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Geometric characterizations of almost Ricci-Bourguignon solitons on Kenmotsu manifolds

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Abstract. The aim of this paper is to find the geometric characterizations of almost Ricci-Bourguignon solitons and gradient almost Ricci-Bourguignon solitons within the background of Kenmotsu manifolds. If (M, g) is a (2n+1)-dimensional Kenmotsu manifold and g represents an almost Ricci-Bourguignon soliton, then we find a sufficient condition under which the manifold M is Einstein (trivial). Next, we show that if g is an almost Ricci-Bourguignon soliton on M and the Reeb vector field ξ leaves $\lambda + \rho r$ invariant, then g reduces to Ricci-Bourguignon soliton on M. Finally, we prove that if g is a gradient almost Ricci-Bourguignon soliton, then the manifold M is either Einstein or g is a gradient $\eta-$Yamabe soliton on M. As a consequence of the results, we obtain several corollaries.

1. Introduction

In the current scenario, geometric flows have been a topic of active research interest in both mathematics and physics. These flows helps us to find the various geometric and topological structures of Riemannian manifolds. Ricci solitons and Yamabe solitons play an important role in geometric flow where they correspond to self-similar solution of the flow. Thus, given a geometric flow, it is natural to study the solitons associated to that flow. As a result of this, in 1982, Hamilton introduced the intrinsic Riemannian geometric flows on a Riemannian manifold called as Ricci flow [20] and Yamabe flow [21]. Also, Ricci-Bourguignon flow is an intrinsic geometric flow, whose fixed points are solitons on Riemannian (or pseudo-Riemannian) manifolds. In [4], Bourguignon introduced Ricci-Bourguignon flow on Riemannian manifold as $\frac{\partial}{\partial t}g(t) = -2(S - \rho r g)$, where ρ is a real constant and r is the scalar curvature of the manifold. This flow can be seen as an interpolation between the Ricci flow and Yamabe flow. Moreover, depending on the choice of ρ, the Ricci-Bourguignon flow may turn to certain celebrated geometric flows, namely, for $\rho = 1/2$ this flow turn to be Einstein flow, for $\rho = 1/2(n-1)$ it will turn to the Schouten flow and for $\rho = 0$ it will turn to the famous Ricci flow.

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Recently, the notion of an almost Ricci-Bourguignon soliton was introduced by Dwivedi [13]. A Riemannian metric g on a smooth manifold M is called a almost Ricci-Bourguignon soliton if there exists a smooth function λ such that its Ricci tensor S satisfies

$$\frac{1}{2}\pounds_V g + S = (\lambda + \rho r)g, \tag{1}$$

where ρ is non-zero real number and r is the scalar curvature of g. The almost Ricci-Bourguignon soliton is said to be expanding, steady and shrinking accordingly as λ is negative, zero and positive, respectively. It is said to be Ricci-Bourguignon soliton if λ is a constant. Note that if V is a Killing vector field, then an almost Ricci-Bourguignon soliton is just a Ricci-Bourguignon soliton as it forces λ to be a constant. If ∇f is the gradient of a smooth function f on M then the $Hess_f$ is defined by

$$\nabla^2 f(X, Y) = Hess_f(X, Y) = g(\nabla_X \nabla f, Y), \quad X, Y \in \chi(M).$$

The $Hess_f$ is symmetric, that is, $g(\nabla_X \nabla f, Y) = g(\nabla_Y \nabla f, X)$. If the vector field V is a gradient of smooth function f on M, that is, $V = \nabla f$ then an almost Ricci-Bourguignon soliton is called a gradient almost Ricci-Bourguignon soliton. The function f is called the potenial function of gradient almost Ricci-Bourguignon soliton. In this case, the equation (1) reduces to

$$\nabla^2 f + S = (\lambda + \rho r)g. \tag{2}$$

The Riemannian metric g satisfies equation (2) is also known as a gradient $\rho-$Einstein soliton [5]. Moreover, Shaikh et al. [27] investigated some aspects of gradient $\rho-$Einstein Ricci soliton in a complete Riemannian manifold and proved that a compact gradient $\rho-$Einstein soliton is isometric to the Euclidean sphere by showing that the scalar curvature becomes constant. For the non-compactness, if the scalar curvature on a gradient ρ-Einstein soliton satisfied some integral condition then it is vanished. Also, gradient $\rho-$einstein solitons within the background of some class of almost Kenmotsu manifolds were studied in [30]. Further, some results for the almost Ricci-Bourguignon solitons were proved by Dwivedi [13] that corresponding generalized results for Ricci solitons and also derived integral formula for compact gradient Ricci-Bourguignon solitons and compact gradient Ricci-Bourguignon almost solitons. Using the integral formula, in [13], it is proved that a compact gradient Ricci-Bourguignon almost soliton is isometric to a Euclidean sphere if it has constant scalar curavture or its associated vector field is conformal. In this series, some remarkable results of Ricci-Bourguignon solitons have been studied in the papers [8–11, 26]. The study of almost Ricci-Bourguignon solitons in the context of contact geometry has been initiated by Dwivedi and Patra [14]. They found some geometric chracterizations of almost $*-$Ricci-Bourguignon solitons on Sasakian manifolds along with the several interesting sufficient conditions under which an almost $*-$Ricci-Bourguignon soliton or a gradient almost $*-$Ricci-Bourguignon soliton on a Sasakina manifold is isometric to an Euclidean sphere or trivial. Recently, Patra et al. [24] gave a characterizations of almost Ricci-Bourguignon soliotns on pseudo-Riemannian manifolds, in particularly, on paracontact metric manifolds and para-Sasakian manifolds.

On the other hand, Kenmotsu manifolds known as not only a special case of almost contcat metric manifolds (see [3]) but also an anolougous of Hermitian manifolds were investigated by several authors. The notion of Kenmotsu manifolds were defined and studied by Kenmotsu [23] in 1972. They set up one of the three classes of almost contact metric manifolds M whose automorphism group attains the maximum dimension (see [28]). For such a manifold, the sectional curvature of plane sections containing a Reeb vector field ξ is constant, say c. Here, (1) if $c > 0$, then M is a homogeneous Sasakian manifold of constant $\phi-$sectional curvature. (2) If $c = 0$, then M is global Riemannian product of a line or a circle with a Kahler manifold of constant holomorphic sectional curvature. (3) If $c < 0$, then M is a warped product space $\mathbb{R} \times_f \mathbb{C}^n$. Kenmotsu [23] characterized the differential geometric properties of manifolds of class (3); the structure so obtained is now known as Kenmotsu structure. A Kenmotsu structure is not Sasakian (see [12, 23]).

In this paper, we study almost Ricci-Bourguignon soliton and gradient almost Ricci-Bourguignon soliton within the framework of Kenmotsu manfolds. The paper is unfold as follows: After preliminaries, in section

3, we study almost Ricci-Bourguignon solitons on Kenmotsu manifolds. Section 4 is devoted to the study of gradient almost Ricci-Bourguignon soliotns on Kenmotsu manifolds. In both the sections we arrive at interesting results.

2. Preliminaries

A smooth manifold M of dimension (2n+1) is called an almost contact manifold (see [3]) if it is equipped with the structure (ϕ, ξ, η) where ϕ is a tensor field of type (1,1), ξ is a vector field (called the characteristic or Reed vector field) and η is a 1-form satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{3}$$

An immediate consequence of (3) follows that

$$\phi\xi = 0, \quad \eta \circ \phi = 0 \quad and \quad rank(\phi) = 2n.$$

In general, a smooth manifold M endowed with an almost contact structure is called an almost contact manifold and it is denoted by (M, ϕ, ξ, η). It is known that a smooth manifold M admits an almost contact structure if and only if the structure group on the tangent bundle of M reduces to $U(n) \times 1$.

Let an almost contac manifold (M, ϕ, ξ, η) admits a Riemannian compatible metric g satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad X, Y \in \chi(M), \tag{4}$$

where $\chi(M)$ is the Lie-algebra of all vector fields on M. Then the manifold is called an almost contact metric manifold and is denoted by (M, ϕ, ξ, η, g) (or, briefly (M, g)). Then from (4) it can be easily deduce that $g(\phi X, Y) = -g(X, \phi Y)$. The fundamental 2-form Φ associated with an almost contact metric structure is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector field $X \in \chi(M)$.

If ∇ is the Levi-Civita connection of g on (M, g), then an almost contact metric manifold (M, g) is said to be Kenmotsu (see [23]) if the structural tensor field ϕ satisfies the identity

$$(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X, \quad X, Y \in \chi(M). \tag{5}$$

According to Kenmotsu [23], the warped product space $\mathbb{R} \times_f K$, where $f(t) = ce^t$ on the real line $\mathbb{R}$ and K is Kahlerian manifold, admits a Kenmotsu structure. Further, on a Kenmotsu manifold M, the following formulae is valid:

$$\nabla_X \xi = X - \eta(X)\xi, \quad (\nabla_\xi \xi = 0), \quad X \in \chi(M). \tag{6}$$

Let R be the Riemannian curvature tensor of the Levi-Civita connection ∇ of g, given by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad X, Y \in \chi(M). \tag{7}$$

For a Kenmotsu manifold, the curvature tensor R satisfies the following:

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{8}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \quad X, Y \in \chi(M). \tag{9}$$

We recall that a symmetric (1, 1)-tensor field Q called as the Ricci operator Q is defined by

$$g(QX, Y) = S(X, Y) = Tr\{Z \to R(Z, X)Y\}, \quad X, Y, Z \in \chi(M), \tag{10}$$

and the scalar curvature of g is a smooth function r and is defined by $r = TrQ$. The gradient of the scalar curvature r is given by

$$(divQ)(X) = \sum_i g((\nabla_{E_i} Q)X, E_i) = \frac{1}{2} g(X, \nabla r) = X(r), \quad X \in \chi(M), \tag{11}$$

where $\{E_i\}$ is a local orthonormal frame on M. Moreover, recalling (8) we can derive the following identities on a Kenmotsu manifold (M, g) of dimension 2n+1 [31]:

$$Q\xi = -2n\xi, \tag{12}$$

$$(\nabla_X Q)\xi = -QX - 2nX, \tag{13}$$

$$(\nabla_\xi Q)X = 2(\nabla_X Q)\xi, \quad X, Y \in \chi(M). \tag{14}$$

If the Ricci tensor of a Kenmotsu manifold (M, g) satisfy

$$S = \alpha g + \beta\eta \otimes \eta, \tag{15}$$

where α and β are smooth functions on M, then we say M is η -Einstein. If $\beta = 0$, then M becomes an Einstein manifold.

3. Almost Ricci-Bourguignon Solitons on Kenmotsu Manifolds

In this section, we investigate the existence of geometry of almost Ricci-Bourguignon solitons on Kenmotsu manifolds. Recently, Ghosh [15] proved that if a metric g of a 3-dimensional Kenmotsu manifold M^3 represents a Ricci soliton, then M^3 is of constant negative curvature -1. Also, note that the condition

$$(£_\xi g)(X, Y) = 2\left\{g(X, Y) - \eta(X)\eta(Y)\right\}, \quad X, Y \in \chi(M) \tag{16}$$

implies ξ is not Killing in a Kenmotsu manifold. Hence, from (16) the metric g of Kenmotsu manifold M admits a Ricci soliton equation:

$$\frac{1}{2}£_V g + S = \lambda g,$$

with the potential vector field V is equal to ξ would become

$$S = -(1 + \lambda)g + \eta \otimes \eta, \tag{17}$$

which means M is an $\eta-$Einstein. But, according to Ghosh (see Theorem 1 of [16] and Theorem 1 of [15]) M must be Einstein, and this will be a contradiction to above equation. Since almost Ricci-Bourguignon soliton is a generaliztion of almost Ricci soliton, it is interesting to study almost Ricci-Bourguignon solitons within the frame-work of Kenmotsu structure. At this moment, first we prove that the potential vector field V being parallel to the Reeb vector field ξ is a sufficient condition under which a Kenmotsu manifold admitting an almost Ricci-Bourguignon soliton is trivial (that is, Einstein). Next, we find one more sufficient condition under which a Kenmotsu manifold admitting an almost Ricci-Bourguignon soliton is Ricci-Bourguignon soliton.

Now, we start with the following:

Theorem 3.1. *If the metric g of a (2n+1)-dimensional ($n > 1$) Kenmotsu manifold M represents an almost Ricci-Bourguignon soliton with the non-zero potential vector field V is parallel to the Reeb vector field ξ (that is, $V = \sigma\xi$ for some smooth function σ), then the manifold is Einstein with constant scalar curvature $-2n(2n + 1)$. Moreover, the gradient of σ is parallel to the Reeb vector field ξ.*

Proof. Let the metric g of M represents an almost Ricci-Bourguignon soliton. Then from (1) we have

$$\frac{1}{2}(£_V g)(X, Y) + S(X, Y) = (\lambda + \rho r)g(X, Y), \quad X, Y \in \chi(M). \tag{18}$$

Since potential vector field V is parallel to ξ, i.e., $V = \sigma\xi$ for a non-zero smooth function σ on M, then we have

$$\nabla_X V = X(\sigma)\xi + \sigma(X - \eta(X)\xi),$$

by the derivative $V = \sigma\xi$ covariantly along $X \in \chi(M)$ and using (6). Thus, the last two equations give

$$2S(X,Y) + X(\sigma)\eta(Y) + Y(\sigma)\eta(X) = 2[(\lambda + \rho r) - \sigma]g(X,Y) + 2\sigma\eta(X)\eta(Y), \quad X, Y \in \chi(M). \tag{19}$$

Now substituting ξ in lien of Y in the foregoing equation and using (12), we find

$$X(\sigma) = [2(\lambda + \rho r) + 4n - \xi(\sigma)]\eta(X), \quad X, \in \chi(M). \tag{20}$$

Setting $X = \xi$ in (20), we obtain

$$\lambda + \rho r = \xi(\sigma) - 2n. \tag{21}$$

Inserting (21) in (20), we achieve

$$X(\sigma) = \xi(\sigma)\xi, \quad X \in \chi(M), \tag{22}$$

where

$$\xi(\sigma) = (\lambda + \rho r) + 2n. \tag{23}$$

In the consequence of (21) and (22), the equation (19) assumes the following form:

$$S(X,Y) = [\xi(\sigma) - \sigma - 2n]g(X,Y) + [\sigma - \xi(\sigma)]\eta(X)\eta(Y), \quad X, Y \in \chi(M). \tag{24}$$

Let $\{e_i\}_{i=1}^{2n+1}$ be a local orthonormal basis on M. Plugging $X = Y = e_i$ in the above equation and then summing over i shows that

$$\xi(\sigma) - \sigma = \frac{r}{2n} + (2n+1). \tag{25}$$

Making use of (25) in (24), we arrive at

$$S(X,Y) = \left(\frac{r}{2n} + 1\right)g(X,Y) - \left(\frac{r}{2n} + 2n + 1\right)\eta(X)\eta(Y). \tag{26}$$

Thus, M is an η-Einstein manifold. With the help of η-Einstein condition, it has been proved that the scalar curvature r of M of $dim > 3$ satisfies the following equation (see, Lemma 3.4 of [31]):

$$Xr = (\xi r)\eta(X), \quad X \in \chi(M). \tag{27}$$

Further, taking the covariant derivative of (26) with respect to Z and then use of (6), we obtain

$$\begin{aligned}(\nabla_Z S)(X,Y) &= -\left(\frac{r}{2n} + 2n + 1\right)[g(X,Z)\eta(Y) + g(Y,Z)\eta(X) - 2\eta(X)\eta(Y)\eta(Z)] \\ &+ \frac{Z(r)}{2n}[g(X,Y) - \eta(X)\eta(Y)], \quad X, Y \in \chi(M).\end{aligned} \tag{28}$$

Contracting (28) over Y and Z, and then using (27) we obtain that

$$\xi(r) = -\frac{2n}{2n-1}[r + 2n(2n+1)]. \tag{29}$$

Also, contraction of (14) gives

$$\xi(r) = -2[r + 2n(2n+1)]. \tag{30}$$

Comparing (29) with (30) yields $r = -2n(2n+1)$. That is, the scalar curvature r of M is constant. By virtue of this, we can conclude from (26) that

$$S(X,Y) = -2ng(X,Y), \quad X, Y \in \chi(M).$$

and hence M is Einstien. Since $r = -2n(2n+1)$, it follows from (25) that $\xi(\sigma) = \sigma$, and this completes the proof of our theorem. □

Next, if we take σ a non-zero constant instead of a function, then from (23), we have $\lambda = -(\rho r + 2n)$ a constant. In this equation, making use of the value of r we get

$$\lambda = 2n[(2n+1)\rho - 1].$$

That is, λ is positive, when ρ is positive and λ is negative, when ρ is negative. Thus, we have the following corollary:

Corollary 3.2. *If the metric g of a (2n+1)-dimensional ($n > 1$) Kenmotsu manifold M represents an almost Ricci-Bourguignon soliton with the non-zero potential vector field V is a constant multiple of ξ, then the soliton is shrinking or expanding according as ρ is positive or negative, respectively.*

Theorem 3.3. *If the metric g of a (2n+1)-dimensional Kenmotsu manifold M represents an almost Ricci-Bourguignon soliton and ξ leaves $\lambda + \rho r$ invariant, then the following relations holds:*

$$(i)\quad div(\nabla(\lambda + \rho r)) = 4n(\lambda + \rho r + 2n) \tag{31}$$

$$(ii)\quad S(Y, \nabla(\lambda + \rho r)) = -3(2n+1)Y(\lambda + \rho r) + g(Y, \nabla_\xi \nabla(\lambda + \rho r)). \tag{32}$$

Proof. First of all, taking the covariant derivative of (1) gives

$$(\nabla_Z £_V g)(X, Y)Z + 2(\nabla_Z S)(X, Y) = 2Z(\lambda + \rho r)g(X, Y). \tag{33}$$

Now, we recall the following commutation formula of Yano [33]:

$$\begin{aligned} &(£_V \nabla_Z g - \nabla_Z £_V g - \nabla_{[V,Z]} g)(X, Y) \\ = \quad & -g((£_V \nabla)(Z, X), Y) - g((£_V \nabla)(Z, Y), X), \quad X, Y, Z \in \chi(M). \end{aligned} \tag{34}$$

Since the Riemannian metric g is parallel, plugging (33) into the above commutation formula yeilds

$$g((£_V \nabla)(Z, X), Y) + g((£_V \nabla)(Z, Y), X) + 2(\nabla_Z S)(X, Y) = 2Z(\lambda + \rho r)g(X, Y).$$

Interchanging the role of X, Y and Z cyclically in the foregoing equation and then using the symmetry of (1,2)-type tensor field, i.e., $(£_V \nabla)(X, Y) = (£_V \nabla)(Y, X)$ we obtain

$$\begin{aligned} g((£_V \nabla)(X, Y), Z) &= (\nabla_Z S)(X, Y) - (\nabla_X S)(Y, Z) - (\nabla_Y S)(Z, X) \\ &+ X(\lambda + \rho r)g(Y, Z) + Y(\lambda + \rho r)g(Z, X) - Z(\lambda + \rho r)g(X, Y), \end{aligned} \tag{35}$$

Substituting ξ in place of Y in the above equation and then using (12), we deduce

$$(£_V \nabla)(X, \xi) = 2QX + 4nX + X(\lambda + \rho r)\xi + \xi(\lambda + \rho r)X - \nabla(\lambda + \rho r)\eta(X)$$

where we have used the symmetric property of Q. Next, keeping in mind the hypothesis: the vector field ξ leaves $\lambda + \rho r$ invarient, i.e., $\xi(\lambda + \rho r) = 0$. we find

$$(£_V \nabla)(X, \xi) = 2QX + 4nX + X(\lambda + \rho r)\xi - \nabla(\lambda + \rho r)\eta(X), \quad X \in \chi(M). \tag{36}$$

Differentiating (36) covariantly along Y and utilizing (6) gives

$$\begin{aligned} &(\nabla_Y £_V \nabla)(X, \xi) + (£_V \nabla)(X, Y) - 2\eta(Y)(QX + 2nX) \\ = \quad & 2(\nabla_Y Q)X + g(X, \nabla_Y \nabla(\lambda + \rho r))\xi + X(\lambda + \rho r)(Y - \eta(Y)\xi) \\ - \quad & \eta(X)\nabla_Y \nabla(\lambda + \rho r) - \nabla(\lambda + \rho r)[g(X, Y) - \eta(X)\eta(Y)]. \end{aligned} \tag{37}$$

Since *Hess* is symmetric and $£_V \nabla$ is symmetric, making use of (37) in the following commutation formula of Yano [33]:

$$(£_V R)(X, Y)Z = (\nabla_X £_V \nabla)(Y, Z) - (\nabla_Y £_V \nabla)(X, Z)$$

we acquire

$$\begin{aligned}(\pounds_V R)(X,Y)\xi &= 2\{(\nabla_X Q)Y - (\nabla_Y Q)X + \eta(X)QY - \eta(Y)QX\} + 4n\{\eta(X)Y - \eta(Y)X\}\\ &+ Y(\lambda+\rho r)X - X(\lambda+\rho r)Y - \eta(Y)\nabla_X\nabla(\lambda+\rho r)\\ &+ \eta(X)\nabla_Y\nabla(\lambda+\rho r), \quad X,Y\in\chi(M).\end{aligned} \tag{38}$$

Inserting $Y=\xi$ in (38) gives

$$(\pounds_V R)(X,\xi)\xi = -X(\lambda+\rho r)\xi - \nabla_X\nabla(\lambda+\rho r) + \eta(X)\nabla_\xi\nabla(\lambda+\rho r). \tag{39}$$

On the other hand, operating $\pounds_V$ to the formula $R(X,\xi)\xi = -X+\eta(X)\xi$ yeilds

$$(\pounds_V R)(X,\xi)\xi + g(X,\pounds_V\xi)\xi - 2\eta(\pounds_V\xi)X = \{(\pounds_V\eta)X\}\xi,$$

which by virtue of (39) becomes

$$-\nabla_X\nabla(\lambda+\rho r) + \eta(X)\nabla_\xi\nabla(\lambda+\rho r) - X(\lambda+\rho r)\xi + g(X,\pounds_V\xi)\xi - 2\eta(\pounds_V\xi)X = \{(\pounds_V\eta)X\}\xi. \tag{40}$$

With the help of (12), the equation (1) takes the form

$$(\pounds_V)g(X,\xi) = 2(\lambda+\rho r+2n)\eta(X). \tag{41}$$

Now, Lie-differentiation of $\eta(X)=g(X,\xi)$ and $g(\xi,\xi)=1$ along V and taking into account of (41) provides

$$(\pounds_V\eta)(X) - g(X,\pounds_V\xi) = 2(\lambda+\rho r+2n)\eta(X),$$

$$\eta(\pounds_V\xi) = -(\lambda+\rho r+2n).$$

Utilizing these equations in (40) yields

$$\nabla_X\nabla(\lambda+\rho r) = \eta(X)\nabla_\xi\nabla(\lambda+\rho r) - X(\lambda+\rho r)\xi + 2(\lambda+\rho r+2n)(X-\eta(X)\xi). \tag{42}$$

Tracing of this provides (*i*). Since g is parallel, taking its covariant derivative along Y and noting that (22), we find

$$\begin{aligned}&\nabla_Y\nabla_X\nabla(\lambda+\rho r)\\ =\;& \eta(\nabla_Y X)\nabla_\xi\nabla(\lambda+\rho r) + [g(X,Y)-\eta(X)\eta(Y)]\nabla_\xi\nabla(\lambda+\rho r)\\ +\;& \eta(X)\nabla_Y\nabla_\xi\nabla(\lambda+\rho r) - YX(\lambda+\rho r)\xi - X(\lambda+\rho r)[Y-\eta(Y)\xi] + 2Y(\lambda+\rho r)[X-\eta(X)\xi]\\ +\;& 2(\lambda+\rho r+2n)[\nabla_Y X - \eta(\nabla_Y X)\xi - \{g(X,Y)-2\eta(X)\eta(Y)\}\xi - \eta(X)Y].\end{aligned} \tag{43}$$

Using the previous equation and (6) in (7), we obtain

$$\begin{aligned}&R(X,Y)\nabla(\lambda+\rho r)\\ =\;& \eta(Y)\nabla_X\nabla_\xi\nabla(\lambda+\rho r) - \eta(X)\nabla_Y\nabla_\xi\nabla(\lambda+\rho r)\\ -\;& 3Y(\lambda+\rho r)[X-\eta(X)\xi] + 3X(\lambda+\rho r)[Y-\eta(Y)\xi]\\ -\;& 2(\lambda+\rho r+2n)[\eta(Y)X-\eta(X)Y].\end{aligned} \tag{44}$$

At this point, contracting (44) over X and using the hypothesis: $\xi(\lambda+\rho r)=0$, we get

$$\begin{aligned}S(Y,\nabla(\lambda+\rho r)) &= \eta(Y)div(\nabla_\xi\nabla(\lambda+\rho r)) - g(\nabla_Y\nabla_\xi\nabla(\lambda+\rho r),\xi)\\ &- (6n+3)Y(\lambda+\rho r) - 4n(\lambda+\rho r+2n)\eta(Y).\end{aligned} \tag{45}$$

Inserting $Y=\xi$ in above equation and using (12), it gives

$$div(\nabla_\xi\nabla(\lambda+\rho r)) = 4n(\lambda+\rho r+2n). \tag{46}$$

Using this value in the equation (45), we get

$$S(Y,\nabla(\lambda+\rho r)) = -(6n+3)Y(\lambda+\rho r) - g(\xi,\nabla_Y\nabla_\xi\nabla(\lambda+\rho r)). \tag{47}$$

It is easy to see that $g(\xi,\nabla_\xi\nabla(\lambda+\rho r)) = 0$, and therefore, by (6) and the symmetry of Hess, we have

$$g(\nabla_Y\nabla_\xi\nabla(\lambda+\rho r),\xi) = -g(Y,\nabla_\xi\nabla(\lambda+\rho r)).$$

Using the above equation in (47), we get the result (ii). □

Theorem 3.4. *If the metric g of a Kenmotsu manifold M represents an almost Ricci-Bourguignon Soliton and ξ leaves $\lambda+\rho r$ invariant, then the metric g of M reduces to Ricci-Bourguignon Soliton.*

Proof. In a Kenmotsu manifold, we now recall the following identity (see, [23]):

$$R(\phi X,\phi Y)Z = R(X,Y)Z + g(Y,Z)X - g(X,Z)Y + g(Y,\phi Z)\phi X - g(X,\phi Z)\phi Y. \tag{48}$$

Inserting X by ϕX and Y by ϕY in (44) and using (48) along with the anti-symmetry of ϕ, we have

$$\begin{aligned} R(X,Y)\nabla(\lambda+\rho r) &= g(X,\nabla(\lambda+\rho r))Y - g(Y,\nabla(\lambda+\rho r))X \\ &- 2[g(\phi Y,\nabla(\lambda+\rho r))\phi X - g(\phi X,\nabla(\lambda+\rho r))\phi Y]. \end{aligned} \tag{49}$$

Further, contracting the above equation over X, we obtain

$$S(Y,\nabla(\lambda+\rho r)) = -2(n+1)Y(\lambda+\rho r). \tag{50}$$

Combining (32) and (50), we obtain

$$g(Y,\nabla_\xi\nabla(\lambda+\rho r)) = (4n+1)Y\nabla(\lambda+\rho r). \tag{51}$$

Further differentiating $g(\xi,\nabla(\lambda+\rho r)) = 0$ along Y and using symmetry of Hess and (6) gives

$$g(Y,\nabla_\xi\nabla(\lambda+\rho r)) = -Y(\lambda+\rho r).$$

In view of this (51) becomes $2(2n+1)Y(\lambda+\rho r) = 0$, and therefore, $Y(\lambda+\rho r) = 0$. Consequently, $(\lambda+\rho r)$ is constant. Thus, g reduces to Ricci-Bourguignon soliton. This proves our theorem. □

4. Gradient Almost Ricci-Bourguignon Solitons on Kenmotsu Manifolds

In this section, we consider the gradient almost Ricci-Bourguignon soliotns on Kenmotsu manifolds. In [32], Wang consider the gradient almost Ricci soliotns on three-dimensional Kenmotsu manifolds and proved that if a metric of a three-dimensional Kenmotsu manifold admits a gradient almost Ricci soliton, then the manifold is of constant negative curvature or the potential vector field is poinwise collinear with the Reeb vector field. Also, in [17], it is shown that if the metric of a Kenmotsu manifold represents a gradient Ricci almost soliton, then it is $\eta-$Einstein and the soliton is expanding. Since the scalar curvature on Kenmotsu manifold is not a constant, gradient almost Ricci-Bourguignon soliton is a generalization of Ricci soliton. Here, we generalize the above result for gradient almost Ricci-Bouguignon soliton.

Here, we prove the following:

Theorem 4.1. *If the metric g of a Kenmotsu manifold M represents an almost Ricci-Bourguignon soliton, then M is either Einstein or M admits a gradient $\eta-$Yamabe soliton.*

Proof. Let the metric g of M represents a gradient almost Ricci-Bourguignon soliton. Then from (2), we have

$$\nabla_X \nabla f + QX = (\lambda + \rho X), \quad X \in \chi(M). \tag{52}$$

By direct computations, the curvature tensor obtained from (52) and (7) satisfies

$$R(X, Y)\nabla f = (\nabla_Y Q)X - (\nabla_X Q)Y + X(\lambda + \rho r)Y - Y(\lambda + \rho r)X, \quad X, Y \in \chi(M). \tag{53}$$

Replacing X by ξ in (53) and then employing (9), we obtain

$$(\xi f)Y - (Yf)\xi = (\nabla_Y Q)\xi - (\nabla_\xi Q)Y + \xi(\lambda + \rho r)Y - Y(\lambda + \rho r)\xi. \tag{54}$$

Next, from (13) and (14) we find

$$(\nabla_Y Q)\xi - (\nabla_\xi Q)Y = QY + 2nY. \tag{55}$$

Inserting (55) into (54) provides

$$\xi(f - (\lambda + \rho r))Y - g(Y, \nabla(f - (\lambda + \rho r)))\xi = QY + 2nY. \tag{56}$$

Choosing inner product of (56) with ξ and using $Q\xi = -2n\xi$ gives

$$Y(f - (\lambda + \rho r)) = \xi(f - (\lambda + \rho r))\eta(Y),$$

from which we have

$$d(f - (\lambda + \rho r)) = \xi(f - (\lambda + \rho r))\eta, \tag{57}$$

Where d is the exterior derivative. This indicates that $f - (\lambda + \rho r)$ is invariant along the distribution $\mathcal{D}$, that is, $Y(f - (\lambda + \rho r)) = 0$ for any vector field $Y \in \mathcal{D}$. Calling (57) into (56), we entails that

$$\xi(f - (\lambda + \rho r))\{\eta(Y)\xi - Y\} = QY + 2nY \tag{58}$$

Contraction of the foregoing equation gives

$$\xi(f - (\lambda + \rho r)) = -\left(\frac{r}{2n} + 2n + 1\right). \tag{59}$$

Making use of (59) in (58), one immediately obtain η-Einstein condition (26). Now, identifying X with ∇f in (26), we deduce

$$S(Y, \nabla f) = \left(\frac{r}{2n} + 1\right)(Yf) - \left(\frac{r}{2n} + 2n + 1\right)(\xi f)\eta(Y). \tag{60}$$

Next, contraction of (53) over X gives

$$S(Y, \nabla f) = \frac{1}{2}Y(r) - 2nY(\lambda + \rho r).$$

Comparing the previous two equations, we can see that

$$Y(r) = 4nY(\lambda + \rho r) + \left(\frac{r}{2n} + 1\right)(Yf) - \left(\frac{r}{2n} + 2n + 1\right)(\xi f)\eta(Y). \tag{61}$$

Replacing Y by ξ in (61) and recalling (59), one can easily get

$$\xi(r) = 2\{r + 2n(2n + 1)\}. \tag{62}$$

Action of d on (57), we get $dr \wedge \eta = 0$, where we used $d = 0$ and $d\eta = 0$. Hence by virtue of (62), we have

$$Y(r) = 2\{r + 2n(2n+1)\}\,\eta(Y), \quad Y \in \chi(M). \tag{63}$$

Suppose that Y in (61) is orthogonal to ξ. Taking into account $f - (\lambda + \rho r)$ being a constant along $\mathcal{D}$ and using (57) and (63), then we get $\{r + 2n(2n+1)\}\,(Yf) = 0$ for any $Y \in \mathcal{D}$. This implies that

$$\{r + 2n(2n+1)\}\,(\nabla f - (\xi f)\xi) = 0. \tag{64}$$

If $r = -2n(2n+1)$, then the equation (26) shows that $S(X,Y) = -2ng(X,Y)$ and hence M is Einstein. On the other hand, suppose $r \neq -2n(2n+1)$ in some open set O of M, then from (64), we have $\nabla f = (\xi f)\xi$. This implies that $df = (\xi f)\eta$. The exterior derivative of this expression gives

$$d^2 f = d(\xi f) \wedge \eta = 0,$$

where $d^2 f = 0$ and $d\eta = 0$ are used. This entails that ξf is constant. The covarient derivative of $\nabla f = (\xi f)\xi$ along the vector field Y gives

$$\nabla_Y \nabla f = Y(\xi f)\xi + (\xi f)\,\{Y - \eta(Y)\xi\} = \xi(f)\,\{Y - \eta(Y)\xi\},$$

since ξ is constant. This equation takes the form

$$Hess(f) = \xi(f)(g - \eta \otimes \eta).$$

A Riemannian manifold M of dimension m is said to be a gradient generalized soliton [2] if there exist $\mathbf{N_1}, \mathbf{N_2}, \mathbf{N_3} \in \mathbb{R}$ such that

$$\nabla\nabla k + \mathbf{N_1} Q = \mathbf{N_2} I + \mathbf{N_3} \eta \otimes \xi,$$

where Q denotes the Ricci operator, I is the identity map, k is a smooth function on M and η is a 1-form associated to ξ, that is, $g(Y,\xi) = \eta(Y)$ for all Y. Particularly, if we choose $\mathbf{N_1} = 0$ in the above equation then the gradient generalized soliton reduces to the gradient $\eta-$Yamabe soliton. From last two equations, we conclude that M is a gradient $\eta-$Yamabe soliton. This completes the proof of the theorem. □

Moreover, in the first case, M is of constant scalar curvature $-2n(2n+1)$. Therefore, it follows that (59) that $\xi f = \xi(\lambda + \rho r)$. Therefore, $\nabla f = \nabla(\lambda + \rho r)$. Thus equation (52) can be exhibited as

$$\nabla_Y \nabla(\lambda + \rho r) = \{(\lambda + \rho r) + 2n\}\,Y, \quad Y \in \chi(M). \tag{65}$$

Recalling the Theorem 2 of Tashiro [29] we obtain that M is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}$, when M is complete. Here, we have the following corollory:

Corollary 4.2. *If the metric g of (2n+1)-dimensionaal complete Kenmotsu manifold M of constant scalar curvature admits a gradient almost Ricci-Bourguignon soliton, then M is locally isometric to a hyperbolic space $\mathbb{H}^{2n+1}$, provided $\nabla f \neq (\xi f)\xi$.*

In the case of Ricci-Bouguignon soliotn, λ is a constant and therefore it is a particular case of almost Ricci-Bourguignon soliton. Thus, in a Kenmotsu manifold (M, g) if g admits a Ricci-Bouguignon soliton and $\nabla f \neq (\xi f)\xi$, then we have $r = -2n(2n+1)$. In view of this fact, equation (2) reduces to

$$\nabla^2 f = \mu g, \tag{66}$$

where $\mu = 2n + \lambda + \rho r$. That is, g reduces to gradient conformal soliton [7]. Thus we state:

Corollary 4.3. *If the metric g of Kenmotsu manifold M of constant scalar curvature admits a gradient Ricci-Bourguignon soliton, then g reduces to gradient conformal soliton.*

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