Filomat 38:31 (2024), 11111–11126
https://doi.org/10.2298/FIL2431111K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Halpern type convergence theorems on a geodesic space with curvature bounded above by a general real number

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Abstract. In this paper, we prove convergence theorems with the Halpern type iterative scheme in the setting of geodesic spaces with curvature bounded above by general real numbers. To obtain the results, we consider another type of convex combination than the canonical one.

1. Introduction

To generate an approximation sequence converging to a fixed point of a mapping, a lot of researchers have introduced many effective iterative schemes. For instance, the Picard type and the Mann type iterative schemes guarantee to generate a weak convergent sequence approximating to some fixed point. On the other hand, Halpern's iterative sequence converges strongly to the closest fixed point to the anchor point. Besides that, there are approximation methods using a sequence of subsets and projections onto them. Fixed point approximation methods above were proposed on Hilbert spaces, and were generalised to those on Banach spaces. For more details about related works, refer to [4, 15, 16] for instance. Recently, they are introduced on a metric space having some convex structures, namely, a geodesic space. Mann's one is investigated by [3, 7] for instance; Halpern's one is also studied by [8–10, 13] for instance.

In a CAT(κ) space, its curvature κ determines the properties of the space. For this reason, many proofs of propositions are based on its curvature. However, according to a function with curvature as a parameter which is proposed by Kajimura and the first author [5], we become able to investigate CAT(κ) spaces without separating cases.

In this paper, we deal with the Halpern type iterative scheme in the setting of geodesic spaces with curvature bounded above by general real numbers. To prove a convergence theorem, we use another notion of convex combination than the canonical one proposed by the first author and Sasaki [8, 9]. Using another convex combination called κ-convex combination, we first prove a convergence theorem for a strongly quasinonexpansive mapping. After that, we get a convergence theorem with the usual convex combination as a direct consequence of one with κ-convex combination. At the end of this paper, we consider the coefficient condition adapted to Halpern type iterative sequences.

2020 *Mathematics Subject Classification*. Primary 47H10; Secondary 58C30.
Keywords. Fixed point approximation, geodesic space, Halpern type iteration.
Received: 19 June 2023; Revised: 23 November 2023; Accepted: 04 December 2023
Communicated by Adrian Petrusel
Research supported by JSPS KAKENHI Grant Number JP21K03316.
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2. Preliminaries

Let (X, d) be a metric space and let $D \in]0, \infty]$. X is called a uniquely D-geodesic space if there exists a unique geodesic for each $x, y \in X$ with $d(x, y) < D$. That is, there is a unique isometric mapping γ_{xy} from $[0, d(x, y)]$ into X such that $\gamma_{xy}(0) = x$ and $\gamma_{xy}(d(x, y)) = y$. In a uniquely D-geodesic space X, for $x, y \in X$ with $d(x, y) < D$ and $t \in [0, 1]$, there is a unique point z such that

$$d(x, z) = (1 - t)d(x, y) \text{ and } d(y, z) = td(x, y).$$

We denote such a point z by $tx \oplus (1 - t)y$, and call it convex combination for x and y.

We define $D_\kappa \in]0, \infty]$ as follows: $D_\kappa = \infty$ if $\kappa \leq 0$; $D_\kappa = \pi/\sqrt{\kappa}$ if $\kappa > 0$. To define a CAT(κ) space, we use a function c_κ from $[0, D_\kappa/2[$ into $[0, \infty[$ defined by

$$c_\kappa(a) = \frac{1}{2}a^2 + \sum_{n=2}^{\infty} \frac{(-\kappa)^{n-1}a^{2n}}{(2n)!} = \begin{cases} \frac{1}{\kappa}\left(1 - \cos\left(\sqrt{\kappa}a\right)\right) & (\kappa > 0); \\ \frac{1}{2}a^2 & (\kappa = 0); \\ \frac{1}{-\kappa}\left(\cosh\left(\sqrt{-\kappa}a\right) - 1\right) & (\kappa < 0) \end{cases}$$

for $a \in [0, D_\kappa/2[$. Then, we know

$$c'_\kappa(a) = \begin{cases} \dfrac{\sin\left(\sqrt{\kappa}a\right)}{\sqrt{\kappa}} & (\kappa > 0); \\ a & (\kappa = 0); \\ \dfrac{\sinh\left(\sqrt{-\kappa}a\right)}{\sqrt{-\kappa}} & (\kappa < 0) \end{cases}$$

and

$$c''_\kappa(a) = \begin{cases} \cos\left(\sqrt{\kappa}a\right) & (\kappa > 0); \\ 1 & (\kappa = 0); \\ \cosh\left(\sqrt{-\kappa}a\right) & (\kappa < 0) \end{cases}$$

for $a \in [0, D_\kappa/2[$. It hold from the definition of c_κ that $c_\kappa(0) = c'_\kappa(0) = 0$ and $c''_\kappa(0) = 1$ for each $\kappa \in \mathbb{R}$. Further,

$$\kappa c_\kappa(a) + c''_\kappa(a) = 1$$

for all $a \in [0, D_\kappa/2[$. For more details about the function c_κ, see [5].

For a metric space (X, d), we define a function ϕ_κ from X^2 into $\mathbb{R}$ by

$$\phi_\kappa(x, y) = c_\kappa(d(x, y))$$

for $x, y \in X$, and we define an adjuster $(\cdot)_l^\kappa$ from $[0, 1]$ onto $[0, 1]$ by

$$(t)_l^\kappa = \begin{cases} \dfrac{c'_\kappa(tl)}{c'_\kappa(l)} & (l \in]0, D_\kappa[); \\ t & (l = 0) \end{cases}$$

for $t \in [0, 1]$. We know the following properties about ϕ_κ:

- $\phi_\kappa(x, y) \geq 0$ for every $x, y \in X$;

- $\phi_\kappa(x,y) = 0$ if and only if $x = y$, where $d(x,y) < 2D_\kappa$;
- $\phi_\kappa(x,y) = \phi_\kappa(y,x)$ for every $x, y \in X$.

Now, we can define a CAT(κ) space. The canonical definition of a CAT(κ) space uses a notion of model spaces and their comparison triangle. However, we can define a CAT(κ) space as follows: Let $\kappa \in \mathbb{R}$ and X a uniquely D_κ-geodesic space. We call X a CAT(κ) space if

$$\begin{aligned}\phi_\kappa(tx \oplus (1-t)y, z) \le{}& (t)_l^\kappa \phi_\kappa(x,z) + (1-t)_l^\kappa \phi_\kappa(y,z)\\ &- (t)_l^\kappa \phi_\kappa(x, tx\oplus(1-t)y) - (1-t)_l^\kappa \phi_\kappa(y, tx \oplus (1-t)y)\end{aligned}$$

for every $x, y, z \in X$ with $d(y,z) + d(z,x) + l < 2D_\kappa$ and $t \in [0,1]$, where $l = d(x,y)$. For more details about this definition, see [11]. Moreover, X is said to be admissible if $d(u,v) < D_\kappa/2$ for any $u, v \in X$.

Let $\kappa \in \mathbb{R}$. We define a function t_κ from $[0, D_\kappa/2[$ into $[0,\infty[$ by

$$t_\kappa(a) = \frac{c'_\kappa(a)}{c''_\kappa(a)} = \begin{cases} \dfrac{\tan(\sqrt{\kappa}a)}{\sqrt{\kappa}} & (\kappa > 0);\\ a & (\kappa = 0);\\ \dfrac{\tanh(\sqrt{-\kappa}a)}{\sqrt{-\kappa}} & (\kappa < 0)\end{cases}$$

for every $a \in [0, D_\kappa/2[$. We know t_κ is continuous, increasing and $t_\kappa(0) = 0$. Since $c''_\kappa(a)^2 + \kappa c'_\kappa(a)^2 = 1$ for $a \in [0, D_\kappa/2[$, the following hold:

$$c'_\kappa(a) = \sqrt{\frac{t_\kappa(a)^2}{1+\kappa t_\kappa(a)^2}} \text{ and } c''_\kappa(a) = \sqrt{\frac{1}{1+\kappa t_\kappa(a)^2}}.$$

Let X be an admissible CAT(κ) space for $\kappa \in \mathbb{R}$. For a real valued function f on X, we denote the set of all minimisers of f by $\operatorname{Argmin}_{u\in X} f(u)$, and defined by

$$\operatorname*{Argmin}_{u\in X} f(u) = \left\{ u \in X \,\middle|\, f(u) = \inf_{x\in X} f(x) \right\}.$$

For $x, y \in X$ and $t \in [0,1]$, a function

$$t\phi_\kappa(x,\cdot) + (1-t)\phi_\kappa(y,\cdot) \colon X \to \mathbb{R}$$

has a unique minimiser. We define κ-convex combination as

$$\left\{ tx \overset{\kappa}{\oplus} (1-t)y \right\} = \operatorname*{Argmin}_{u\in X} \Big(t\phi_\kappa(x,u) + (1-t)\phi_\kappa(y,u) \Big).$$

Moreover, we know

$$\begin{aligned}&tx \overset{\kappa}{\oplus} (1-t)y\\ &= \frac{1}{d(x,y)} t_\kappa^{-1}\left(\frac{tc'_\kappa(d(x,y))}{1-t+tc''_\kappa(d(x,y))} \right) x \oplus \frac{1}{d(x,y)} t_\kappa^{-1}\left(\frac{(1-t)c'_\kappa(d(x,y))}{t+(1-t)c''_\kappa(d(x,y))} \right) y\end{aligned}$$

for $x, y \in X$ with $x \neq y$ and $t \in [0,1]$. If $x = y$, then $tx \overset{\kappa}{\oplus} (1-t)y = x = y$. For more details about κ-convex combination, see [8, 9, 11] for instance.

Theorem 2.1 (Kimura–Sudo [11]). *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$. Then,*

$$\phi_\kappa(tx \overset{\kappa}{\oplus} (1-t)y, z) \leq \frac{t\phi_\kappa(x,z) + (1-t)\phi_\kappa(y,z) - t\phi_\kappa(x, tx \overset{\kappa}{\oplus} (1-t)y) - (1-t)\phi_\kappa(y, tx \overset{\kappa}{\oplus} (1-t)y)}{\sqrt{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(d(x,y))}}$$

and

$$\phi_\kappa(tx \overset{\kappa}{\oplus} (1-t)y, z) \leq t\phi_\kappa(x,z) + (1-t)\phi_\kappa(y,z)$$

for every $x, y, z \in X$ and $t \in [0,1]$.

Let X be a metric space and T a mapping from X into itself. $\operatorname{Fix} T$ stands for the set of all fixed points of T. Further, T is said to be quasinonexpansive if $\operatorname{Fix} T$ is nonempty and $d(p, Tx) \leq d(p, x)$ for every $p \in \operatorname{Fix} T$ and $x \in X$. Moreover, on a CAT(κ) space X, we say T is strongly quasinonexpansive if it is quasinonexpansive, and for a sequence $\{x_n\}$ of X, it holds that $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ whenever there exists a fixed point $p \in \operatorname{Fix} T$ such that $\sup_{n\in\mathbb{N}} d(p, Tx_n) < D_\kappa/2$ and that

$$\lim_{n\to\infty}(d(p, x_n) - d(p, Tx_n)) = 0.$$

Let X be an admissible CAT(κ) space for $\kappa \in \mathbb{R}$ and C a subset of X. We say C is convex if $tx \oplus (1-t)y \in C$ for every $x, y \in C$ and $t \in [0,1]$. A fixed point set of a quasinonexpansive mapping on admissible CAT(κ) spaces is closed and convex.

Let C be a nonempty closed convex subset of an admissible complete CAT(κ) space X. Then, for $x \in X$, there exists a unique point $p_x \in C$ such that

$$d(x, p_x) = \inf_{y\in C} d(x, y).$$

We call such a mapping P_C defined by $P_C x = p_x$ a metric projection onto C. Notice that metric projections are quasinonexpansive with the fixed point set $\operatorname{Fix} P_C = C$.

Let X be a metric space and $\{x_n\}$ a bounded sequence of X. We call $z \in X$ an asymptotic centre of $\{x_n\}$ if

$$z \in \operatorname*{Argmin}_{u\in X}\left(\limsup_{n\to\infty} d(u, x_n)\right) = \operatorname*{Argmin}_{u\in X}\left(\limsup_{n\to\infty} \phi_\kappa(u, x_n)\right).$$

Let $\{x_n\}$ be a sequence of X and $x_0 \in X$. We say $\{x_n\}$ Δ-converges to a Δ-limit x_0 if x_0 is a unique asymptotic centre of any subsequence $\{x_{n_i}\}$ of $\{x_n\}$. A sequence $\{x_n\}$ of an admissible CAT(κ) space X for $\kappa \in \mathbb{R}$ is said to be κ-bounded if

$$\inf_{x\in X}\limsup_{n\to\infty} d(x, x_n) < \frac{D_\kappa}{2}.$$

We know the following lemmas about Δ-convergence:

Lemma 2.2 (Bačák [1], Espínola–Fernández-León [2], Kirk–Panyanak [12]). *Let X be a complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and $\{x_n\}$ a κ-bounded sequence of X. Then, $\{x_n\}$ has a unique asymptotic centre and it has a Δ-convergent subsequence.*

Lemma 2.3 (Bačák [1], He–Fang–Lopez–Li [3]). *Let (X, d) be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$. Then,*

$$d(x_0, z) \leq \liminf_{n\to\infty} d(x_n, z)$$

for all $z \in X$ whenever a κ-bounded sequence $\{x_n\}$ Δ-converges to $x_0 \in X$.

Let X be an admissible CAT(κ) space for $\kappa \in \mathbb{R}$ and T a mapping on X. We say T is Δ-demiclosed if $x_0 \in X$ is a fixed point of T whenever $\lim_{n\to\infty} d(x_n, Tx_n) = 0$ for some κ-bounded sequence $\{x_n\}$ of X which Δ-converges to x_0.

3. Lemmas to prove convergence theorems

In this section, we obtain some lemmas to prove a convergence theorem.

Lemma 3.1. *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$. Then,*

$$\phi_\kappa(tx \overset{\kappa}{\oplus} (1-t)y, z) \le \frac{t\phi_\kappa(x,z) + (1-t)\phi_\kappa(y,z)}{M} - \frac{2t(1-t)\phi_\kappa(x,y)}{M(1+M)}$$

for every $x, y, z \in X$ and $t \in [0,1]$, where

$$M = \sqrt{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(d(x,y))}.$$

Proof. When $x = y$ or $\kappa = 0$, we easily obtain the desired inequality. We assume that $x \neq y$ and $\kappa \neq 0$. From Theorem 2.1,

$$\begin{aligned}
&\phi_\kappa(tx \overset{\kappa}{\oplus} (1-t)y, z) \\
&\le \frac{t\phi_\kappa(x,z) + (1-t)\phi_\kappa(y,z) - t\phi_\kappa(x, tx \overset{\kappa}{\oplus} (1-t)y) - (1-t)\phi_\kappa(y, tx \overset{\kappa}{\oplus} (1-t)y)}{M}.
\end{aligned}$$

We prove the following identity:

$$t\phi_\kappa(x, tx \overset{\kappa}{\oplus} (1-t)y) + (1-t)\phi_\kappa(y, tx \overset{\kappa}{\oplus} (1-t)y) = \frac{2t(1-t)\phi_\kappa(x,y)}{1+M}.$$

Let $l = d(x,y) \neq 0$ and

$$\sigma = \frac{1}{l} t_\kappa^{-1}\left(\frac{tc'_\kappa(l)}{1-t+tc''_\kappa(l)}\right).$$

We remark that $tx \overset{\kappa}{\oplus} (1-t)y = \sigma x \oplus (1-\sigma)y$. Then, we obtain

$$\begin{aligned}
c''_\kappa(\sigma l) &= c''_\kappa\left(t_\kappa^{-1}\left(\frac{tc'_\kappa(l)}{1-t+tc''_\kappa(l)}\right)\right) = \sqrt{\frac{(1-t+tc''_\kappa(l))^2}{(1-t+tc''_\kappa(l))^2 + \kappa t^2 c'_\kappa(l)^2}} \\
&= \frac{1-t+tc''_\kappa(l)}{\sqrt{(1-t+tc''_\kappa(l))^2 + t^2\kappa c'_\kappa(l)^2}} \\
&= \frac{1-t+tc''_\kappa(l)}{\sqrt{(1-t)^2 + 2t(1-t)c''_\kappa(l) + t^2c''_\kappa(l)^2 + t^2\kappa c'_\kappa(l)^2}} \\
&= \frac{1-t+tc''_\kappa(l)}{\sqrt{t^2 + (1-t)^2 + 2t(1-t)c''_\kappa(l)}} = \frac{1-t+tc''_\kappa(l)}{M}.
\end{aligned}$$

Similarly, we get

$$c''_\kappa((1-\sigma)l) = \frac{t + (1-t)c''_\kappa(l)}{M}.$$

Thus,

$$\begin{aligned}
&t\phi_\kappa(x, tx \overset{\kappa}{\oplus} (1-t)y) + (1-t)\phi_\kappa(y, tx \overset{\kappa}{\oplus} (1-t)y) \\
&= \frac{1}{\kappa}\left(1 - tc''_\kappa((1-\sigma)l) - (1-t)c''_\kappa(\sigma l)\right)
\end{aligned}$$

$$= \frac{1}{\kappa}\left(1 - \frac{t^2 + (1-t)^2 + 2t(1-t)c_\kappa''(l)}{M}\right) = \frac{1-M}{\kappa} = \frac{1-M^2}{\kappa(1+M)}$$
$$= \frac{1 - t^2 - (1-t)^2 - 2t(1-t)c_\kappa''(l)}{\kappa(1+M)} = \frac{2t(1-t)(1-c_\kappa''(l))}{\kappa(1+M)} = \frac{2t(1-t)\phi_\kappa(x,y)}{1+M}.$$

Hence, we obtain the desired result. □

Using this result, we get the following:

Lemma 3.2. *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$. Let $x, y, z \in X$, and $\alpha \in]0,1[$. Set*

$$M = \sqrt{\alpha^2 + (1-\alpha)^2 + 2\alpha(1-\alpha)c_\kappa''(d(x,y))} \text{ and } \beta = 1 - \frac{1-\alpha}{M} \neq 0.$$

Then,

$$\phi_\kappa(\alpha x \overset{\kappa}{\oplus} (1-\alpha)y, z)$$
$$\leq (1-\beta)\phi_\kappa(y,z) + \beta\left(\frac{(M+1-\alpha)((1+M)\phi_\kappa(x,z) - 2(1-\alpha)\phi_\kappa(x,y))}{(1+M)(\alpha + 2(1-\alpha)c_\kappa''(d(x,y)))}\right).$$

Proof. From the previous theorem, we get

$$\phi_\kappa(\alpha x \overset{\kappa}{\oplus} (1-\alpha)y, z) \leq (1-\beta)\phi_\kappa(y,z) + \frac{\alpha\phi_\kappa(x,z)}{M} - \frac{2\alpha(1-\alpha)\phi_\kappa(x,y)}{M(1+M)}.$$

Then,

$$\frac{\alpha\phi_\kappa(x,z)}{M} - \frac{2\alpha(1-\alpha)\phi_\kappa(x,y)}{M(1+M)} = \beta\left(\frac{1}{\beta}\right)\left(\frac{\alpha\phi_\kappa(x,z)}{M} - \frac{2\alpha(1-\alpha)\phi_\kappa(x,y)}{M(1+M)}\right)$$
$$= \beta\left(\frac{M}{M-(1-\alpha)}\right)\left(\frac{\alpha\phi_\kappa(x,z)}{M} - \frac{2\alpha(1-\alpha)\phi_\kappa(x,y)}{M(1+M)}\right)$$
$$= \beta\left(\frac{\alpha\phi_\kappa(x,z)}{M-(1-\alpha)} - \frac{2\alpha(1-\alpha)\phi_\kappa(x,y)}{(1+M)(M-(1-\alpha))}\right)$$
$$= \beta\left(\frac{\alpha(M+1-\alpha)\phi_\kappa(x,z)}{M^2-(1-\alpha)^2} - \frac{2\alpha(1-\alpha)(M+1-\alpha)\phi_\kappa(x,y)}{(1+M)(M^2-(1-\alpha)^2)}\right)$$
$$= \beta\left(\frac{\alpha(M+1-\alpha)\phi_\kappa(x,z)}{\alpha^2 + 2\alpha(1-\alpha)c_\kappa''(d(x,y))} - \frac{2\alpha(1-\alpha)(M+1-\alpha)\phi_\kappa(x,y)}{(1+M)(\alpha^2 + 2\alpha(1-\alpha)c_\kappa''(d(x,y)))}\right)$$
$$= \beta\left(\frac{(M+1-\alpha)\phi_\kappa(x,z)}{\alpha + 2(1-\alpha)c_\kappa''(d(x,y))} - \frac{2(1-\alpha)(M+1-\alpha)\phi_\kappa(x,y)}{(1+M)(\alpha + 2(1-\alpha)c_\kappa''(d(x,y)))}\right)$$
$$= \beta\left(\frac{(1+M)(M+1-\alpha)\phi_\kappa(x,z) - 2(1-\alpha)(M+1-\alpha)\phi_\kappa(x,y)}{(1+M)(\alpha + 2(1-\alpha)c_\kappa''(d(x,y)))}\right).$$

It completes the proof. □

Moreover, we get the following lemmas:

Lemma 3.3. *Let $\kappa \in \mathbb{R}$ and $\{l_n\}$ a bounded real sequence of $[0, D_\kappa/2[$. Let $\{\alpha_n\}$ be a real sequence of $]0,1[$ such that $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$. Define a real sequence $\{\beta_n\}$ of $]0,1[$ by*

$$\beta_n = 1 - \frac{1-\alpha_n}{\sqrt{\alpha_n^2 + (1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)c_\kappa''(l_n)}}$$

for each $n \in \mathbb{N}$. Further, assume one of the following:

(a) $\sup_{n\in\mathbb{N}} l_n < D_\kappa/2$;

(b) $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$.

Then, $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=1}^{\infty} \beta_n = \infty$.

Proof. Since $\{l_n\}$ is bounded, there exists $L > 0$ such that $c''_\kappa(l_n) \le L$ for any $n \in \mathbb{N}$. We first show $\lim_{n\to\infty} \beta_n = 0$. From the definition of $\{\beta_n\}$,

$$0 \le \beta_n \le 1 - \frac{1-\alpha_n}{\sqrt{\alpha_n^2 + (1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)L}}.$$

Since $\alpha_n \to 0$, we have $\beta_n \to 0$ as $n \to \infty$. We next show $\sum_{n=1}^{\infty} \beta_n = \infty$. Since $\lim_{n\to\infty} \alpha_n = 0$, there exists $n_0 \in \mathbb{N}$ such that $1 - \alpha_n \ge 1/2$ for any $n \ge n_0$. Note that

$$\alpha_n^2 + (1-\alpha_n)^2 \le 2 \text{ and } \alpha_n(1-\alpha_n) \le 1$$

for any $n \in \mathbb{N}$. Let

$$M_n = \sqrt{\alpha_n^2 + (1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)c''_\kappa(l_n)}$$

for each $n \in \mathbb{N}$. We remark that $M_n \le \sqrt{2(1+L)}$ for all $n \in \mathbb{N}$. Then, for $n \ge n_0$, we obtain

$$\begin{aligned}\beta_n &= 1 - \frac{1-\alpha_n}{M_n} = \frac{M_n - (1-\alpha_n)}{M_n} = \frac{M_n^2 - (1-\alpha_n)^2}{M_n(M_n + 1 - \alpha_n)} \\ &= \frac{\alpha_n^2 + 2\alpha_n(1-\alpha_n)c''_\kappa(l_n)}{M_n(M_n + 1 - \alpha_n)} \ge \frac{\alpha_n^2 + c''_\kappa(l_n)\alpha_n}{\sqrt{2(1+L)}\left(\sqrt{2(1+L)} + 1\right)}.\end{aligned}$$

From (a) or (b), we get $\sum_{n=1}^{\infty} \beta_n = \infty$. □

Lemma 3.4. *Let* $\kappa \in \mathbb{R}$, $\{l_n\}$ *a bounded real sequence of* $[0, D_\kappa/2[$ *and* $l \in [0, D_\kappa/2[$. *Let* $\{\alpha_n\}$ *be a real sequence of* $]0, 1[$ *which converges to* 0. *Let*

$$M_n = \sqrt{\alpha_n^2 + (1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)c''_\kappa(l_n)}$$

and

$$t_n = \frac{(M_n + 1 - \alpha_n)((1+M_n)c_\kappa(l) - 2(1-\alpha_n)c_\kappa(l_n))}{(1+M_n)(\alpha_n + 2(1-\alpha_n)c''_\kappa(l_n))}$$

for each $n \in \mathbb{N}$. *Then,* $\limsup_{n\to\infty} t_n \le 0$ *whenever* $\liminf_{n\to\infty} l_n \ge l$.

Proof. Note that $M_n \to 1$ as $n \to \infty$ since $\alpha_n \to 0$ and $\{l_n\}$ is bounded. We can take a subsequence $\{t_{n_i}\}$ of $\{t_n\}$ such that

$$\lim_{i\to\infty} t_{n_i} = \limsup_{n\to\infty} t_n.$$

Moreover, there exists a subsequence $\{l_{n_{i_j}}\}$ of $\{l_{n_i}\}$ which converges to $l_0 = \liminf_{i\to\infty} l_{n_i}$. Henceforth, we denote n_{i_j} by j simply. We remark that

$$l_0 = \lim_{j\to\infty} l_j = \liminf_{i\to\infty} l_{n_i} \ge \liminf_{n\to\infty} l_n \ge l.$$

Since $\alpha_j \to 0$ as $j \to \infty$, we have

$$\limsup_{n\to\infty} t_n = \lim_{j\to\infty} \frac{(M_j + 1 - \alpha_j)((1+M_j)c_\kappa(l) - 2(1-\alpha_j)c_\kappa(l_j))}{(1+M_j)(\alpha_j + 2(1-\alpha_j)c''_\kappa(l_j))}$$

$$= \lim_{j\to\infty} \frac{4c_\kappa(l) - 4c_\kappa(l_j)}{4c''_\kappa(l_j)} = \lim_{j\to\infty} \frac{c_\kappa(l) - c_\kappa(l_j)}{c''_\kappa(l_j)}.$$

Whenever $c''_\kappa(l_0) \neq 0$, we obtain the desired result. In what follows, we assume that $\kappa > 0$ and $l_0 = D_\kappa/2$. Then,

$$\begin{aligned}\limsup_{n\to\infty} t_n &= \lim_{j\to\infty} \frac{c_\kappa(l) - c_\kappa(l_j)}{c''_\kappa(l_j)} = \lim_{j\to\infty} \frac{1 - c''_\kappa(l) - 1 + c''_\kappa(l_j)}{\kappa c''_\kappa(l_j)} \\ &= \lim_{j\to\infty} \frac{c''_\kappa(l_j) - c''_\kappa(l)}{\kappa c''_\kappa(l_j)} = \frac{1}{\kappa} - \lim_{j\to\infty} \frac{c''_\kappa(l)}{\kappa c''_\kappa(l_j)} = -\infty < 0.\end{aligned}$$

It completes the proof. □

4. Halpern type convergence theorems

In this section, we prove convergence theorems to a fixed point a mapping. To obtain the results, we use the following lemma:

Lemma 4.1 (Kimura–Saejung [6], Saejung–Yotkaew [14])**.** *Let $\{s_n\}$ be a real sequence of $[0, \infty[$ and $\{t_n\}$ a real sequence. Let $\{\beta_n\}$ be a real sequence of $]0, 1]$ such that $\sum_{n=1}^{\infty} \beta_n = \infty$. Suppose that*

$$s_{n+1} \leq (1 - \beta_n)s_n + \beta_n t_n$$

for all $n \in \mathbb{N}$ and that $\limsup_{i\to\infty} t_{n_i} \leq 0$ for every subsequence $\{s_{n_i}\}$ of $\{s_n\}$ satisfying that

$$\limsup_{i\to\infty} (s_{n_i} - s_{n_i+1}) \leq 0.$$

Then, $\lim_{n\to\infty} s_n = 0$.

We first obtain the following convergence theorems with the Halpern type iterative scheme:

Theorem 4.2. *Let X be an admissible complete $\mathrm{CAT}(\kappa)$ space for $\kappa \in \mathbb{R}$ and T a strongly quasinonexpansive and Δ-demiclosed mapping on X. Let $\{\alpha_n\}$ be a real sequence of $]0, 1[$ such that $\lim_{n\to\infty} \alpha_n = 0$ and that $\sum_{n=1}^{\infty} \alpha_n = \infty$. For an anchor point $u \in X$ and an initial point $x_1 \in X$, generate a sequence $\{x_n\}$ of X as follows:*

$$x_{n+1} = \alpha_n u \overset{\kappa}{\oplus} (1 - \alpha_n)Tx_n$$

for each $n \in \mathbb{N}$. Further, assume one of the following:

(a) $\sup_{n\in\mathbb{N}} d(u, Tx_n) < D_\kappa/2$;

(b) $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$.

Then, $\{x_n\}$ converges to a fixed point $P_{\mathrm{Fix}\,T}u$, where $P_{\mathrm{Fix}\,T}$ is a metric projection onto $\mathrm{Fix}\,T$.

Proof. Set $p = P_{\mathrm{Fix}\,T}u$. Since T is quasinonexpansive, for each $n \in \mathbb{N}$,

$$\begin{aligned}\phi_\kappa(p, x_{n+1}) &\leq \alpha_n\phi_\kappa(p, u) + (1 - \alpha_n)\phi_\kappa(p, Tx_n) \\ &\leq \alpha_n\phi_\kappa(p, u) + (1 - \alpha_n)\phi_\kappa(p, x_n) \leq \max\{\phi_\kappa(p, u), \phi_\kappa(p, x_n)\}.\end{aligned}$$

Therefore, for all $n \in \mathbb{N}$,

$$d(p, Tx_n) \leq d(p, x_n) \leq \max\{d(p, u), d(p, x_1)\} < \frac{D_\kappa}{2},$$

which implies that $\{x_n\}$ is κ-bounded and $\sup_{n\in\mathbb{N}} d(p, Tx_n) < D_\kappa/2$. Further, for any $n \in \mathbb{N}$,

$$d(u, Tx_n) \le d(u, p) + d(p, Tx_n) \le \max\{2d(u, p), d(u, p) + d(p, x_1)\} < D_\kappa$$

and thus $\{d(u, Tx_n)\}$ is bounded. Fix $n \in \mathbb{N}$. Let $l = d(u, p)$, $l_n = d(u, Tx_n)$,

$$M_n = \sqrt{\alpha_n^2 + (1-\alpha_n)^2 + 2\alpha_n(1-\alpha_n)c_\kappa''(l_n)} \text{ and } \beta_n = 1 - \frac{1-\alpha_n}{M_n}.$$

Further, set

$$\begin{aligned} t_n &= \frac{(M_n + 1 - \alpha_n)((1+M_n)\phi_\kappa(p,u) - 2(1-\alpha_n)\phi_\kappa(u, Tx_n))}{(1+M_n)(\alpha_n + 2(1-\alpha_n)c_\kappa''(d(u, Tx_n)))} \\ &= \frac{(M_n + 1 - \alpha_n)((1+M_n)c_\kappa(l) - 2(1-\alpha_n)c_\kappa(l_n))}{(1+M_n)(\alpha_n + 2(1-\alpha_n)c_\kappa''(l_n))} \end{aligned}$$

and $s_n = \phi_\kappa(p, x_n)$. From Lemma 3.2,

$$\begin{aligned} s_{n+1} &= \phi_\kappa(p, x_{n+1}) = \phi_\kappa(p, \alpha_n u \overset{\kappa}{\oplus} (1-\alpha_n)Tx_n) \le (1-\beta_n)\phi_\kappa(p, Tx_n) + \beta_n t_n \\ &\le (1-\beta_n)\phi_\kappa(p, x_n) + \beta_n t_n = (1-\beta_n)s_n + \beta_n t_n. \end{aligned}$$

Since one of (a) and (b) holds, from Lemma 3.3, we have

$$\lim_{n\to\infty} \beta_n = 0 \text{ and } \sum_{n=1}^{\infty} \beta_n = \infty.$$

Let $\{s_{n_i}\}$ be a subsequence of $\{s_n\}$ such that

$$\limsup_{i\to\infty} (s_{n_i} - s_{n_i+1}) \le 0,$$

and we show $\limsup_{i\to\infty} t_{n_i} \le 0$. Then, we get

$$\begin{aligned} 0 &\ge \limsup_{i\to\infty} (s_{n_i} - s_{n_i+1}) = \limsup_{i\to\infty} \left(\phi_\kappa(p, x_{n_i}) - \phi_\kappa(p, x_{n_i+1})\right) \\ &= \limsup_{i\to\infty} \left(\phi_\kappa(p, x_{n_i}) - \phi_\kappa(p, \alpha_{n_i} u \overset{\kappa}{\oplus} (1-\alpha_{n_i})Tx_{n_i})\right) \\ &\ge \limsup_{i\to\infty} \left(\phi_\kappa(p, x_{n_i}) - \alpha_{n_i}\phi_\kappa(p, u) - (1-\alpha_{n_i})\phi_\kappa(p, Tx_{n_i})\right) \\ &= \limsup_{i\to\infty} \left(\phi_\kappa(p, x_{n_i}) - \phi_\kappa(p, Tx_{n_i})\right) \ge \liminf_{i\to\infty} \left(\phi_\kappa(p, x_{n_i}) - \phi_\kappa(p, Tx_{n_i})\right) \ge 0 \end{aligned}$$

and thus $\lim_{i\to\infty} \left(\phi_\kappa(p, x_{n_i}) - \phi_\kappa(p, Tx_{n_i})\right) = 0$. We notice that c_κ^{-1} is uniformly continuous on a compact interval $[0, c_\kappa(\sup_{n\in\mathbb{N}} d(p, x_n))]$ and that

$$\lim_{i\to\infty} \left|c_\kappa(d(p, x_{n_i})) - c_\kappa(d(p, Tx_{n_i}))\right| = 0.$$

Therefore,

$$\lim_{i\to\infty} (d(p, x_{n_i}) - d(p, Tx_{n_i})) = \lim_{i\to\infty} \left|c_\kappa^{-1}(c_\kappa(d(p, x_{n_i}))) - c_\kappa^{-1}(c_\kappa(d(p, Tx_{n_i})))\right| = 0.$$

Since T is strongly quasinonexpansive, we have

$$\lim_{i\to\infty} d(Tx_{n_i}, x_{n_i}) = 0.$$

Take a subsequence $\{w_j\}$ of $\{x_{n_i}\}$ such that

$$\lim_{j\to\infty} d(u, w_j) = \liminf_{i\to\infty} d(u, x_{n_i})$$

and that it Δ-converges to some $w \in X$. Since T is Δ-demiclosed, we get $w \in \operatorname{Fix} T$. Further, since

$$d(u, x_{n_i}) \le d(u, Tx_{n_i}) + d(Tx_{n_i}, x_{n_i}) \le d(u, x_{n_i}) + 2d(Tx_{n_i}, x_{n_i}),$$

we have

$$\liminf_{i\to\infty} l_{n_i} = \liminf_{i\to\infty} d(u, Tx_{n_i}) = \liminf_{i\to\infty} d(u, x_{n_i}).$$

Hence, from Lemma 2.3,

$$\liminf_{i\to\infty} l_{n_i} = \liminf_{i\to\infty} d(u, x_{n_i}) = \lim_{j\to\infty} d(u, w_j) \ge d(u, w) \ge d(u, p) = l.$$

From Lemma 3.4, we have

$$\limsup_{i\to\infty} t_{n_i} \le 0.$$

Consequently, from Lemma 4.1, we obtain $\lim_{n\to\infty} s_n = 0$. It means that $\{x_n\}$ converges to $P_{\operatorname{Fix} T} u$. □

Theorem 4.3. *Let X, T, $\{\alpha_n\}$ and $\{x_n\}$ be the same as the previous theorem, and assume one of the following:*

(A) $\sup_{y\in X} d(u, y) < D_\kappa/2$;

(B) $d(u, P_{\operatorname{Fix} T} u) < D_\kappa/4$ *and* $d(u, P_{\operatorname{Fix} T} u) + d(x_1, P_{\operatorname{Fix} T} u) < D_\kappa/2$;

(C) $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$.

Then, $\{x_n\}$ converges to a fixed point $P_{\operatorname{Fix} T} u$, where $P_{\operatorname{Fix} T}$ is a metric projection onto $\operatorname{Fix} T$.

Proof. It is sufficient to prove $\sup_{n\in\mathbb{N}} d(u, Tx_n) < D_\kappa/2$ when (A) or (B) hold. If (A) holds, then we easily get the desired inequality. Assume (B) holds. We know

$$d(P_{\operatorname{Fix} T} u, Tx_n) \le \max\{d(u, P_{\operatorname{Fix} T} u), d(x_1, P_{\operatorname{Fix} T} u)\}$$

for all $n \in \mathbb{N}$. Then,

$$\begin{aligned} \sup_{n\in\mathbb{N}} d(u, Tx_n) &\le \sup_{n\in\mathbb{N}} (d(u, P_{\operatorname{Fix} T} u) + d(Tx_n, P_{\operatorname{Fix} T} u)) \\ &\le d(u, P_{\operatorname{Fix} T} u) + \max\{d(u, P_{\operatorname{Fix} T} u), d(x_1, P_{\operatorname{Fix} T} u)\} \\ &= \max\{2d(u, P_{\operatorname{Fix} T} u), d(u, P_{\operatorname{Fix} T} u) + d(x_1 P_{\operatorname{Fix} T} u)\} < \frac{D_\kappa}{2}. \end{aligned}$$

It completes the proof. □

In Theorem 4.2, to get convergence of the sequence, we need to use κ-convex combination. To obtain a convergence theorem with the usual convex combination as a direct consequence of Theorem 4.2, we need the following lemmas:

Lemma 4.4 (Kimura–Sudo [11]). *Let X be an admissible* CAT(κ) *space for $\kappa \in \mathbb{R}$. Then,*

$$tx \oplus (1-t)y = \left(\frac{(t)_l^\kappa}{(t)_l^\kappa + (1-t)_l^\kappa} \right) x \overset{\kappa}{\oplus} \left(\frac{(1-t)_l^\kappa}{(t)_l^\kappa + (1-t)_l^\kappa} \right) y$$

for every $x, y \in X$ and $t \in [0,1]$, where $l = d(x, y)$.

Lemma 4.5. *Let $\kappa \in \mathbb{R}$. Let $\{\alpha_n\}$ be a real sequence of $]0,1[$ which converges to 0 and let $\{l_n\}$ be a bounded real sequence of $[0, D_\kappa/2[$. Let*

$$\sigma_n = \frac{(\alpha_n)^\kappa_{l_n}}{(\alpha_n)^\kappa_{l_n} + (1-\alpha_n)^\kappa_{l_n}}$$

for each $n \in \mathbb{N}$. Then, there exist positive real numbers r_1 and r_2, and $n_0 \in \mathbb{N}$ such that

$$r_1\alpha_n \le \sigma_n \le r_2\alpha_n$$

for all $n \in \mathbb{N}$ with $n \ge n_0$.

Proof. We first show that there exist a real number r_2 and $n_0 \in \mathbb{N}$ such that $\sigma_n \le r_2\alpha_n$ for all $n \in \mathbb{N}$ with $n \ge n_0$. Fix $n \in \mathbb{N}$ arbitrarily. If $l_n = 0$, then

$$\sigma_n = \frac{(\alpha_n)^\kappa_0}{(\alpha_n)^\kappa_0 + (1-\alpha_n)^\kappa_0} = \alpha_n.$$

Suppose $l_n \ne 0$. Then,

$$\sigma_n = \frac{(\alpha_n)^\kappa_{l_n}}{(\alpha_n)^\kappa_{l_n} + (1-\alpha_n)^\kappa_{l_n}} = \frac{c'_\kappa(\alpha_n l_n)}{c'_\kappa(\alpha_n l_n) + c'_\kappa((1-\alpha_n)l_n)}.$$

We notice that

$$\frac{\tau l_n}{c'_\kappa(\tau l_n)} \le 2$$

for any $\kappa \in \mathbb{R}$ and $\tau \in]0,1[$. Therefore,

$$c'_\kappa(\alpha_n l_n) + c'_\kappa((1-\alpha_n)l_n) \ge \frac{\alpha_n l_n}{2} + \frac{(1-\alpha_n)l_n}{2} = \frac{l_n}{2}$$

and hence

$$\sigma_n \le \frac{2c'_\kappa(\alpha_n l_n)}{l_n} = 2\alpha_n \cdot \frac{c'_\kappa(\alpha_n l_n)}{\alpha_n l_n}.$$

Since $\{l_n\}$ is bounded and $\lim_{n\to\infty} \alpha_n = 0$, we have

$$\lim_{n\to\infty} \frac{c'_\kappa(\alpha_n l_n)}{\alpha_n l_n} = 1.$$

Therefore, setting $r_2 = 4$, we obtain the desired evaluation. We next show that there exists a real number r_1 such that $r_1\alpha_n \le \sigma_n$ for all $n \in \mathbb{N}$. Let $l_0 = \sup_{n\in\mathbb{N}} l_n$. If $l_0 = 0$, then setting $r_1 = 1$, we obtain the desired result. Fix $n \in \mathbb{N}$ arbitrarily. Suppose $l_n \ne 0$. If $\kappa > 0$, then

$$\sigma_n = \frac{(\alpha_n)^\kappa_{l_n}}{(\alpha_n)^\kappa_{l_n} + (1-\alpha_n)^\kappa_{l_n}} \ge \frac{(\alpha_n)^\kappa_{l_n}}{2} = \frac{c'_\kappa(\alpha_n l_n)}{2c'_\kappa(l_n)} \ge \frac{\alpha_n c'_\kappa(l_n)}{2c'_\kappa(l_n)} = \frac{\alpha_n}{2}.$$

On the other hand, if $\kappa \le 0$, then

$$\sigma_n = \frac{(\alpha_n)^\kappa_{l_n}}{(\alpha_n)^\kappa_{l_n} + (1-\alpha_n)^\kappa_{l_n}} \ge (\alpha_n)^\kappa_{l_n} = \frac{c'_\kappa(\alpha_n l_n)}{c'_\kappa(l_n)} \ge \frac{\alpha_n l_n}{c'_\kappa(l_n)}.$$

Since

$$c'_\kappa(a) \leq \frac{c'_\kappa(l_0)}{l_0} a$$

for all $a \in [0, l_0]$, we have

$$\sigma_n \geq \frac{\alpha_n l_n}{c'_\kappa(l_n)} \geq \frac{\alpha_n l_0}{c'_\kappa(l_0)}.$$

Set $r_1 = \min\{1/2, l_0/c'_\kappa(l_0)\}$. Then, for any $\kappa \in \mathbb{R}$, we have $\sigma_n \geq r_1 \alpha_n$ whenever $l_n \neq 0$. Note that this inequality holds even if $l_n = 0$. Consequently, we obtain the desired result. □

Using lemmas above, we obtain the following result:

Corollary 4.6. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and T a strongly quasinonexpansive and Δ-demiclosed mapping on X. Let $\{\alpha_n\}$ be a real sequence of* $]0, 1[$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and that* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *For an anchor point $u \in X$ and an initial point $x_1 \in X$, generate a sequence $\{x_n\}$ of X as follows:*

$$x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T x_n$$

for each $n \in \mathbb{N}$. Further, assume one of the following:

(a) $\sup_{n\in\mathbb{N}} d(u, Tx_n) < D_\kappa/2$;

(b) $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$.

Then, $\{x_n\}$ converges to a fixed point $P_{\mathrm{Fix}\,T} u$, where $P_{\mathrm{Fix}\,T}$ is a metric projection onto Fix T.

5. An improvement of coefficient condition

In Theorem 4.3, we need to assume one of the following conditions:

(A) $\sup_{y\in X} d(u, y) < D_\kappa/2$;

(B) $d(u, P_{\mathrm{Fix}\,T} u) < D_\kappa/4$ and $d(u, P_{\mathrm{Fix}\,T} u) + d(x_1, P_{\mathrm{Fix}\,T} u) < D_\kappa/2$;

(C) $\sum_{n=1}^{\infty} \alpha_n^2 = \infty$.

When $\kappa \leq 0$, the condition (B) always holds in the situation of Theorem 4.3. However, the condition (A) is too strong to assume for each $\kappa \in \mathbb{R}$, and the condition (C) is barely proper for actual calculation when $\kappa > 0$. In what follows, we consider an improvement of the coefficient condition for a Halpern type convergence theorem. Focusing on the proofs of Lemma 3.3 and Theorem 4.2, we obtain the following theorem:

Theorem 5.1. *Let X be an admissible complete* CAT(κ) *space for $\kappa \in \mathbb{R}$ and T a strongly quasinonexpansive and Δ-demiclosed mapping on X. Let $\{\varepsilon_n\}$ be a real sequence of* $]0, 1[$ *such that* $\lim_{n\to\infty} \varepsilon_n = 0$ *and that* $\sum_{n=1}^{\infty} \varepsilon_n = \infty$. *For an anchor point $u \in X$ and an initial point $x_1 \in X$, generate a sequence $\{x_n\}$ of X as follows:*

$$\gamma_n \in \left]0, \frac{c''_\kappa(d(u, Tx_n))}{2}\right];$$
$$\alpha_n \in \left[\sqrt{\varepsilon_n + \gamma_n^2} - \gamma_n, \sqrt{\varepsilon_n}\right] \subset]0, 1[\,;$$
$$x_{n+1} = \alpha_n u \overset{\kappa}{\oplus} (1 - \alpha_n) T x_n$$

for each $n \in \mathbb{N}$. Then, $\{x_n\}$ converges to a fixed point $P_{\mathrm{Fix}\,T} u$, where $P_{\mathrm{Fix}\,T}$ is a metric projection onto Fix T.

Proof. For $\varepsilon \in]0,1[$ and $\gamma > 0$, we know

$$0 < \sqrt{\varepsilon + \gamma^2} - \gamma < \sqrt{\varepsilon} < 1.$$

Indeed,

$$\sqrt{\varepsilon + \gamma^2} - \gamma > \sqrt{\gamma^2} - \gamma = 0$$

and

$$\left(\sqrt{\varepsilon + \gamma^2} - \gamma\right)^2 = \varepsilon + 2\gamma^2 - 2\gamma\sqrt{\varepsilon + \gamma^2} < \varepsilon + 2\gamma^2 - 2\gamma\sqrt{\gamma^2} = \varepsilon < 1.$$

Therefore, the sequence $\{x_n\}$ is well-defined and $\lim_{n\to\infty} \alpha_n = 0$. Moreover, we obtain

$$\alpha_n^2 + 2\gamma_n\alpha_n \geq \varepsilon_n + 2\gamma_n^2 - 2\gamma_n\sqrt{\varepsilon_n + \gamma_n^2} + 2\gamma_n\left(\sqrt{\varepsilon_n + \gamma_n^2} - \gamma_n\right) = \varepsilon_n$$

and hence

$$\sum_{n=1}^{\infty}(\alpha_n^2 + c_\kappa''(d(u, Tx_n))\alpha_n) \geq \sum_{n=1}^{\infty}(\alpha_n^2 + 2\gamma_n\alpha_n) \geq \sum_{n=1}^{\infty} \varepsilon_n = \infty.$$

In what follows, we prove convergence of the sequence $\{x_n\}$. Although we use the same fashions as Lemma 3.3 and Theorem 4.2, we give the proof for the sake of completeness. Set $p = P_{\mathrm{Fix}\,T}u$. Since T is quasinonexpansive, for each $n \in \mathbb{N}$,

$$\begin{aligned}\phi_\kappa(p, x_{n+1}) &\leq \alpha_n\phi_\kappa(p, u) + (1 - \alpha_n)\phi_\kappa(p, Tx_n)\\ &\leq \alpha_n\phi_\kappa(p, u) + (1 - \alpha_n)\phi_\kappa(p, x_n) \leq \max\{\phi_\kappa(p, u), \phi_\kappa(p, x_n)\}.\end{aligned}$$

Therefore, for all $n \in \mathbb{N}$,

$$d(p, Tx_n) \leq d(p, x_n) \leq \max\{d(p, u), d(p, x_1)\} < \frac{D_\kappa}{2},$$

which implies that $\{x_n\}$ is κ-bounded. Further, for any $n \in \mathbb{N}$,

$$d(u, Tx_n) \leq d(u, p) + d(p, Tx_n) \leq \max\{2d(u, p), d(u, p) + d(p, x_1)\}$$

and thus $\{d(u, Tx_n)\}$ is bounded. Fix $n \in \mathbb{N}$. Let $l = d(u, p)$, $l_n = d(u, Tx_n)$,

$$M_n = \sqrt{\alpha_n^2 + (1 - \alpha_n)^2 + 2\alpha_n(1 - \alpha_n)c_\kappa''(l_n)} \text{ and } \beta_n = 1 - \frac{1 - \alpha_n}{M_n}.$$

Further, set

$$\begin{aligned}t_n &= \frac{(M_n + 1 - \alpha_n)((1 + M_n)\phi_\kappa(p, u) - 2(1 - \alpha_n)\phi_\kappa(u, Tx_n))}{(1 + M_n)(\alpha_n + 2(1 - \alpha_n)c_\kappa''(d(u, Tx_n)))}\\ &= \frac{(M_n + 1 - \alpha_n)((1 + M_n)c_\kappa(l) - 2(1 - \alpha_n)c_\kappa(l_n))}{(1 + M_n)(\alpha_n + 2(1 - \alpha_n)c_\kappa''(l_n))}\end{aligned}$$

and $s_n = \phi_\kappa(p, x_n)$. From Lemma 3.2,

$$\begin{aligned}s_{n+1} &= \phi_\kappa(p, x_{n+1}) = \phi_\kappa(p, \alpha_n u \overset{\kappa}{\oplus} (1 - \alpha_n)Tx_n) \leq (1 - \beta_n)\phi_\kappa(p, Tx_n) + \beta_n t_n\\ &\leq (1 - \beta_n)\phi_\kappa(p, x_n) + \beta_n t_n = (1 - \beta_n)s_n + \beta_n t_n.\end{aligned}$$

We next show that $\sum_{n=1}^{\infty} \beta_n = \infty$. Since $\{l_n\}$ is bounded, there exists $L > 0$ such that $c_\kappa''(l_n) \leq L$. Note that $M_n \leq \sqrt{2(1+L)}$ for all $n \in \mathbb{N}$. Moreover, there exists $n_0 \in \mathbb{N}$ such that $1 - \alpha_n \geq 1/2$ for all $n \in \mathbb{N}$ with $n \geq n_0$ since $\lim_{n\to\infty} \alpha_n = 0$. Therefore, we have

$$\beta_n = \frac{\alpha_n^2 + 2\alpha_n(1-\alpha_n)c_\kappa''(l_n)}{M_n(M_n + 1 - \alpha_n)} \geq \frac{\alpha_n^2 + c_\kappa''(l_n)\alpha_n}{\sqrt{2(1+L)}\left(\sqrt{2(1+L)}+1\right)}$$

and hence $\sum_{n=1}^{\infty} \beta_n = \infty$. Let $\{s_{n_i}\}$ be a subsequence of $\{s_n\}$ such that

$$\limsup_{i\to\infty} (s_{n_i} - s_{n_i+1}) \leq 0,$$

and we show $\limsup_{i\to\infty} t_{n_i} \leq 0$. Then, we get

$$\begin{aligned}
0 &\geq \limsup_{i\to\infty} (s_{n_i} - s_{n_i+1}) = \limsup_{i\to\infty} \left(\phi_\kappa(p, x_{n_i}) - \phi_\kappa(p, x_{n_i+1})\right) \\
&= \limsup_{i\to\infty} \left(\phi_\kappa(p, x_{n_i}) - \phi_\kappa(p, \alpha_{n_i} u \overset{\kappa}{\oplus} (1-\alpha_{n_i}) T x_{n_i})\right) \\
&\geq \limsup_{i\to\infty} \left(\phi_\kappa(p, x_{n_i}) - \alpha_{n_i}\phi_\kappa(p, u) - (1-\alpha_{n_i})\phi_\kappa(p, T x_{n_i})\right) \\
&= \limsup_{i\to\infty} \left(\phi_\kappa(p, x_{n_i}) - \phi_\kappa(p, T x_{n_i})\right) \geq \liminf_{i\to\infty} \left(\phi_\kappa(p, x_{n_i}) - \phi_\kappa(p, T x_{n_i})\right) \geq 0
\end{aligned}$$

and thus $\lim_{i\to\infty} \left(\phi_\kappa(p, x_{n_i}) - \phi_\kappa(p, T x_{n_i})\right) = 0$. We notice that c_κ^{-1} is uniformly continuous on a compact interval $[0, c_\kappa(\sup_{n\in\mathbb{N}} d(p, x_n))]$ and that

$$\lim_{i\to\infty} \left| c_\kappa(d(p, x_{n_i})) - c_\kappa(d(p, T x_{n_i})) \right| = 0.$$

Therefore,

$$\lim_{i\to\infty} (d(p, x_{n_i}) - d(p, T x_{n_i})) = 0.$$

Since T is strongly quasinonexpansive, we have

$$\lim_{i\to\infty} d(T x_{n_i}, x_{n_i}) = 0.$$

Take a subsequence $\{w_j\}$ of $\{x_{n_i}\}$ such that

$$\lim_{j\to\infty} d(u, w_j) = \liminf_{i\to\infty} d(u, x_{n_i})$$

and that it Δ-converges to some $w \in X$. Since T is Δ-demiclosed, we get $w \in \operatorname{Fix} T$. Further, since

$$d(u, x_{n_i}) \leq d(u, T x_{n_i}) + d(T x_{n_i}, x_{n_i}) \leq d(u, x_{n_i}) + 2d(T x_{n_i}, x_{n_i}),$$

we have

$$\liminf_{i\to\infty} l_{n_i} = \liminf_{i\to\infty} d(u, T x_{n_i}) = \liminf_{i\to\infty} d(u, x_{n_i}).$$

Hence, from Lemma 2.3,

$$\liminf_{i\to\infty} l_{n_i} = \liminf_{i\to\infty} d(u, x_{n_i}) = \lim_{j\to\infty} d(u, w_j) \geq d(u, w) \geq d(u, p) = l.$$

From Lemma 3.4, we have

$$\limsup_{i\to\infty} t_{n_i} \leq 0.$$

Consequently, from Lemma 4.1, we obtain $\lim_{n\to\infty} s_n = 0$. It means that $\{x_n\}$ converges to $P_{\operatorname{Fix} T} u$. □

According to Theorem 5.1, the generated sequence $\{x_n\}$ converges to the closest fixed point to u even if $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sup_{y \in X} d(u, y) = D_\kappa/2$. Furthermore, we know the following facts: In the iteration of Theorem 5.1, we can take $\{\alpha_n\}$ as $\{\sqrt{\varepsilon_n}\}$ for each $\kappa \in \mathbb{R}$. We further suppose that $c_\kappa''(d(u, Tx_n)) > 1 - \varepsilon_n$ for $n \in \mathbb{N}$. Then, we can take γ_n as

$$\gamma_n \in \left] \frac{1 - \varepsilon_n}{2}, \frac{c_\kappa''(d(u, Tx_n))}{2} \right[.$$

It implies that $1 - 2\gamma_n < \varepsilon_n$ and thus $\varepsilon_n - 2\gamma_n \varepsilon_n < \varepsilon_n^2$. Then, we know

$$\varepsilon_n + \gamma_n^2 < \varepsilon_n^2 + 2\gamma_n \varepsilon_n + \gamma_n^2 = (\varepsilon_n + \gamma_n)^2$$

and therefore $\sqrt{\varepsilon_n + \gamma_n^2} - \gamma_n < \varepsilon_n$. It means that

$$\varepsilon_n \in \left[\sqrt{\varepsilon_n + \gamma_n^2} - \gamma_n, \sqrt{\varepsilon_n} \right].$$

It implies that we can set $\alpha_n = \varepsilon_n$ for $n \in \mathbb{N}$ whenever

$$c_\kappa''(d(u, Tx_n)) > 1 - \varepsilon_n.$$

If $\kappa \leq 0$, we can always take $\{\alpha_n\}$ as $\{\varepsilon_n\}$. Indeed, when $\kappa \leq 0$, we get

$$c_\kappa''(d(u, Tx_n)) \geq 1 > 1 - \varepsilon_n$$

for any $n \in \mathbb{N}$. Consequently, Theorem 5.1 is a result with an improvement of the coefficient condition.

6. Conclusion

In this paper, we introduced an iteration method with an improvement of the coefficient condition. In previous research, we should consider proofs which are dependent on curvature parameters. However, using Lemma 3.1, we can prove Halpern type convergence theorems without separating cases with κ and in a manner of Banach spaces. They imply that some techniques in this paper can be applied to other issues in geodesic spaces with general curvatures. They also may let us study geodesic spaces such as flat, spherical and hyperbolical surfaces in the same ways.

Acknowledgement. This work was partially supported by JSPS KAKENHI Grant Number JP21K03316.

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