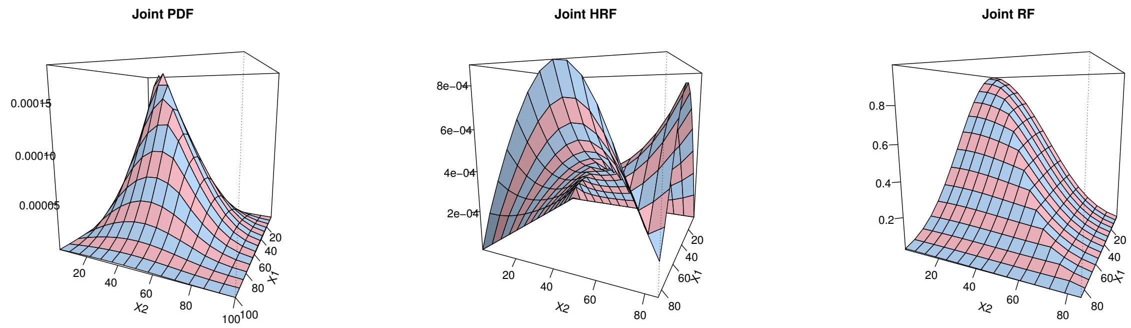

Figure 13. The joint PDF, joint HRF, and joint RF for the real data.

8. Concluding Remarks and Future Work

In this article, a new flexible bivariate generator of distributions has been introduced, in the so-called bivariate odd generalized exponential-G (BOGE-G) family. The marginal distributions of the BOGE class are OGE-G families. Both the joint CDF and the joint PDF of the BOGE-G family have simple forms; therefore, this new bivariate class can be easily applied in practice to model bivariate data constrained in the interval $(0, \infty)$. Some distribution statistics have been derived and discussed in detail. It was found that the BOGE-G family can be used to model asymmetric data under various forms of failure rates. Further, the stress-strength model was not based on the baseline function, but only on the parameters of the bivariate generator. Regarding estimating the family parameters, the MLE approach has been used for this purpose. The simulation results have indicated that the MLE technique works quite satisfactorily and can be applied to estimate the family parameters. A real data set has been analyzed to provide the capacity and highlight of the new generator. We can conclude this article by reporting a multivariate extension of the OGE-G family. Let $X_1, X_2, ..., X_{n+1}$ be independent RVs with $X_i \sim OGE - G\,(\alpha, \xi, \beta_i)$, such that $i = 1, 2, ..., n+1$. Define $Y_j = \max\{X_j, X_{n+1}\}$ for $j = 1, 2, ..., n$. Hence, the joint CDF of $Y_1, Y_2, ..., Y_n$ can be formulated as

$$\begin{aligned} F_{Y_1,Y_2,...Y_n}(y_1, y_2, ..., y_n) &= \Pr\left[Y_1 \le y_1, Y_2 \le y_2, ..., Y_n \le y_n, Y_{n+1} \le y\right] \\ &= F(y; \alpha, \xi, \beta_{n+1}) \prod_{j=1}^{n} F(y_j; \alpha, \xi, \beta_j), \end{aligned}$$

for $(y_1, y_2, ..., y_n) \in (0, \infty)^n$, where $y = \min\{y_1, y_2, ..., y_n\}$. Clearly, the BOGE generator arises from this multivariate OGE-G class by taking $n = 2$. As a future work, we will discuss in detail the multivariate extension of the OGE-G family, as it has many applications in lifelong analysis, environmental sciences, economics, engineering, and medical sciences. Finally, we hope that our new bivariate family will attract a wider range of applications in areas such as survival and life data, economics, engineering, hydrology, and others.

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