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On initial inverse problem for Caputo fractional elliptic equation

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Abstract. In this work, we are interested to study the initial inverse problem for elliptic equation with Caputo derivative. We obtained some different results regarding the well-posedness of the solution corresponding to the given input data. In addition, we also derive the upper bound of the derivative of the mild solution. The principal analysis is based on the estimation of the Mittag-Leffler function, combined with main analysis in Hilbert scale authorized, and we also use fixed point techniques to prove the inequalities in this paper.

1. Introduction

Fractional calculus has extensive use in several sectors, including biology, engineering, physics, and more. While there are many different kinds of minor derivatives, the community focuses mostly on two types of derivatives: Riemann-Liouville and Caputo. Referring to some fascinating studies on Caputo or Riemann-Liouville, we have the following [16, 1, 12, 13, 14, 15, 25, 26, 27, 28].

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with sufficiently smooth boundary $\partial\Omega, T$ be a positive number. In this paper, we examined the following model

$$\begin{cases} {}_C D_t^{\alpha} u + \Delta u = F(x,t), & \text{in} \quad \Omega \times (0,T], \\ u|_{\partial\Omega} = 0, & \text{in} \quad \Omega, \\ u_t(x,0) = 0, & \text{in} \quad \Omega, \end{cases} \tag{1}$$

with the terminal condition

$$u(x,T) = g(x), \qquad x \in \Omega. \tag{2}$$

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Here g is the function defined later. The symbol ${}_C D_t^\alpha z$ is the Caputo derivative, (see [18, 6])

$$\begin{cases} {}_C D_t^\alpha z(x,t) = \dfrac{1}{\Gamma(2-\alpha)} \displaystyle\int_0^t (t-r)^{1-\alpha} \dfrac{\partial^2 z}{\partial r^2}(x,r)\mathrm{d}r, & \text{for } 1<\alpha<2, \\ {}_C D_t^\alpha z(t,x) = \dfrac{\partial^\alpha z}{\partial t^\alpha}(x,t), & \text{for } \alpha = 1,2, \end{cases} \tag{3}$$

and Γ is the Gamma function. This model mentioned above is an approximation of the elliptic equation. Indeed, if $\alpha \to 2^-$ then the solution of Problem (1) tends to the solution of the elliptic equation. The elliptic equation has many applications, see in [7]. There are several results on Cauchy elliptic equations with classical derivatives such as [17, 10, 12] and references therein. In some phenomena related to memory, the fractional derivative model is often used instead of the classical derivative. It explains a number of phenomena related to past distribution or viscosity models. In [8], Binh-Thang-Phuong [8] considered the elliptic equation under the Caputo derivative on the plane

$$\begin{cases} {}_C D_t^\alpha u + \Delta u = F(x,t), & \text{in } \mathbb{R}^2 \times (0,T], \\ u(x,0) = 0, & \text{in } \mathbb{R}^2, \\ u_t(x,0) = 0, & \text{in } \mathbb{R}^2. \end{cases} \tag{4}$$

They obtained several regularity results for the mild solution based on various assumptions of the input data. The principal techniques of the analysis is based on the bound of the Mittag-Leffler functions, combined with analysis in Hilbert scales space. In [9], the authors studied an initial value problem for a class of 2D time-fractional diffusion evolution equations with Riemann–Liouville fractional derivative.

They focused the existence and ill-posedness result (in the sense of Hadamard) in some cases of the source terms. By Fourier truncation method, the authors founded the regularized problems, and shown the error estimate between the exact solution and the approximate solution. Next, in [11], the authors considered a Cauchy problem for a semi linear fractional elliptic equation. They known the approximated solution through truncation method, the order logarithmic is error estimate between the regularized solution and the sought solution.

This type of backward problem in the fractional diffusion equation has an important application foundation and is receiving more and more attention. The final value problem is an inverse problem that requires redefining the distribution at the initial time when the distribution at the past time is known. Backward problem appear in many applications, such as image deblurring and inpainting. Some typical papers on backward problem can be listed as follows, see in [19, 12, 22, 23, 24].

This paper is organized as follows. Under homogeneous case, we obtain the well-posedness of the mild solution. Under the inhomogeneous linear source term, we also get the regularity results of the mild solution with some fixed point techniques used.

2. Preliminaries

Definition 2.1. *The spectral problem*

$$\begin{cases} \Delta e_n(x) = -\lambda_n e_n(x), & x \in \Omega, \\ e_n(x) = 0, & x \in \partial\Omega, \end{cases}$$

admits the eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \dots$, *and corresponding eigenfunctions* $e_n \in H_0^1(\Omega)$.

Definition 2.2. *The Hilbert scale space* $\mathbb{H}^s(\Omega)$ *given as follows*

$$\mathbb{H}^s(\Omega) = \Big\{ f \in L^2(\Omega) \,\Big|\, \sum_{n=1}^{\infty} \lambda_n^{2s} \Big(\int_\Omega f(x) e_n(x) \mathrm{d}x \Big)^2 < \infty \Big\}, \tag{5}$$

for any $s \geq 0$, with the the norm

$$\|f\|_{\mathbb{H}^s(\Omega)} = \Big(\sum_{n=1}^{\infty} \lambda_n^{2s} \Big(\int_\Omega f(x) e_n(x) \mathrm{d}x \Big)^2 \Big)^{1/2}, \quad f \in \mathbb{H}^s(\Omega). \tag{6}$$

Definition 2.3. *The Mittag-Leffler function*

$$E_{\alpha,\theta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \theta)}, \quad z \in \mathbb{C}, \tag{7}$$

where $\alpha > 0$ and $\theta \in \mathbb{R}$ are arbitrary constants.

Lemma 2.4 (See [21]). *Let $1 < \alpha_0 < \alpha_1 < 2$ and $\alpha \in [\alpha_0, \alpha_1]$. Then there exists a constant $m_1, m_2, \overline{m}_1, \overline{m}_2 > 0$ and $z > 0$ such that*

$$\frac{m_1}{\alpha} \exp(z^{\frac{1}{\alpha}}) \leq E_{\alpha,1}(z) \leq \frac{m_2}{\alpha} \exp(z^{\frac{1}{\alpha}}), \tag{8}$$

and

$$\frac{\overline{M}_1}{\alpha} \exp(z^{\frac{1}{\alpha}}) \leq z^{\frac{\alpha-1}{\alpha}} E_{\alpha,\alpha}(z) \leq \frac{\overline{M}_2}{\alpha} \exp(z^{\frac{1}{\alpha}}). \tag{9}$$

Lemma 2.5. *Let $z \in \mathbb{R}$. Then we have*

$$\frac{d}{dz} E_{\alpha,1}(z) = \frac{E_{\alpha,\alpha}(z)}{\alpha}, \frac{d}{dz} E_{\alpha,1}(\lambda z^\alpha) = \frac{1}{z} E_{\alpha,0}(\lambda z^\alpha). \tag{10}$$

Lemma 2.6. *Let $\alpha \in (1,2)$, and $\lambda > 0$ and $\mu > 0$, we get*

$$\frac{E_{\alpha,1}(\mu) - E_{\alpha,1}(-\mu)}{2\mu} = E_{2\alpha,\alpha+1}(\mu^2),$$

$$\frac{E_{\rho,\mu-1}(z) + (1 - \rho + \mu) E_{\rho,\mu}(z)}{\rho} = E_{\rho,\mu}(z^2),$$

$$E_{\alpha,\mu}(z) = \frac{1}{\Gamma(\mu)} + z E_{\alpha,\mu+\alpha}(z), \quad \lambda > 0, \mu > 0. \tag{11}$$

Lemma 2.7. *For $z > 0$, $\alpha \in (1,2)$, it gives (see p.5, [20])*

$$E_{2,\alpha}(z^2) = \frac{1}{2} (E_\alpha(z) + E_\alpha(-z)), \tag{12}$$

and

$$\frac{d^m}{dz^m} \left[z^{\beta-1} E_{\alpha,\beta}(z^\alpha) \right] = z^{\beta-m-1} E_{\alpha,\beta-m}(z^\alpha). \tag{13}$$

Lemma 2.8. *Let $h, k > 0$, then we receive*

$$\frac{1}{\Gamma(k)} \int_0^t (t-r)^{k-1} E_{\alpha,\beta}(\lambda r^\alpha) r^{\beta-1} dr = t^{\beta+k-1} E_{\alpha,\beta+k}(\lambda t^\alpha). \tag{14}$$

We can see the proof in Vol. 1, pp. 269–295 [3].

Lemma 2.9. *For $1 < \alpha < 2$, $\lambda > 0$, and $t > 0$, then we have*

a) $\partial_t E_{\alpha,1}(\lambda t^\alpha) = \lambda t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha)$, *and* $\partial_t (t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)) = t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha)$;

b) $\partial_t^\alpha E_{\alpha,1}(-\lambda t^\alpha) = -\lambda E_{\alpha,1}(-\lambda t^\alpha)$, *and* $\partial_t^\alpha (t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)$.

Definition 2.10. *The Gevrey class of functions of order $\Psi > 0$ and index $\Upsilon > 0$, defined by the spectrum of the Laplacian is denoted by*

$$\mathbb{G}^{\Psi}_{\alpha,\Upsilon}(\Omega) := \Big\{ f \in L^2(\Omega) : \sum_{j=1}^{\infty} \lambda_j^{2\Psi} \exp(2\Upsilon \lambda_j^{\frac{1}{\alpha}}) u_j^2 < \infty \Big\}, \tag{15}$$

and its norm given by

$$\|f\|_{\mathbb{G}^{\Psi}_{\alpha,\Upsilon}} = \Big(\sum_{j=1}^{\infty} \lambda_j^{2\Psi} \exp(2\Upsilon \lambda_j^{\frac{1}{\alpha}}) u_j^2 \Big)^{\frac{1}{2}}. \tag{16}$$

Lemma 2.11. *Let $1 < \alpha < 2$, then we get*

$$\frac{m_1}{\alpha} \exp(\lambda_j^{\frac{1}{\alpha}} t) \le E_{\alpha,1}\left(\lambda_j t^\alpha\right) \le \frac{m_2}{\alpha} \exp(\lambda_j^{\frac{1}{\alpha}} t). \tag{17}$$

and

$$\frac{\overline{m}_1}{\alpha} \lambda_j^{\frac{1}{\alpha}-1} t^{1-\alpha} \exp(t\lambda_j^{\frac{1}{\alpha}}) \le E_{\alpha,\alpha}(\lambda_j t^\alpha) \le \frac{\overline{m}_2}{\alpha} \lambda_j^{\frac{1}{\alpha}-1} t^{1-\alpha} \exp(t\lambda_j^{\frac{1}{\alpha}}). \tag{18}$$

Using Lemma (2.4) with $z = \lambda_j^{\frac{1}{\alpha}} t$, (17) and (18) is proved.

Lemma 2.12. *Let $1 < \alpha < 2$, then we receive*

$$\frac{m_1}{m_2} \exp\left(\lambda_j^{\frac{1}{\alpha}}(t-T)\right) \le \frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)}{E_{\alpha,1}(\lambda_j T^\alpha)} \le \frac{m_2}{m_1} \exp(\lambda_j^{\frac{1}{\alpha}}(t-T)). \tag{19}$$

Using Lemma (2.4), we derive that

$$\frac{m_1}{\alpha} \exp(\lambda_j^{\frac{1}{\alpha}} t) \le E_{\alpha,1}\left(\lambda_j t^\alpha\right) \le \frac{m_2}{\alpha} \exp(\lambda_j^{\frac{1}{\alpha}} t). \tag{20}$$

Since (20), one has

$$\frac{m_1}{m_2} \exp(\lambda_j^{\frac{1}{\alpha}}(t-T)) \le \frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)} \le \frac{m_2}{m_1} \exp(\lambda_j^{\frac{1}{\alpha}}(t-T)). \tag{21}$$

Lemma 2.13. *Let $1 < \alpha < 2$, then we have*

$$\frac{\overline{m}_1}{m_2} t^{1-\alpha} \lambda_j^{\frac{1}{\alpha}-1}((t-T)\lambda_j^{\frac{1}{\alpha}}) \le \frac{E_{\alpha,\alpha}(\lambda_j t^\alpha)}{E_{\alpha,1}(\lambda_j T^\alpha)} \le \frac{\overline{m}_2}{m_1} t^{1-\alpha} \lambda_j^{\frac{1}{\alpha}-1}((t-T)\lambda_j^{\frac{1}{\alpha}}). \tag{22}$$

Using (9), for any $0 \le t \le T$, we have

$$E_{\alpha,\alpha}(\lambda_j t^\alpha) \le \frac{\overline{m}_2}{\alpha} \exp((\lambda_j t^\alpha)^{\frac{1}{\alpha}}) \left(\lambda_j t^\alpha\right)^{\frac{1-\alpha}{\alpha}} = \frac{\overline{m}_2}{\alpha} \lambda_j^{\frac{1}{\alpha}-1} t^{1-\alpha} \exp\left(t\lambda_j^{\frac{1}{\alpha}}\right). \tag{23}$$

This inequality together with (20) yields to

$$\frac{E_{\alpha,\alpha}(\lambda_j t^\alpha)}{E_{\alpha,1}(\lambda_j T^\alpha)} \le \frac{\overline{m}_2}{m_1} t^{1-\alpha} \lambda_j^{\frac{1}{\alpha}-1}\left((t-T)\lambda_j^{\frac{1}{\alpha}}\right), \tag{24}$$

by a similar techniques, one has

$$\frac{E_{\alpha,\alpha}(\lambda_j t^\alpha)}{E_{\alpha,1}(\lambda_j T^\alpha)} \ge \frac{\overline{m}_1}{m_2} t^{1-\alpha} \lambda_j^{\frac{1}{\alpha}-1} \exp\left((t-T)\lambda_j^{\frac{1}{\alpha}}\right). \tag{25}$$

3. Homogeneous case $F = 0$

We consider the following problem

$$\begin{cases} {}_C D_t^\alpha w + \Delta w = 0, & \text{in} \quad \Omega \times (0,T], \\ w|_{\partial\Omega} = 0, & \text{in} \quad \Omega, \\ w_t(x,0) = 0, & \text{in} \quad \Omega, \end{cases} \tag{26}$$

with the terminal condition (2).

Theorem 3.1. *Let the terminal data $g \in \mathbb{H}^{q-\frac{\rho}{\alpha}}(\Omega)$ for $q \geq 0$ and $0 < \rho < 1$. Then problem* (26)-(2) *has a unique solution $w \in L^s(0,T;\mathbb{H}^q(\Omega))$ for any $1 < s < \frac{1}{\rho}$. In addition, we get*

$$\left\|w\right\|_{L^s(0,T;\mathbb{H}^q(\Omega))} \leq C(\rho,s,q) T^{\frac{1}{s}-\rho} \left\|g\right\|_{\mathbb{H}^{q-\frac{\rho}{\alpha}}(\Omega)}. \tag{27}$$

If $g \in \mathbb{H}^{q+\frac{1-\beta}{\alpha}-1}(\Omega)$ for $0 < \beta < 1$ then $w \in C^{1-\beta}([0,T];\mathbb{H}^q(\Omega))$. In addiiton, we get that

$$\left\|w\right\|_{C^{1-\beta}([0,T];\mathbb{H}^q(\Omega))} \leq \frac{C_\beta \overline{m}_2 T^{\alpha-1}}{m_1(1-\beta)} \theta^{1-\beta} \left\|g\right\|_{\mathbb{H}^{q+\frac{1-\beta}{\alpha}-1}(\Omega)}. \tag{28}$$

Let us assume that the problem (26) with the terminal condition (2) has a solution w, then we have

$$w_j(t) = E_{\alpha,1}\left(\lambda_j t^\alpha\right) w_j(0). \tag{29}$$

Taking the derivative of w_j, we get

$$w_j'(t) = \frac{d}{dt}\Big(E_{\alpha,1}\left(\lambda_j t^\alpha\right)\Big) w_j(0). \tag{30}$$

From (30), we know that

$$w_j(T) = E_{\alpha,1}(\lambda_j T^\alpha) w_j(0) = g_j. \tag{31}$$

Hence, we derive that

$$w_j(t) = \frac{E_{\alpha,1}(\lambda_j t^\alpha)}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)} \Big(\int_\Omega g(x) e_j(x) dx \Big). \tag{32}$$

This implies that

$$w(x,t) = \sum_{j=1}^\infty \frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)} \Big(\int_\Omega g(x) e_j(x) dx \Big) e_j(x). \tag{33}$$

Thank to the inequality $e^{-y} \leq C_\rho y^{-\rho}$ for any $\rho > 0$, one has

$$\left(\frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)} \right)^2 \leq \frac{m_2^2}{m_1^2} \exp(2\lambda_j^{\frac{1}{\alpha}}(t-T)) \leq |C_\rho|^2 \frac{m_2^2}{m_1^2} (T-t)^{-2\rho} \lambda_j^{\frac{-2\rho}{\alpha}}. \tag{34}$$

where we have used Lemma (2.12). This follows from (33) that

$$\begin{aligned} \left\|w(.,t)\right\|^2_{\mathbb{H}^q(\Omega)} &= \sum_{j=1}^\infty \lambda_j^{2q} \Big(\frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)} \Big)^2 \Big(\int_\Omega g(x) e_j(x) dx \Big)^2 \\ &\leq |C_\rho|^2 \frac{m_2^2}{m_1^2} (T-t)^{-2\rho} \sum_{j=1}^\infty \lambda_j^{2q-\frac{2\rho}{\alpha}} \Big(\int_\Omega g(x) e_j(x) dx \Big)^2. \end{aligned} \tag{35}$$

Therefore, we give the following bound

$$\|w(.,t)\|_{\mathbb{H}^q(\Omega)} \le C(\rho, m_2, m_1)(T-t)^{-\rho} \left\|g\right\|_{\mathbb{H}^{q-\frac{\rho}{\alpha}}(\Omega)}, \tag{36}$$

for any $\rho < 1$ and $C(\rho, m_2, m_1)$ depends on ρ, m_2, m_1. Since $1 < s < \frac{1}{\rho}$, then the integral $\int_0^T (T-t)^{-\rho s} dt$ is convergent. This allows us to get that $w \in L^s(0,T;\mathbb{H}^q(\Omega))$ since the following observation

$$\|w\|_{L^s(0,T;\mathbb{H}^q(\Omega))} = \left(\int_0^T \|w(.,t)\|^s_{\mathbb{H}^q(\Omega)} dt\right)^{1/s} \le C(\rho, m_1, m_1, s, q) T^{\frac{1}{s}-\rho} \|g\|_{\mathbb{H}^{q-\frac{\rho}{\alpha}}(\Omega)}. \tag{37}$$

From (33) and we take $t, t+\theta$ such that $0 \le t \le t+\theta \le T$, and $\theta > 0$. Then we get

$$\begin{aligned} w(x,t+\theta) - w(x,t) &= \sum_{j=1}^{\infty} \frac{E_{\alpha,1}\left(\lambda_j (t+\theta)^\alpha\right) - E_{\alpha,1}\left(\lambda_j t^\alpha\right)}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)} \left(\int_\Omega g(x) e_j(x) \mathrm{d}x\right) e_j(x) \\ &= \sum_{j=1}^{\infty} \frac{\lambda_j \int_t^{t+\theta} r^{\alpha-1} E_{\alpha,\alpha}(\lambda_j r^\alpha) dr}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)} \Big(\int_\Omega g(x) e_j(x) \mathrm{d}x\Big) e_j(x). \end{aligned} \tag{38}$$

Here in the last above equality, we have used Lemma 2.5 in order to obtain that

$$\frac{d}{dt} E_{\alpha,1}(\lambda_j t^\alpha) = \frac{dE_{\alpha,1}(\lambda_j t^\alpha)}{d(\lambda_j t^\alpha)} \frac{d(\lambda_j t^\alpha)}{dt} = \lambda_j t^{\alpha-1} E_{\alpha,\alpha}(\lambda_j t^\alpha). \tag{39}$$

In view of Lemma (2.13), we derive that for any $0 < r < T$

$$\frac{E_{\alpha,\alpha}(\lambda_j r^\alpha)}{E_{\alpha,1}(\lambda_j T^\alpha)} \le \frac{\overline{m}_2}{m_1} t^{1-\alpha} \lambda_j^{\frac{1}{\alpha}-1} \exp((r-T)\lambda_j^{\frac{1}{\alpha}}) \le \frac{C_\beta \overline{m}_2}{m_1} t^{1-\alpha} \lambda_j^{\frac{1-\beta}{\alpha}-1} (T-r)^{-\beta}, \tag{40}$$

where we used the inequality $e^{-z} \le C_\beta z^{-\beta}$ for any $z > 0$ and $\beta > 0$. Thus, if we choose $0 < \beta < 1$, one has

$$\frac{\lambda_j \int_t^{t+\theta} r^{\alpha-1} E_{\alpha,\alpha}(\lambda_j r^\alpha) dr}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)} \le \frac{C_\beta \overline{m}_2}{m_1} t^{1-\alpha} \lambda_j^{\frac{1-\beta}{\alpha}} \int_t^{t+\theta} r^{\alpha-1} (T-r)^{-\beta} dr. \tag{41}$$

It is obvious to see that

$$\begin{aligned} \int_t^{t+\theta} r^{\alpha-1} (T-r)^{-\beta} dr &\le T^{\alpha-1} \int_t^{t+\theta} (T-r)^{-\beta} dr \\ &= T^{\alpha-1} \frac{(T-t)^{1-\beta} - (T-t-\theta)^{1-\beta}}{1-\beta} \le \frac{T^{\alpha-1}\theta^{1-\beta}}{1-\beta}. \end{aligned} \tag{42}$$

Combining (40) and (42), we infer that

$$\frac{\lambda_j \int_t^{t+\theta} r^{\alpha-1} E_{\alpha,\alpha}(\lambda_j r^\alpha) dr}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)} \le \frac{C_\beta \overline{m}_2 T^{\alpha-1}}{m_1(1-\beta)} \lambda_j^{\frac{1-\beta}{\alpha}-1} \theta^{1-\beta}, \quad 0 < \beta < 1. \tag{43}$$

This inequality together with (38) yields that

$$\begin{aligned} \|w(x,t+\theta) - w(x,t)\|^2_{\mathbb{H}^q(\Omega)} &= \sum_{j=1}^{\infty} \lambda_j^{2q} \Big(\frac{\lambda_j \int_t^{t+\theta} r^{\alpha-1} E_{\alpha,\alpha}(\lambda_j r^\alpha) dr}{E_{\alpha,1}(\lambda_j T^\alpha)}\Big)^2 \Big(\int_\Omega g(x) e_j(x) dx\Big)^2 \\ &\le \Big(\frac{C_\beta \overline{m}_2 T^{\alpha-1}}{m_1(1-\beta)}\Big)^2 \theta^{2-2\beta} \sum_{j=1}^{\infty} \lambda_j^{2q+\frac{2-2\beta}{\alpha}-2} \left(\int_\Omega g(x) e_j(x) \mathrm{d}x\right)^2. \end{aligned} \tag{44}$$

By Parseval's equality, we derive that

$$\left\|w(x,t+\theta)-w(x,t)\right\|_{\mathbb{H}^q(\Omega)} \le \frac{C_\beta \overline{m}_2 T^{\alpha-1}}{m_1(1-\beta)}\theta^{1-\beta}\|g\|_{\mathbb{H}^{q+\frac{1-\beta}{\alpha}-1}(\Omega)}. \tag{45}$$

This implies that $w \in C^{1-\beta}\left([0,T];\mathbb{H}^q(\Omega)\right)$.

4. Linear inhomogeneous case $F = F(x,t)$

We consider the following problem

$$\begin{cases} {}_C D_t^\alpha w + \Delta w = F(x,t), & \text{in} \quad \Omega \times (0,T], \\ w|_{\partial\Omega} = 0, & \text{in} \quad \Omega, \\ w_t(x,0) = 0, & \text{in} \quad \Omega, \end{cases} \tag{46}$$

with the terminal condition (2). Our main aim in this section is to show that the well-posedness of the mild solution.

Theorem 4.1. *Let $g \in \mathbb{H}^{q-\frac{\rho}{\alpha}(\Omega)}$ for $q \ge 0, 0<\rho<1$, let $F \in L^2(0,T;\mathbb{G}_{\alpha,2T}^{2q-\frac{2\rho}{\alpha}+\frac{1}{\alpha}-2}(\Omega))$, then*

$$\|u(.,t)\|_{\mathbb{H}^q(\Omega)} \le C_\rho (T-t)^{-\rho}\left\|g\right\|_{\mathbb{H}^{q-\frac{\rho}{\alpha}(\Omega)}} + (T-t)^{-\rho}\left\|F\right\|_{L^2(0,T;\mathbb{G}_{\alpha,2T}^{2q-\frac{2\rho}{\alpha}+\frac{1}{\alpha}-2}(\Omega))} + \sqrt{\frac{m_2\overline{m}_2}{\alpha^2}}\left\|F\right\|_{L^2(0,T;\mathbb{G}_{\alpha,2T}^{2q+\frac{1}{\alpha}-2}(\Omega))}. \tag{47}$$

Let $g \in \mathbb{H}^{q-\frac{\rho}{\alpha}(\Omega)}$ and $F \in L^\infty(0,T;\mathbb{H}^q(\Omega))$, then we obtain

$$\|u(.,0)\|_{\mathbb{H}^q(\Omega)} \le C_\rho T^{-\rho}\|g\|_{\mathbb{H}^{q-\frac{\rho}{\alpha}(\Omega)}} + \sqrt{\left(\frac{\overline{m}_2}{m_1}\right)^2 C_\alpha \lambda_1^{\frac{3-3\alpha}{\alpha}}\frac{T^{2-\alpha}}{2-\alpha}}\ \left\|F\right\|_{L^\infty(0,T;\mathbb{H}^q(\Omega))}. \tag{48}$$

From [8], one gets

$$w_j(t) = E_{\alpha,1}\left(\lambda_j t^\alpha\right)w_j(0) + \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_j(t-\tau)^\alpha\right)F_j(s)ds. \tag{49}$$

By let $t=T$, we find that

$$g_j = w_j(T) = E_{\alpha,1}\left(\lambda_j T^\alpha\right)w_j(0) + \int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_j(T-\tau)^\alpha\right)F_j(\tau)d\tau. \tag{50}$$

Hence, one has the following equality

$$w_j(0) = \frac{g_j}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)} - \frac{\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_j(T-\tau)^\alpha\right)F_j(\tau)d\tau}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)}. \tag{51}$$

From (49) and (51), one gets

$$\begin{aligned} u_j(t) =& \frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)g_j}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)} + \frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_j(T-\tau)^\alpha\right)F_j(s)ds}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)} \\ &+ \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_j(t-\tau)^\alpha\right)F_j(\tau)d\tau. \end{aligned} \tag{52}$$

Hence, one has

$$u(x,t) = \sum_{j=1}^{\infty} \frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)} g_j e_j(x) - \sum_{j=1}^{\infty} \Big(\frac{E_{\alpha,1}(\lambda_j t^\alpha) \int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_j (T-\tau)^\alpha\right) F_j(s) ds}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)} \Big) e_j(x)$$
$$+ \sum_{j=1}^{\infty} \Big(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_j (t-\tau)^\alpha\right) F_j(\tau) d\tau \Big) e_j(x)$$
$$= \mathbb{K}_1(x,t) + \mathbb{K}_2(x,t) + \mathbb{K}_3(x,t). \tag{53}$$

Part 1. The case $t > 0$. Under this case, we get that

$$u(x,t) = \mathbb{K}_1(x,t) + \mathbb{K}_2(x,t) + \mathbb{K}_3(x,t). \tag{54}$$

By a similar techniques as in (35), we find that

$$\|\mathbb{K}_1(.,t)\|^2_{\mathbb{H}^q(\Omega)} \le C(\rho, m_2, m_1)(T-t)^{-2\rho} \sum_{j=1}^{\infty} \lambda_j^{2q - \frac{2\rho}{\alpha}} \Big(\int_\Omega g(x) e_j(x) dx \Big)^2. \tag{55}$$

Thus, we get that

$$\|\mathbb{K}_1(.,t)\|_{\mathbb{H}^q(\Omega)} \le C(\rho, m_2, m_1)(T-t)^{-\rho} \|g\|_{\mathbb{H}^{q-\frac{\rho}{\alpha}}(\Omega)}, \quad 0 < \rho < 1. \tag{56}$$

Let us continue to estimate the second term $\mathbb{K}_2$. It is obvious to see that

$$\|\mathbb{K}_2(.,t)\|^2_{\mathbb{H}^q(\Omega)} \le C_\rho (T-t)^{-2\rho} \sum_{j=1}^{\infty} \lambda_j^{2q - \frac{2\rho}{\alpha}} \Big(\int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_j (T-\tau)^\alpha\right) F_j(s) ds \Big)^2. \tag{57}$$

Using Hölder inequality, we find that

$$\Big(\int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_j (T-\tau)^\alpha\right) F_j(\tau) d\tau \Big)^2$$
$$\le \Big(\int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_j (T-\tau)^\alpha\right) ds \Big) \Big(\int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_j (T-\tau)^\alpha\right) F_j^2(s) ds \Big)$$
$$= \frac{E_{\alpha,1}\left(\lambda_j T^\alpha\right) - 1}{\lambda_j} \Big(\int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_j (T-\tau)^\alpha\right) F_j^2(s) ds \Big) = \mathbb{L}_1. \tag{58}$$

Here we note that

$$\int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda_j (T-\tau)^\alpha) ds = \frac{E_{\alpha,1}(\lambda_j T^\alpha) - 1}{\lambda_j}.$$

In view of Lemma (2.11), we get

$$E_{\alpha,1}(\lambda_j T^\alpha) \le \frac{m_2}{\alpha} \exp\left(\lambda_j^{\frac{1}{\alpha}} T\right), \quad E_{\alpha,\alpha}(\lambda_j t^\alpha) \le \frac{\overline{m}_2}{\alpha} \lambda_j^{\frac{1}{\alpha}-1} t^{1-\alpha} \exp(t \lambda_j^{\frac{1}{\alpha}}). \tag{59}$$

This together with (57) yields to

$$\left\|\mathbb{K}_2(.,t)\right\|^2_{\mathbb{H}^q(\Omega)} \le \frac{C_\rho (T-t)^{-2\rho} m_2 \overline{m}_2}{\alpha^2} \sum_{j=1}^{\infty} \lambda_j^{2q - \frac{2\rho}{\alpha} + \frac{1}{\alpha} - 2} \exp\left(t \lambda_j^{\frac{1}{\alpha}}\right) \int_0^T \exp((T-\tau)\lambda_j^{\frac{1}{\alpha}}) F_j^2(\tau) d\tau. \tag{60}$$

It is obvious to see that

$$\sum_{j=1}^{\infty} \lambda_j^{2q-\frac{2\rho}{\alpha}+\frac{1}{\alpha}-2} \exp\left(t\lambda_j^{\frac{1}{\alpha}}\right) \int_0^T \exp\left((T-\tau)\lambda_j^{\frac{1}{\alpha}}\right) F_j^2(\tau) d\tau \le \int_0^T \left\|F(.,\tau)\right\|^2_{\mathbb{G}_{\alpha,2T}^{2q-\frac{2\rho}{\alpha}+\frac{1}{\alpha}-2}(\Omega)} d\tau.$$

Thus, one gets

$$\left\|\mathbb{K}_2(x,t)\right\|_{\mathbb{H}^q(\Omega)} \le \sqrt{\frac{C_\rho (T-t)^{-2\rho} m_2 \overline{m}_2}{\alpha^2}} \left\|F\right\|_{L^2(0,T;\mathbb{G}_{\alpha,2T}^{2q-\frac{2\rho}{\alpha}+\frac{1}{\alpha}-2}(\Omega))}. \tag{61}$$

In order to estimate the third term $\mathbb{K}_3$, we observe that its evaluation is similar to the evaluation for the component $\mathbb{L}_1$. Thus, we obtain

$$\begin{aligned} &\left(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_j (t-\tau)^\alpha\right) F_j(\tau) d\tau\right)^2 \\ &\qquad \le \frac{m_2 \overline{m}_2}{\alpha^2} \lambda_j^{\frac{1}{\alpha}-2} \exp\left(\lambda_j^{\frac{1}{\alpha}} t\right) \int_0^t \exp\left((t-\tau)\lambda_j^{\frac{1}{\alpha}}\right) F_j^2(\tau) d\tau. \end{aligned} \tag{62}$$

This implies that

$$\begin{aligned} \|\mathbb{K}_3(.,t)\|^2_{\mathbb{H}^q(\Omega)} &= \sum_{j=1}^{\infty} \lambda_j^{2q} \Big(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda_j (t-\tau)^\alpha) F_j(\tau) d\tau\Big)^2 \\ &\le \frac{m_2 \overline{m}_2}{\alpha^2} \sum_{j=1}^{\infty} \lambda_j^{2q+\frac{1}{\alpha}-2} \int_0^t \exp((t-\tau)\lambda_j^{\frac{1}{\alpha}}) F_j^2(\tau) d\tau. \end{aligned} \tag{63}$$

Thus, we obtain that

$$\left\|\mathbb{K}_3(.,t)\right\|_{\mathbb{H}^q(\Omega)} \le \sqrt{\frac{m_2 \overline{m}_2}{\alpha^2}} \left\|F\right\|_{L^2(0,T;\mathbb{G}_{\alpha,2T}^{2q+\frac{1}{\alpha}-2}(\Omega))}. \tag{64}$$

By collecting some previous results, we find that

$$\begin{aligned} \left\|u(.,t)\right\|_{\mathbb{H}^q(\Omega)} &\le \sum_{j=1}^{3} \left\|\mathbb{K}_j(.,t)\right\|_{\mathbb{H}^q(\Omega)} \le C_\rho (T-t)^{-\rho} \left\|g\right\|_{\mathbb{H}^{q-\frac{\rho}{\alpha}}(\Omega)} \\ &\quad + (T-t)^{-\rho} \left\|F\right\|_{L^2(0,T;\mathbb{G}_{\alpha,2T}^{2q-\frac{2\rho}{\alpha}+\frac{1}{\alpha}-2}(\Omega))} + \sqrt{\frac{m_2 \overline{m}_2}{\alpha^2}} \, ||F||_{L^2(0,T;\mathbb{G}_{\alpha,2T}^{2q+\frac{1}{\alpha}-2}(\Omega))}. \end{aligned} \tag{65}$$

Part 2. The case $t = 0$. Under this case, we get that

$$u(x,0) = \mathbb{K}_1(x,0) + \mathbb{K}_2(x,0).$$

Using the similar techniques as in (56), we know that

$$\left\|\mathbb{K}_1(x,0)\right\|_{\mathbb{H}^q(\Omega)} \le C_\rho T^{-\rho} \left\|g\right\|_{\mathbb{H}^{q-\frac{\rho}{\alpha}}(\Omega)}, \quad 0 < \rho < 1. \tag{66}$$

In addition, one gets

$$\mathbb{K}_2(x,0) = -\sum_{j=1}^{\infty} \Big(\frac{\int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda_j (T-\tau)^\alpha) F_j(\tau) d\tau}{E_{\alpha,1}(\lambda_j T^\alpha)}\Big) e_j(x). \tag{67}$$

Using Lemma (2.13), we get that

$$E_{\alpha,\alpha}(\lambda_j(T-\tau)^\alpha)\left(E_{\alpha,1}(\lambda_j T^\alpha)\right)^{-1} \le \frac{\overline{m}_2}{m_1}(T-\tau)^{1-\alpha}\lambda_j^{\frac{1}{\alpha}-1}\exp(-\tau\lambda_j^{\frac{1}{\alpha}}). \tag{68}$$

Hence, we obtain

$$\left|\frac{\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_j(T-\tau)^\alpha\right)F_j(\tau)d\tau}{E_{\alpha,1}(\lambda_j T^\alpha)}\right| \le \frac{\overline{m}_2}{m_1}\sum_{j=1}^\infty \lambda_j^{\frac{1}{\alpha}-1}\Big(\int_0^T \exp(-\tau\lambda_j^{\frac{1}{\alpha}})F_j(\tau)d\tau\Big). \tag{69}$$

Using Hölder inequality, we obtain

$$\begin{aligned}\left\|\mathbb{K}_2(.,0)\right\|^2_{\mathbb{H}^q(\Omega)} &= \sum_{j=1}^\infty\Big(\lambda_j^{2q}\frac{\int_0^T (T-\tau)^{\alpha-1}E_{\alpha,\alpha}\left(\lambda_j(T-\tau)^\alpha\right)F_j(\tau)d\tau}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)}\Big)^2\\ &\le \Big(\frac{\overline{m}_2}{m_1}\Big)^2\sum_{j=1}^\infty \lambda_j^{2q}\lambda_j^{\frac{2}{\alpha}-2}\Big(\int_0^T \exp\Big(-\tau\lambda_j^{\frac{1}{\alpha}}\Big)F_j(\tau)d\tau\Big)^2\\ &\le \Big(\frac{\overline{m}_2}{m_1}\Big)^2\sum_{j=1}^\infty \lambda_j^{2q}\lambda_j^{\frac{2}{\alpha}-2}\Big(\int_0^T \exp(-2\tau\lambda_j^{\frac{1}{\alpha}})|F_j(\tau)|^2d\tau\Big).\end{aligned} \tag{70}$$

Using the inequality $e^{-y} \le C_\mu y^{-\mu}$ with $\mu = \frac{\alpha-1}{2}$ and $y = \tau\lambda_j^{\frac{1}{\alpha}}$, we get

$$\exp\Big(-2\tau\lambda_j^{\frac{1}{\alpha}}\Big) \le C_\alpha \tau^{1-\alpha}\lambda_j^{\frac{1-\alpha}{\alpha}}.$$

This implies that

$$\sum_{j=1}^\infty \lambda_j^{2q}\lambda_j^{\frac{2}{\alpha}-2}\Big(\int_0^T \exp(-2\tau\lambda_j^{\frac{1}{\alpha}})|F_j(\tau)|^2d\tau\Big) \le C_\alpha\int_0^T \tau^{1-\alpha}\Big(\sum_{j=1}^\infty \lambda_j^{2q}\lambda_j^{\frac{3-3\alpha}{\alpha}}|F_j(\tau)|^2\Big)d\tau. \tag{71}$$

Combining (70) and (71), and noting that $\lambda_j^{\frac{3-3\alpha}{\alpha}} \le \lambda_1^{\frac{3-3\alpha}{\alpha}}$, one gets

$$\begin{aligned}\left\|\mathbb{K}_2(.,0)\right\|^2_{\mathbb{H}^q(\Omega)} &\le \Big(\frac{\overline{m}_2}{m_1}\Big)^2 C_\alpha\lambda_1^{\frac{3-3\alpha}{\alpha}}\int_0^T \tau^{1-\alpha}\left\|F(\tau)\right\|^2_{\mathbb{H}^q(\Omega)}d\tau\\ &\le \Big(\frac{\overline{m}_2}{m_1}\Big)^2 C_\alpha\lambda_1^{\frac{3-3\alpha}{\alpha}}\left\|F\right\|^2_{L^\infty(0,T;\mathbb{H}^q(\Omega))}\Big(\int_0^T \tau^{1-\alpha}d\tau\Big)\\ &= \Big(\frac{\overline{m}_2}{m_1}\Big)^2 C_\alpha\lambda_1^{\frac{3-3\alpha}{\alpha}}\frac{T^{2-\alpha}}{2-\alpha}\left\|F\right\|^2_{L^\infty(0,T;\mathbb{H}^q(\Omega))}.\end{aligned} \tag{72}$$

Thus, we deduce that

$$||\mathbb{K}_2(.,0)||_{\mathbb{H}^q(\Omega)} \le \Big(\Big(\frac{\overline{m}_2}{m_1}\Big)^2 C_\alpha\lambda_1^{\frac{3-3\alpha}{\alpha}}\frac{T^{2-\alpha}}{2-\alpha}\Big)^{\frac{1}{2}}\left\|F\right\|_{L^\infty(0,T;\mathbb{H}^q(\Omega))}. \tag{73}$$

Combining (66) and (73), we deduce that

$$\begin{aligned}\left\|u(.,0)\right\|_{\mathbb{H}^q(\Omega)} &\le \left\|\mathbb{K}_1(.,0)\right\|_{\mathbb{H}^q(\Omega)} + \left\|\mathbb{K}_2(.,0)\right\|_{\mathbb{H}^q(\Omega)}\\ &\le C_\rho T^{-\rho}\left\|g\right\|_{\mathbb{H}^{q-\frac{\rho}{\alpha}}(\Omega)} + \Big(\Big(\frac{\overline{m}_2}{m_1}\Big)^2 C_\alpha\lambda_1^{\frac{3-3\alpha}{\alpha}}\frac{T^{2-\alpha}}{2-\alpha}\Big)^{\frac{1}{2}}\left\|F\right\|_{L^\infty(0,T;\mathbb{H}^q(\Omega))}.\end{aligned} \tag{74}$$

Theorem 4.2. *Let $F(x,t) \equiv F(x)$, let us assume that $g \in \mathbb{H}^{q-\frac{\rho}{\alpha}}(\Omega)$ and $F \in \mathbb{H}^{q-1}(\Omega)$ for any $\rho > 0, q \geq 0$, then we obtain*

$$\left\|u(.,t)\right\|_{\mathbb{H}^q(\Omega)} \leq C(\rho, m_1, m_2)(T-t)^{-\rho}\Big(\|g\|_{\mathbb{H}^{q-\frac{\rho}{\alpha}}(\Omega)} + \|F\|_{\mathbb{H}^{q-\frac{\rho}{\alpha}-1}(\Omega)}\Big) + \|F\|_{\mathbb{H}^{q-1}(\Omega)}. \tag{75}$$

If $g \in \mathbb{H}^{q+1-\frac{\rho}{\alpha}}(\Omega)$ and $F \in \mathbb{H}^{q-\frac{\rho}{\alpha}}(\Omega)$ then

$$\left\|D_t^\alpha u(.,t)\right\|_{\mathbb{H}^q(\Omega)} \leq C(\rho, m_1, m_2)(T-t)^{-\rho}\Big(\|g\|_{\mathbb{H}^{q+1-\frac{\rho}{\alpha}}(\Omega)} + \|F\|_{\mathbb{H}^{q-\frac{\rho}{\alpha}}(\Omega)}\Big). \tag{76}$$

Since $F(x,t) \equiv F(x)$, we know that $F_j(\tau) = F_j$ for any $0 \leq \tau \leq T$, where

$$F_j = \int_\Omega F(x) e_j(x) dx.$$

Thus, we have immediately that

$$\frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)\int_0^T (T-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_j (T-\tau)^\alpha\right) F_j(s) ds}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)} = \frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)}{E_{\alpha,1}(\lambda_j T^\alpha)} \frac{E_{\alpha,1}(\lambda_j T^\alpha) - 1}{\lambda_j}. \tag{77}$$

Hence, we get the following equality

$$\mathbb{K}_2(x,t) = \sum_{j=1}^\infty \frac{-E_{\alpha,1}\left(\lambda_j t^\alpha\right) F_j}{\lambda_j} e_j(x) + \sum_{j=1}^\infty \frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right) F_j}{\lambda_j E_{\alpha,1}\left(\lambda_j T^\alpha\right)} e_j(x), \tag{78}$$

and

$$\mathbb{K}_3(x,t) = \sum_{j=1}^\infty \left(\int_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}\left(\lambda_j (t-\tau)^\alpha\right) F_j(\tau) d\tau\right) e_j(x) = \sum_{j=1}^\infty \frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right) - 1}{\lambda_j} F_j e_j(x). \tag{79}$$

By collecting some previous results, we obtain the fomula of the mild solution

$$u(x,t) = \sum_{j=1}^\infty \frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)}\Big(g_j + \frac{F_j}{\lambda_j}\Big) e_j(x) - \sum_{j=1}^\infty \frac{F_j}{\lambda_j} e_j(x). \tag{80}$$

Using Parseval's equality, one has

$$\begin{aligned}\left\|u(.,t)\right\|^2_{\mathbb{H}^q(\Omega)} &= \sum_{j=1}^\infty \lambda_j^{2q}\Big(\frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)}\Big)^2\Big(g_j + \frac{F_j}{\lambda_j}\Big)^2 + \sum_{j=1}^\infty \lambda_j^{2q}\Big(\frac{F_j}{\lambda_j}\Big)^2 \\ &\leq 2\sum_{j=1}^\infty \lambda_j^{2q}\Big(\frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)}\Big)^2 g_j^2 + 2\sum_{j=1}^\infty \lambda_j^{2q}\Big(\frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)}\Big)^2\Big(\frac{F_j}{\lambda_j}\Big)^2 + \sum_{j=1}^\infty \lambda_j^{2q}\Big(\frac{F_j}{\lambda_j}\Big)^2.\end{aligned} \tag{81}$$

By noting (34), we get

$$\sum_{j=1}^\infty \lambda_j^{2q}\Big(\frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)}\Big)^2 g_j^2 \leq |C_\rho|^2 \frac{m_2^2}{m_1^2}(T-t)^{-2\rho} \sum_{j=1}^\infty \lambda_j^{2q-\frac{2\rho}{\alpha}} g_j^2, \tag{82}$$

and

$$\sum_{j=1}^\infty \lambda_j^{2q}\Big(\frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)}\Big)^2\Big(\frac{F_j}{\lambda_j}\Big)^2 \leq |C_\rho|^2 \frac{m_2^2}{m_1^2}(T-t)^{-2\rho} \sum_{j=1}^\infty \lambda_j^{2q-\frac{2\rho}{\alpha}-2} F_j^2, \tag{83}$$

and

$$\sum_{j=1}^{\infty} \lambda_j^{2q} \Big(\frac{F_j}{\lambda_j}\Big)^2 = \Big\|F\Big\|^2_{\mathbb{H}^{q-1}(\Omega)}. \tag{84}$$

Combining (88), (82), (89), (84), we derive that

$$\Big\|u(.,t)\Big\|^2_{\mathbb{H}^q(\Omega)} \le 2|C_\rho|^2 \frac{m_2^2}{m_1^2}(T-t)^{-2\rho}\Big\|g\Big\|^2_{\mathbb{H}^{q-\frac{\rho}{\alpha}}(\Omega)} + 2|C_\rho|^2 \frac{m_2^2}{m_1^2}(T-t)^{-2\rho}\Big\|F\Big\|^2_{\mathbb{H}^{q-\frac{\rho}{\alpha}-1}(\Omega)} + \Big\|F\Big\|^2_{\mathbb{H}^{q-1}(\Omega)}. \tag{85}$$

This implies that

$$\|u(.,t)\|_{\mathbb{H}^q(\Omega)} \le C(\rho, m_1, m_2)(T-t)^{-\rho}\Big(\Big\|g\Big\|_{\mathbb{H}^{q-\frac{\rho}{\alpha}}(\Omega)} + \Big\|F\Big\|_{\mathbb{H}^{q-\frac{\rho}{\alpha}-1}(\Omega)}\Big) + \Big\|F\Big\|_{\mathbb{H}^{q-1}(\Omega)}. \tag{86}$$

Let us now to provide the proof of estimate for the derivative of the mild solution. The proof is similar to the above but with other modifications. In fact, it yields

$$D_t^\alpha E_{\alpha,1}\left(\lambda_j t^\alpha\right) = \lambda_j E_{\alpha,1}\left(\lambda_j t^\alpha\right)$$

and based on (80), we get the following equalities

$$D_t^\alpha u(x,t) = \sum_{j=1}^{\infty} \lambda_j \frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)}\Big(g_j + \frac{F_j}{\lambda_j}\Big)e_j(x). \tag{87}$$

Using Parseval's equality and the inequality $(a+b)^2 \le 2a^2 + 2b^2$, one has

$$\begin{aligned} \Big\|D_t^\alpha u(.,t)\Big\|^2_{\mathbb{H}^q(\Omega)} &= \sum_{j=1}^{\infty} \lambda_j^{2q+2}\Big(\frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)}\Big)^2\Big(g_j + \frac{F_j}{\lambda_j}\Big)^2 \\ &\le 2\sum_{j=1}^{\infty} \lambda_j^{2q+2}\Big(\frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)}\Big)^2 g_j^2 + 2\sum_{j=1}^{\infty} \lambda_j^{2q+2}\Big(\frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)}\Big)^2\Big(\frac{F_j}{\lambda_j}\Big)^2. \end{aligned} \tag{88}$$

In view of (82), we know that

$$\sum_{j=1}^{\infty} \lambda_j^{2q+2}\Big(\frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)}\Big)^2 g_j^2 \le |C_\rho|^2 \frac{m_2^2}{m_1^2}(T-t)^{-2\rho}\sum_{j=1}^{\infty} \lambda_j^{2q+2-\frac{2\rho}{\alpha}} g_j^2. \tag{89}$$

In view of (83), we know that

$$\sum_{j=1}^{\infty} \lambda_j^{2q+2}\Big(\frac{E_{\alpha,1}\left(\lambda_j t^\alpha\right)}{E_{\alpha,1}\left(\lambda_j T^\alpha\right)}\Big)^2\Big(\frac{F_j}{\lambda_j}\Big)^2 \le |C_\rho|^2 \frac{m_2^2}{m_1^2}(T-t)^{-2\rho}\sum_{j=1}^{\infty} \lambda_j^{2q-\frac{2\rho}{\alpha}} F_j^2. \tag{90}$$

Combining three latter results, we obtain that

$$\Big\|D_t^\alpha u(.,t)\Big\|^2_{\mathbb{H}^q(\Omega)} \le 2|C_\rho|^2 \frac{m_2^2}{m_1^2}(T-t)^{-2\rho}\Big\|g\Big\|^2_{\mathbb{H}^{q+1-\frac{\rho}{\alpha}}(\Omega)} + 2|C_\rho|^2 \frac{m_2^2}{m_1^2}(T-t)^{-2\rho}\Big\|F\Big\|^2_{\mathbb{H}^{q-\frac{\rho}{\alpha}}(\Omega)}. \tag{91}$$

This completes the proof of (76).

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