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Geometric realizations of homotopic paths over curved surfaces

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Abstract. This paper introduces geometric realizations of homotopic paths over simply-connected surfaces with non-zero curvature as a means of comparing and measuring paths between antipodes with either a Feynman path integral or Woodhouse contour integral, resulting in a number of extensions of the Borsuk Ulam Theorem. All realizations of homotopic paths reside on a Riemannian surface S, which is simply-connected and has non-zero curvature at every point in S. A fundamental result in this paper is that for any pair of antipodal surface points, a path can be found that begins and ends at the antipodal points. The realization of homotopic paths as arcs on a Riemannian surface leads to applications in Mathematical Physics in terms of Feynman path integrals on trajectory-of-particle curves and Woodhouse countour integrals for antipodal vectors on twistor curves. Another fundamental result in this paper is that the Feynman trajectory of a particle is a homotopic path geometrically realizable as a Lefschetz arc.

1. Introduction

This paper introduces a path-Borsuk-Ulam Theorem, stemming from three main forms of paths over curved surfaces that have been identified, namely,

1^o **Poincaré Contour paths** were introduced by Poincaré in 1892 in his analysis situs paper [17]. In a contour path, each subpath is an infinitely small contour on a manifold [17, p. 240]. Recently, N.M.J. Woodhouse [23] introduced contour integrals defined on twistor curves on a complex manifold.

2^o **Whitehead Homotopic paths** were introduced during the late 1940s by J.H.C. Whitehead [21, 22] and S. Lefschetz [6], elaborated in [14–16]. For Whitehead, a *path* is a continuous map $h : [0, 1] \to S$, *i.e.*, a map from the unit interval to a space S. For Lefschetz, a *homotopic path* h in an arcwise connected space S is

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simply a map of a directed (= oriented) closed arc $\widehat{v_0, v_1}$ into S [6, p. 158]. A space is arcwise connected provided every vector in the space S is on a path containing an initial vector and a terminal vector such as the arcs in Figure 1.

3^o **Feynman paths** were introduced by R.P. Feynman in his thesis completed in 1942 [3, p. xiv]. A *Feynman path* is a trace of the trajectory of a particle between fixed endpoints [3, p. xiv], providing a framework for a path integral, also introduced by Feynman[3] and elaborated by R.P. Feynman and A.R. Hibbs in [4]. A *Penrose path* over a twistor curve (from R. Penrose's 1968 paper [13]) and its refinement by R.S. Ward in his 1977 thesis [20] supervised by Penrose, is a form of Feynman path in which the trajectory of a particle is over a twistor curve.

The original Borsuk-Ulam Theorem (BUT) [2] from K. Borsuk in 1933 is given in terms of antipodal vectors $\vec{p}, -\vec{p}$ on the surface of an n-dimensional Euclidean sphere S^n, defined by

$$S^n = \left\{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}\ ,\ x_1^2 + \cdots + x_{n+1}^2 = 1,\ n \geq 2\right\}.$$

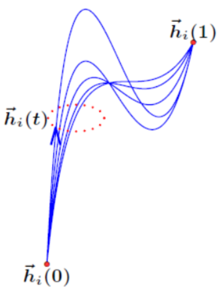

Figure 1: Discrete paths $h : I_d \to S$ with all $t \in I_d$.

Points on the surface of a sphere are *antipodal* provided the points are diametrically opposite each other. Examples of antipodal vectors are the poles on the surface of a planet.

In 1933, K. Borsuk introduced the following theorem.

Theorem 1.1. (**Borsuk-Ulam Theorem**) [2, p. 178] *For every continuous map $f : S^n \to \mathbb{R}^n$, there exists $\vec{p} \in S^n$ such that $f(\vec{p}) = f(-\vec{p})$.*

Remark 1.2. *Theorem 1.1 is a translation from German, which is given by J. Matoussěk [9, p. 21].*

Remark 1.3. *The basis for Theorem 1.1 came from K. Borsuk's thesis completed in 1930 [1]. Ulam is credited by Borsuk (in a footnote [2, p. 178]) with the idea codified in Theorem 1.1, which Ulam stated as a conjecture. In effect, Borsuk proved Ulam's conjecture in 1933. In 1930, L. Lusternik and S. Shnirel'man introduced the nonvoid intersection of sets of closed surface curves that have antipodal vectors in common.*

Theorem 1.4. (**Lusternik-Shnirel'man Theorem**) [7] *For any cover $F_1, \ldots, F_{n+1}$ of the sphere S^n by $n+1$ closed sets, there is at least one set containing a pair of antipodal points common to $F_i, -F_i$ (i.e., $F_i \cap -F_i \neq \emptyset$).*

Remark 1.5. *Theorem 1.4 is a translation from Russian, which is given by J. Matoussěk [9, p. 21].*

Theorem 1.4 contrasts with Theorem 1.1. In the Lusternik-Shnirel'man Theorem 1.4, there is a closed set F_i that is a cover of a sphere S^n and that has an opposite set $-F_i$, in which the sets $F_i, -F_i$ contain antipodal points such that $F_i \cap -F_i \neq \emptyset$. This sharply contrasts with the Borsuk-Ulam Theorem, which asserts there is a continuous map f from S^n into $\mathbb{R}^n$ over a surface containing antipodal surface vectors $\vec{p}, -\vec{p}$ such that $f(\vec{p}) = f(-\vec{p})$. Also, Theorem 1.4 concludes with the observation that the intersection of $F_i, -F_i$ is nonvoid but the values of the shared antipodal points are not given. In the LS theorem formulation, it is possible that the antipodal points in $F_i \cap -F_i$ have different values. By contrast, in the Theorem 1.1 formulation, it is asserted that the antipodal points map to the same value.

Given a path $h : I \to S^n$, let $T = \{t_i\}$ be an ordered and countable subset of I, where $0 < t_i < t_{i+1} < 1$ such that $h(t_i) \neq h(t_{i+1})$. We then have $I_d = \{0, 1\} \cup T$, which is called a *discrete unit interval*.

Example 1.6. *Given a path* $h : I \to S^n$, *let* $T_{0.0001} = \{t_i\}$ *be a countable and ordered subset of* I *such that* $0 < t_i < t_j < 1$ *for all* $i < j$, *and* $|h(t_i) - h(t_{i+1})| = 0.00001$ *for all* i. *Then* $I_d = \{0, 1\} \cup T_{0.00001}$ *is a discrete unit interval.*

2. Preliminaries

More recent versions of the Borsuk-Ulam Theorem (see, *e.g.*, [11, §68,p.405], [19, p.266],[9, §2.1,p. 23]) require the map $f : S^n \to \mathbb{R}^n$ to be continuous. The map f is continuous provided for each subset $E \subset S^n$, if a point $\vec{p}$ is arbitrarily close to E (*i.e.*, $\inf_{\vec{e}\in E} |\vec{p} - \vec{e}| = 0$), then $f(\vec{p})$ is arbitrarily close to $f(E)$. However, in keeping with an interest in the geometric realization of discrete paths as surface arcs containing points with gaps between them, we consider discrete maps.

Definition 2.1. *Let* S *be a Riemannian surface. Given a path* $h : I \to S$, *a discrete path is a map* $h : I_d \to S$ *where* I_d *is a discrete unit interval of* I. *(We will also denote the discrete path by* h.*) Here* $\vec{h}(0)$ *and* $\vec{h}(1)$ *are the initial and terminal points in* S, *respectively, and* $\vec{h}(t) \in S$ *for all* $t \in I_d$.

Example 2.2. *Discretely close surface points* $\vec{p}, \vec{q}$ *such as close water molecules always have a minute gap between them.*

Example 2.3. *The discrete unit interval* I_d *is a collection of discretely close points* $t, t' \in I_d$ *such that* $t' = t_{i\pm1}$.

Definition 2.4. *A map* $f : S^n \to \mathbb{R}^n$ *is said to be discrete provided for each subset* $E \subset S^n$, *if a point* $\vec{p}$ *is discretely close to* E, *then* $f(\vec{p})$ *is close to* $f(E)$.

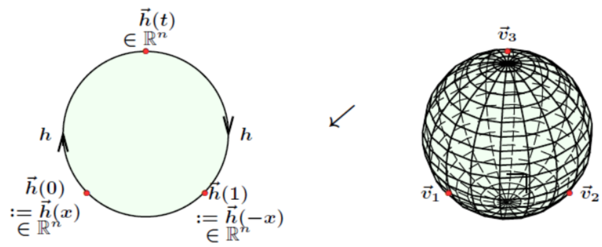

Figure 2: The left-slanting arrow ↙ reads *collapses to*, *e.g*, ◂ ↙ ▹, *i.e.*, collapse a left-pointing solid triangle to its boundary. For example, collapse a sphere S to a circle containing a discrete path $h : I_d \to S$ with $\vec{h}(0) = \vec{h}(x) \in \mathbb{R}^2$, antipodal to $\vec{h}(1) = \vec{h}(-x) \in \mathbb{R}^2$, with $\vec{h}(t) \in \mathbb{R}^2$ for $t \in I_d \setminus \{0, 1\}$.

Example 2.5. *A sample discrete path* $h : I_d \to S$ *on the surface of a Riemannian sphere is shown in Figure 2. This path begins at vector* $\vec{h}(0) \in \mathbb{R}^n$ *at* $\vec{v}_1$ *on the surface of* S *and ends at vector* $\vec{h}(1) \in \mathbb{R}^n$, *which is the value of antipode of* $\vec{v}_1$. *The assumption made here is that* $\vec{h}(0)$ *and* $\vec{h}(1)$ *have the same value such as identical temperature.*

That is, a *discrete path* $h : I_d \to S$ is a map from the discrete unit interval $I_d \subset I$ (for $I = [0, 1]$) to a bounded, simply connected surface S with non-zero curvature. Path h is discrete, since there are gaps between all points $\vec{h}(t) \in S$ between 0 and 1 in $I_d \subset [0, 1]$. The surface S is *simply connected* provided every path h has end points $h(0), h(1) \in S$ and h has no self-loops.

Paths either lie entirely on a surface in the planar case or lie on a surface and, possibly, puncture a surface in the non-planar case. Paths that puncture a surface are called cross-cuts. A *cross cut path* P (also called an *ideal arc* [10, §3, p.11]) has both ends in P and path interior in the interior of S.

Remark 2.6. *Homotopic paths were introduced by J.H.C. Whitehead [21]. For Whitehead, a path* $h : [0, 1] \to X$ *is a continuous map from the unit interval to a cell complex X. In the pursuit of discrete paths in a curved space, the focus is on* 0*-cells (single points) and* 1*-cells (arcs) in an n-dimensional Riemannian space S. A single surface vector is a 0-cell.*

Definition 2.7. *[5] An arc is a curvilinear line segment attached to a pair of 0-cells.*

Definition 2.8. *A pair of vectors* v_0, v_1 *is path-connected provided there is a sequence of 0-cells starting with* v_0 *and ending with* v_1 *in such a way that* v_0, v_1 *are attached to a Lefschetz arc. If such an arc exists between a pair of* 0*-cells* v_0 *and* v_1 *in this sequence (i.e., each pair* v_0 *and* v_1 *in the sequence of 0-cells are path connected), a collection of Leftschetz arcs corresponding to this sequence is called a discrete Lefschetz arc. We will denote the discrete Lefschetz arc between* v_0 *and* v_1 *by* $\widehat{v_0, v_1}$.

Proposition 2.9. *There is a discrete Lefschetz arc between each pair of 0-cells.*

Proof. Immediate from Definition 2.8. □

Example 2.10. *All vectors on the circle in Figure 2 are path-connected, since, from Proposition 2.9, there is a Lefschetz arc between each pair of vectors.*

3. Antipodal and Non-Antipodal Path Borsuk-Ulam Theorem

This section introduces results for the geometric realization of homotopic paths in surface arcs.

Lemma 3.1. *Every discrete path constructs a discrete Lefschetz arc.*

Proof. Given a path $h : I \to X$, let $h : I_d \to X$ be a discrete path. Then the collection

$$\{h(0), h(1)\} \cup \{h(t_i) \ : \ t_i \in I_d\}$$

forms a sequence of path connected 0-cells in X, hence it forms a discrete Lefschetz arc between $h(0)$ and $h(1)$. □

Theorem 3.2. *The endpoints of a discrete Lefschetz arc can be the same.*

Proof. Given two path connected 0-cells $\vec{v}_0$ and $\vec{v}_1$, we know that there is a discrete Lefschetz arc from $\vec{v}_0$ to $\vec{v}_1$. One can reverse the direction of arcs (since it can be considered to be a discrete path) so that the union of the discrete Lefschetz arcs $\widehat{v_0, v_1}$ and $\widehat{v_1, v_0}$ will form a discrete Lefschetz arc $\widehat{v_0, v_0}$. □

Next, consider the geometric realization of discrete homotopic path as a discrete arc and which constructs a vector field.

Theorem 3.3. *Every discrete path constructs a vector field.*

Proof. Let $h : I_d \to S$ be a discrete path. From Lemma 3.1, h constructs a discrete arc $\widehat{h(0), h(1)}$ on a surface S. Consequently, each $\vec{h}(t) \in \widehat{h(0), h(1)}$ has a location $(x_1, \dots) \in S$ with its own magnitude and direction S, *i.e.*, every $\vec{h}(t)$ is a vector in S. Hence, h constructs a vector field. □

Lemma 3.4. *Let $\vec{v}_1, \vec{v}_2$ be antipodal vectors on the surface of an n-sphere S^n. There exists a discrete path h with vectors that are antipodal on a surface S^n.*

Proof. Let $\vec{v}_1, \vec{v}_2$ be antipodal vectors on the surface of an n-sphere S^n. Since S^n is path connected, there is a discrete Lefschetz arc $\widehat{v_1, v_2}$. The collection of Lefschetz arcs (hence the discrete Lefschetz arc itself) forms a discrete path $h : I_d \to S^n$ with $\vec{h}(0) = \vec{v}_1$ and $\vec{h}(1) = \vec{v}_2$. Hence, a discrete path can be defined for every pair of antipodal points on S^n. □

From what we have observed about discrete paths on the surface of a sphere, we obtain the following theorem.

Theorem 3.5. (**Path-Borsuk-Ulam Theorem**) *Given a continuous map $f : S^n \to \mathbb{R}^n$ (hence a discrete map), there exist a discrete path $h : I_d \to \mathbb{R}^n$ and a point $\vec{p} \in S^n$ such that $h(0) = f(\vec{p}) = f(-\vec{p})$. In fact, h forms a discrete loop based at $f(\vec{p})$.*

Proof. It is obvious that a continuous map $f : S^n \to \mathbb{R}^n$ is also a discrete map. From Theorem 1.1 (Borsuk-Ulam Theorem), we know that there is a point $\vec{p} \in S^n$ such that $f(\vec{p}) = f(-\vec{p})$. Consider a sequence of points $\{\vec{v}_t\} \subset S^n$ indexed over a discrete interval I_d such that $\vec{v}_0 = \vec{p}$, $\vec{v}_1 = -\vec{p}$, and two consequtive terms $\vec{v}_t, \vec{v}_{t+1}$ are discretely close for all $t \in I_d$. Then consider the image of this sequence $\{f(\vec{v}_t)\}_{t \in I_d}$. This set can be considered as the image of the discrete path $h : I_d \to \mathbb{R}^n$ defined by $h(t) = f(\vec{v}_t)$. In fact, h is a discrete loop. □

Remark 3.6. *An immediate consequence of Theorem 3.5 is that, for any pair of antipodal surface points, we can always introduce a discrete path h that begins and ends at the antipodal points such as places that have same latitude and longitude. For example, the antipode of Winnipeg, Manitoba, Canada with coordinates 49°.53'N, 97°.8'W is Port-aux-Français, Kerguelen, French Southern Territories.*

Example 3.7. *An example of a discrete path that begins and ends at antipodal surface points is shown in Figure 2.*

Observe that a path can be constructed between any pair of surface vectors. This observation leads to more general form of Theorem 3.5.

Theorem 3.8. (**Non-antipodal path-BUT**) *Let the discrete unit interval I_d be an index set for vectors $v_0, \dots, v_t, \dots, v_1$, $t \in I_d$ in S^n in a continuous map $f : S^n \to \mathbb{R}^n$ such that $f(v_0) = f(v_1)$ for some $v_0, v_1 \in S^n$. There is a discrete path $k : I_d \to \mathbb{R}^n$ with endpoints $f(v_0), f(v_1)$ that are values in $\mathbb{R}^n$ such that $k(0) = k(1)$.*

Proof. Let $h : I \to S^n$ be a path from v_0 to v_1 and $h : I_d \to S^n$ be its associated discrete path. Then the composition $k = f \circ h$ is a discrete path in $\mathbb{R}^n$ with endpoints $f(v_0)$ and $f(v_1)$ so that $k(0) = k(1)$. □

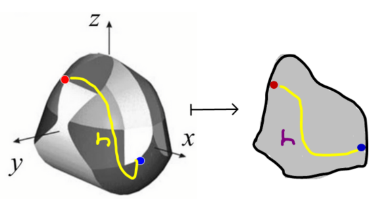

Figure 3: 2D and 3D views of discrete paths on a Gomboc Riemannian surface.

Example 3.9. *An example of a discrete path that begins and ends at antipodal surface vectors on a bumpy Riemannian sphere (aka Gomboc sphere) is shown in Figure 3.*

Example 3.10. *An example of a discrete path $h : S^2 \to \mathbb{R}^3$ on a 3D Gomboc Riemannian surface is shown in Figure 3. The same path is also depicted on a 2D slice of the 3D surface. In keeping with Theorem 3.8, each vector $\vec{h}(v_t)$ is a signal value from the path h. For example, if we let the discrete path be an optical field flow containing a stream of photons reflected from a Riemannian surface, then there are number of possible signal values for $\vec{h}(v_t)$,* e.g.,

1^o *wavelength of* $\vec{h}(v_t)$.
2^o *frequency of* $\vec{h}(v_t)$.
3^o *electron voltage of* $\vec{h}(v_t)$.
4^o *lumens (luminosity) of* $\vec{h}(v_t)$.
5^o *gradient of* $\vec{h}(v_t), t \in I_d$, *which would be perpendicular to the surface at* (x, y, z), *defined by*

$$grad(\vec{h}(v_t)) = \frac{\partial \vec{h}}{x} i + \frac{\partial \vec{h}}{y} j + \frac{\partial \vec{h}}{z} k.$$

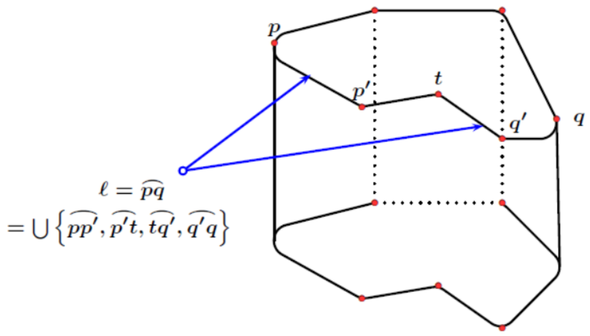

Figure 4: Trajectory of a particle over twistor curve realizable as the union of a sequence of sub-arcs on Lefschetz arc $\ell = \widehat{pq}$ on a R.S. Ward hypersurface $\mathbb{C}S$ [20, p.62].

4. Feynman Trajectories of a Particle

This section introduces particle trajectories as continuous paths over the curvature of space-time, which leads to the counterpart of the discrete path results already given. The transition from discrete paths results from the geometry of space-time generated by quantum processes [8], which is in keeping with the observation by R. Penrose [13] that the link between space-time curvature and quantum processes such as those found in Feynman trajectory of a particle is supplied by the use of twistors. A *twistor space* is a complex manifold $\mathbb{C}M$. For example, a Lefschetz arc in curved space-time is a R.S. Ward hypersurface $\mathcal{S}$ twistor [20, p.56], which is a complex curve ℓ in $\mathbb{C}\mathcal{S}$.

Example 4.1. *A sample twistor curve* $\ell \in \mathbb{C}S$ *is shown in Fig. 4, which is a geometric realization of a Feynman trajectory of a particle (see Def. 4.2), which leads to a space-time view of a Lefschetz arc (see Def. 4.2 and Lemma 4.5).*

Definition 4.2. *The trajectory of a particle in a 2-plane in curved space-time is a map*

$$h : \mathbb{R}^2 \times S^2 \to \mathbb{R}^2 \times S^2$$

defined by

$$h(t_{t_x}) = \widehat{t_{t_x}, t_{t_{x_i}}} \cup \bigcup \widehat{t_{t_{x_i}}, t'_{t'_{x'_i}}} \quad t, t' \in \mathbb{R}, x_i \in S^2, i \in I,$$

in which each $\widehat{t_{t_{x_i}}, t'_{t'_{x'_i}}}$ *is a space-time line segment in a curve* ℓ *starting with subarc* $\widehat{t_{t_{x_0}}, t_{t_{x_i}}}$ *in a Lefschetz arc at times* t_t *(instant t in region time t) with index i in the unit interval* $I = [0, 1]$ *is mapped to an arcwise-connected set,* i.e, *the line segments in the trajectory are attached to each other and starting with* $\widehat{t_{t_{x_0}}, t_{t'_{x_i}}}$, *there is a path from any subarc a sequence of subarcs can be traversed to reach an ending subarc* $\widehat{t_{t_{x_n}}, t'_{t'_{x_{n'}}}}$ *in a N.M.J. Woodhouse [23] twistor space* $\mathbb{R}^2 \times S^2$ *with metric signature* $+ + --$.

Remark 4.3. *From Definition 4.2, the vectors in $h(t_{x_i})$ are J.H.C. Whitehead zero cells [21] in an arcwise-connected space $\mathbb{R}^2 \times S^2$.*

Definition 4.4. *A Lefschetz arc E is a curve ℓ attached between a pair of 0-cells p, p'. We assume the curve ℓ is dense and the points in ℓ are path-connected, i.e., between every pair of points q, q' in ℓ, there is a sequence of sub-arcs traversable between q and q'.*

Lemma 4.5. *A trajectory of a particle is realizable as a Lefschetz arc.*

Proof. From Definition 4.2, a trajectory h is a curve ℓ that starts and ends with a 0-cell and is the union of subarcs in an arcwise-connected space. Hence, from Definition 4.4, the trajectory h is realizable as a Lefschetz arc. □

Example 4.6. *A sample trajectory of a particle as a Lefschetz arc over a twistor curve realized as a Lefschetz arc $\ell = \widehat{pq}$ with endpoints (0-cells) $\vec{p}, \vec{q}$ and which is the union of sub-arcs is shown in Figure 4.*

Definition 4.7. *The unit $I = [0,1] \in \mathbb{R}$ is the set of all real values in the closed interval with initial value 0 and ending 1 and an unbounded number of consecutive everywhere dense subintervals of real values between 0 and 1. That is, every real number x in a subinterval of $A \subset I$ has another real number $x' \in A$ that is arbitrarily close to x.*

Lemma 4.8. *The trajectory of a particle is continuous.*

Proof. From Definition 4.7, I is dense and is the index set for the points in the trajectory of a particle. A particle moving along the Lefschetz curve can be observed at any real value in the unit interval $I = [0,1]$ (see J.J. Sakurai and J. Napolitano [18, p. 37]). Let h be the trajectory of a particle t_{t_x}. One can consider this trajectory as a curve $\ell : I \to Im\ h$ defined by $\ell(t) = t_{t_{x_i}}$ with $\ell(0) = t_{t_x}$. Since ℓ is continuous, for any close pair i, j in I will be mapped to close pair $t_{t_{x_i}}$ and $t_{t_{x_j}}$ and hence close points in $\mathbb{R}^2 \times S^2$ will be mapped to two close trajectories. Hence, h is continuous. Then if $i, i' \in I$ are close, then $t_{x_i}, t_{x_{i'}}$ are close. Hence, the trajectory h is continuous. □

Remark 4.9. *In the proof of Lemma 4.8, we considered a trajectory of a particle as a curve, parametrized on the closed interval $[0,1]$. However, in 1-D Quantum Mechanics, this is not the case, i.e.. The points of the trajectory may have an infinite number of possible values so that they may not be limited in $[0,1]$ but rather are lying in $(-\infty, \infty)$. For more details, see J.J. Sakurai and J. Napolitano [18, pp. 37-42].*

Example 4.10. *Given a trajectory h, consider the set $J = \{t_i\}_{i \in I}$ of the instants of time of occurrence of the points in the trajectory of a particle over a vector field. The map $g : I \to \mathbb{R}$ defined by $g(i) = t_i$ is continuous, since for every arbitrarily close pair i and j, t_i and t_j are also arbitrarily close.*

5. Feynman Path Integral

In this section, it is observed that a Feynman path is continuous (Lemma 5.2), which leads to the results in Theorem 5.4 and Theorem 5.5 for Feynman paths, which are consequences of the Borsuk-Ulam Theorem.

Definition 5.1. *[4, p. 31] A Feynman path is a function $H : \mathbb{R}^2 \times S^2 \to S^2$ defined by $H(t_{t_x}) = x$ for a particle at point x at time t_t.*

Lemma 5.2. *Every Feynman path is continuous.*

Proof. Let $H : \mathbb{R}^2 \times S^2 \to S^2$ be a Feynman path, defined by $H(t_{t_{x_a}}) = x_a$ which is the trajectory h of a particle at point x_a at time t_t. Let ℓ represent that a particle travels over during its trajectory and let $H(t_{t_{x_a}}) = x_a$ be a point in ℓ. For simplicity, the curve ℓ is referred to as the trajectory of a particle. During the passage of a particle over ℓ, ℓ has no gaps in it. Since a trajectory map h is continuous, two close points $t_{t_{x_a}}$ and $t_{t_{x_b}}$ will lead us two close points x_a and x_b in ℓ at time t_t. Hence, a Feynman path h is continuous. □

Remark 5.3. *In Lemma 5.2, the continuity of a Feynman path H is explained in terms of the closeness (nearness) paradigm from [12, §1.5, p. 8], instead of the abstract (less intuitive) $\epsilon - \delta$ view of continuity. This approach befits the character of the trajectory of a particle over a curve ℓ, where the trajectory of a particle and the curve ℓ (without gaps) are traced by the particle in its trajectory. Just as pairs of points in the curve ℓ can be arbitraily close, so too, from Lemma 4.8, the vectors $H(t_{t_{x_a}}), H(t_{t_{x_b}})$ in the trajectory of a particle can be arbitrarily close.*

The value of a path between points a and b on a curve ℓ (the positions of a particle trajectory at times t_a, t_b, respectively), is $K(b, a)$, defined in a complex space $\mathbb{C}S$ with respect to Planck's constant $\hbar$ by Feynman and Hibbs [4, p. 45] by

$$\begin{aligned}
V(x,t) &= \textit{Potential energy of particle with mass } m.\\
L &= \frac{m}{2}\dot{x}^2 - V(x,y) \text{ (\textit{Lagragian} for the system)}.\\
S[b,a] &= \int_{t_a}^{t_b} L(\dot{x},x,t)dt\\
a,b &= \textit{ points on a twistor curve.}\\
K(b,a) &= \int_a^b e^{\left(\frac{i}{\hbar}\right)S[b,a]}\mathcal{D}x(t).
\end{aligned}$$

A Feynman path $H : \mathbb{R}^2 \times S^2 \to S^2$ over a curved space S^2 can be considered as $H = pr_2 \circ h$, the composition of its corresponding trajectory map $h : \mathbb{R}^2 \times S^2 \to \mathbb{R} \times S^2$ and the second projection map $pr_2 : \mathbb{R}^2 \times S^2 \to S^2$. Given a fixed point b_h on ℓ, define $\alpha : S^2 \to \mathbb{R}^2$ by $\alpha(\vec{a}) = K(b_h, a)$ where $K(b_h, a)$ is the value of the trajectory h containing points b_h, a in a segment $\widehat{b_h, a}$ in a curve ℓ starting at a and terminating at b_h.

Theorem 5.4. (**Feynman Path Theorem**) *Given a map $\alpha : S^2 \to \mathbb{R}^2$, there exists $\vec{a}$ in S^2 such that $\alpha(\vec{a}) = \alpha(-\vec{a})$.*

Proof. From Lemma 5.2, a Feynman path H is continuous and so that α is also continuous. Hence, from Theorem 1.1, we obtain the desired result, $\alpha(\vec{a}) = \alpha(-\vec{a})$ for antipodal points $a, -a$ in a Feynman path H. □

Theorem 5.5. (**Feynman Trajectory-of-Particle Theorem**) *The Feynman trajectory of a particle satisfies Borsuk-Ulam Theorem 1.1. Let $H : S^2 \to \mathbb{R}^2$ be the trajectory of a particle on the surface of sphere. There is at least one pair vectors $\vec{p}, \vec{p'} \in S^2$ such that $H(\vec{p}) = H(\vec{p'})$.*

Proof. From Lemma 5.2, a Feynman trajectory is continuous. Hence, from Theorem 1.1, we obtain the desired result for antipodal points $\vec{p}, -\vec{p} \in S^n$ in the Feynman trajectory h. □

Theorem 5.6. (**Feynman Path Integral Theorem**) *There exists a Feynman path with an initial path integral $K(b_h, a)$ for an initial vector $\vec{a}$ that equals the path integral $K(b_h, -a)$ for a later vector $-\vec{a}$, which may or may not be the antipode of vector $\vec{a}$.*

Proof. $K(b_h, a)$ are Feynman path integrals that resonate (have values) for a particle that has gradients on either two different surface curvatures along a surface curve ℓ or on the same surface curvature on a path ℓ' for a boomerang trajectory that follows a path that is a cycle. In either case, choose an intermediate point b_h in the path between $\vec{a}$ and b_h so that the two segments on ℓ have the same length. In that case, $K(b_h, a) = K(b_h, -a)$. □

Remark 5.7. *The significance of Theorem 5.6 is that the endpoints on a particle trajectory curve ℓ need not be antipodal points. That is, Theorem 5.6 is more general than Theorem 5.4.*

6. Woodhouse Borsuk-Ulam Theorem

This section gives three results for N.M.J. Woodhouse contour integrals [23, p. 198], defined with respect to the set of all real α-planes that has topology $\mathbb{R}^2 \times S^1$, which is compactified by adding S^1 representing α-planes that lie in the null cone at ∞. First, consider

$$\xi = x_1 + ix_2 \text{ and } \tau = t_1 + it_2, \text{ representing } \alpha\text{-planes as surfaces, with}$$
$$w = \xi + \bar{z}\tau,\ \bar{w} = \bar{\xi} + z\bar{\tau},\ \text{constant for } z = e^{i\theta},\ \text{where}$$
$$z = e^{i\theta} \text{ determines orientation of } \alpha\text{-plane.}$$
$$\phi(w, \bar{w}, z) = \frac{1}{2\pi}\oint_{|z|=1} f(w, \bar{w}, z)\frac{dz}{z}, \text{ expanded to obtain}$$
$$\phi(w, \bar{w}, z) = \frac{1}{2\pi}\oint_{|z|=1} f(\xi + \bar{z}\tau, \bar{\xi} + z\bar{\tau}, z)\frac{dz}{z}.$$

Let $\Phi : S^2 \to \mathbb{C}$ be a map defined by $\Phi(\vec{p}) = \phi(w_{\vec{p}}, \bar{w}_{\vec{p}}, z)$, where $w_{\vec{p}}$ is the point representing $\vec{p}$ on the equilateral circle $S_{\vec{p}}$ on S^2 which is passing through $\vec{p}$. The function Φ can be realized as a function $\Phi : S^2 \to \mathbb{R}^2$ as $\mathbb{C}$ and $\mathbb{R}^2$ are homeomorphic.

Definition 6.1. *The contour integral $\Phi : S^2 \to \mathbb{R}^2$ is a smooth function, since ϕ is a smooth function on a twistor space [23]. That is, Φ is continuous.*

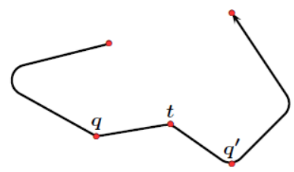

Figure 5: Woodhouse contour integrals on sub-twistor curve antipodes q, q' with $\oint_{\widehat{qt}()} = \oint_{\widehat{tq'}()}$.

Theorem 6.2. *The contour integral Φ satisfies the Borsuk-Ulam Theorem.*

Proof. From Definition 6.1, the contour integral $\Phi : S^2 \to \mathbb{R}^2$ is a continuous function. The result follows from Theorem 1.1, *i.e.*, there exist antipodes $\vec{p}, -\vec{p}$ on a twistor curve in $\mathbb{R}^2 \times S^2$ such that $\Phi(p) = \Phi(-p)$. □

Corollary 6.3. *The map Φ also satisfies the path-Borsuk-Ulam Theorem given in Theorem 3.5.*

Proof. Take $n = 2$ and replace the continuous map $f : S^n \to \mathbb{R}^n$ with $\Phi : S^2 \to \mathbb{R}^2$ in the proof of Theorem 3.5. □

Example 6.4. *Sample contour integrals on sub-twistor vectors that are antipodal are shown in Figure 5.*

Theorem 6.5. *Let ϕ, ϕ' be the Woodhouse contour integrals over a twistor curve ℓ and let p, p' be any two distinct points on ℓ. Then there are Φ, Φ' such that $\Phi(p) = \Phi'(p')$.*

Proof. Replace the Feynman path integral with the Woodhouse contour integral in the proof of Theorem 5.6, and the desired result follows. That is, we can always find a point q between p, p' on the twistor ℓ such that $\Phi(p) = \Phi'(p')$. □

Remark 6.6. *Theorem 6.5 covers a broader spectrum of twistor length measurements than Theorem 6.2. That is, for any pair of distinct vectors on a twistor curve, we can always find an intermediate vector so that the contour integrals over the resulting twistor sub-arcs have equal value.*

Example 6.7. *Sample contour integrals on sub-twistor curves $\widehat{v_1, v_2}, \widehat{v_2, v_3}$ with end points that may or may not be non-antipodal are shown in Figure 6.*

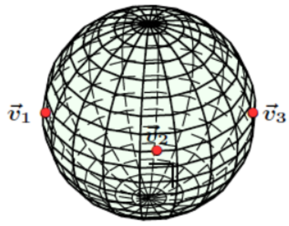

Figure 6: Woodhouse contour integrals on sub-twistor curves $\widehat{v_1, v_2}, \widehat{v_2, v_3}$ with $\oint_{\widehat{v_1,v_2}} () = \oint_{\widehat{v_2,v_3}} ()$.

7. Concluding Remarks

The focus in path Borsuk-Ulam Theorem 3.5 is on a homotopic path between antipodes on the surface of a sphere S^n mapped to real values in $\mathbb{R}^n$. The geometry underlying the Borsuk-Ulam Theorem looms up, for example, in the realization of a homotopic path as an arc stretching over a planetary curved surface between one location and another location at varying space-times with the same latitude and longitude. In this paper, the Borsuk-Ulam Theorem is an emperor with new clothes, namely,

1^o *How to look:* consider either a discrete or continuous homotopic paths between antipodes.
2^o *Geometric realization:* endpoints of twistor curves that are either antipodal or non-antipodal.
3^o *Length-of-arc measure: e.g.*, measure with either a Feynman path integral or Woodhouse contour integral over arcs having antipodal endpoints.

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