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Topological and differentiable sphere theorems for 4− dimensional CR-warped product submanifolds of 6-dimensional unit sphere

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Abstract. In this paper, taking into account that the 6-dimensional unit sphere is nearly Kaehler manifold, 4-dimensional CR-warped product submanifolds of the sphere are studied. First, an interesting relation is obtained among the warping function of the CR-warped product submanifold, the scalar curvature of the fibers and the components of the second fundamental form.Using this relation, topological and differential sphere theorems are given and totally geodesicity of CR-warped product submanifold of the 6-dimensional sphere is obtained. Moreover, a result is presented about homology groups of a CR-warped submanifold.

1. Introduction

Sphere theorems are the most important area of study in geometry and topology. Spherical theorems have attracted the attention of many authors [22–24]. Therefore, many sphere theorems were obtained by putting lower and upper limits on geometric invariants of the submanifold, Indeed, In [33], the author obtained sphere theorems depending on the mean curvature vector field (parallelity, constant, etc.) and curvatures. Recently, in [29] the authors investigated topological and differentiable structures of submanifolds under extrinsic restrictions. They obtained topological sphere theorem for compact submanifolds in a Riemannian manifold and proved an optimal differentiable sphere theorem for 4-dimensional complete submanifolds in a space form as a partial solution for Poincare conjecture. They also proved some new differentiable sphere theorems $n-$ dimensional submanifolds in a Riemannian manifolds.

One of the most interesting structures in manifold theory is the 6-dimensional unit sphere. Its interestingness is due to the complex structure on it. The almost complex structure constructed with the help of Cayley numbers on the 6-dimensional sphere is not a Kaehler structure. This structure is the nearly Kaehler structure well known in the theory of complex manifolds. For this reason, the submanifolds of the 6-dimensional sphere, or more generally the nearly Kaehler manifolds, have been also studied by many authors [10, 11, 14, 17, 26–28]. The submanifolds of a complex manifold are defined with respect to complex structure of the manifold. Holomorphic submanifolds and anti-invariant submanifolds of complex manifolds were well known in the literature [20], [31]. In 1978, Bejancu [3] defined CR-submanifolds containing these two notions. Chen defined the notion of CR-product [8] and obtained interesting characterizations [9]. A CR-product is essentially a local product manifold of holomorphic and anti-invariant submanifolds.

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In 2001, Chen generalized CR-products by introducing the concept of the CR-warped product submanifold [6, 7] with the notion of warped product manifolds [4]. Results on CR-warped product submanifolds can be seen in [5].

The geometry of CR-warped product submanifolds of a nearly Kaehler manifold has been studied by many authors [1, 15, 16, 25]. A nice survey on the geometry of submanifolds of nearly Kaehler manifolds were given in [2]. In this paper, our aim is to investigate the topological properties and differential properties of the 4-dimensional CR-warped product submanifolds of the 6-dimensional sphere, which is a nearly Kaehler manifold. For this purpose, we obtain totally geodesicity of 4-dimensional CR-warped product submanifolds under certain conditions (Theorem 3.2, Theorem 3.6). We also obtain topological sphere theorems (Theorem 3.5, Theorem 3.7) and differential sphere theorems (Theorem 3.3, Theorem 3.4) for such submanifolds in 6− dimensional the unit sphere. Moreover, a vanishing theorem (Theorem 3.8) on homology groups of CR-warped product submanifold is given.

2. Preliminaries

Let (B, g_1) and (M_2, g_2) be Riemannian manifolds, f a positive function on M_1 π and σ projections from $M_1 \times M_2$ to M_1 and M_2, respectively. The warped product [4] $\mathcal{N} = M_1 \times_f M_2$ is the manifold $M_1 \times M_2$ equipped with the Riemannian structure such that

$$g_{\mathcal{N}}(\xi_1, \xi_2) = g_1(\pi_*\xi_1, \pi_*\xi_2) + (f \circ \pi)^2 g_2(\sigma_*\xi_1, \sigma_*\xi_2)$$

for ξ_1, ξ_2 on $\mathcal{N}$. The function f is named the warping function. Let ξ_1, ξ_2 be vector fields on M_1 and $\zeta_1, \zeta_2, \zeta_3$ vector fields on M_2, then from Lemma 7.3 of [4], we get

$$\nabla_{\xi_1}\zeta_2 = \nabla_{\zeta_2}\xi_1 = (\frac{\xi_1 f}{f})\zeta_2. \tag{1}$$

where ∇ is the metric connection on $\mathcal{N}$, $\overset{M_2}{R}$ is curvature tensor of M_2 and ∇f is the gradient of f. We denote the curvature tensor fields of $\mathcal{N}$ and M_2 by R and $\overset{M_2}{R}$. Then we get

$$R(\zeta_2, \zeta_3)\zeta_1 = \overset{M_2}{R}(\zeta_2, \zeta_3)\zeta_1 + \frac{g_{\mathcal{N}}(\nabla f, \nabla f)}{f^2}\{g_{\mathcal{N}}(\zeta_2, \zeta_1)\zeta_3 - g_{\mathcal{N}}(\zeta_3, \zeta_1)\zeta_2\}. \tag{2}$$

It is well known that an almost complex structure is defined by the vector product of Cayley numbers on the 6-dimensional sphere as a hypersurface of 7-dimensional Euclidean space. This nearly complex structure, together with the metric induced from 7− dimensional Euclidean space, defines a nearly Kaehler structure that is not Kaehler. Thus, there are relations

$$(\nabla_{\xi_1} J)\xi_1 = 0 \tag{3}$$

and

$$\bar{g}(J\xi_1, J\xi_2) = \bar{g}(\xi_1, \xi_2) \tag{4}$$

for the vector fields ξ_1 and ξ_2 on the 6− dimensional unit sphere. Thus S^6 can be considered as a nearly Kaehler manifold of constant curvature 1 [32].

A submanifold M of a nearly Kaehler manifold $\bar{M}$ is named a CR-submanifold if there exists a differentiable distributions D^T and $D^\perp$ such that $J(D^T) = D^T$ and $J(D^\perp) \subseteq TM^\perp$. A submanifold M of a nearly Kaehler manifold $\bar{M}$ is named CR-warped product [16] if it is the warped product manifold.

We denote the set of tangent vector fields and normal vector fields on a submanifold of an even dimensional sphere $\mathcal{S}^n$ by $\Gamma(TM)$ and $\Gamma(TM^\perp)$, respectively. We also denote the second fundamental form and the shape operator by h and A_ξ, respectively. Gauss and Weingarten formulas of a submanifold M are given

$$\bar{\nabla}_{\xi_1}\xi_2 = \nabla_{\xi_1}\xi_2 + h(\xi_1, \xi_2), \nabla_{\xi_1}\xi_2 = -A_V\xi_1 + \nabla_{\xi_1}^{\perp}V$$

respectively, for $\xi_1, \xi_2 \in \Gamma(TM)$, $V \in \Gamma(TM^{\perp})$. Let $\bar{M}$ be a Riemannian manifold and M a submanifold of $\bar{M}$. Then, for vector fields $\xi_1, \xi_3, \xi_3, \xi_4$ on M, the Gauss equation is

$$g(\bar{R}(\xi_1, \xi_2)\xi_3, \xi_4) = g(R(\xi_1, \xi_2)\xi_3, \xi_4) - g(h(\xi_1, \xi_4), h(\xi_2, \xi_3)) + g(h(\xi_1, \xi_3), h(\xi_2, \xi_4)), \tag{5}$$

where $\bar{R}$ and R are the curvature tensor fields of $\bar{M}$ and M, respectively.

3. 4− dimensional CR-warped product submanifolds of a 6− dimensional sphere

In this section, we first obtain an relation between the warping function, scalar curvature, and second fundamental form of a CR-warped product submanifold. Then, using the results given on the sphere theorems before, topological and differential sphere theorems are given. At the end of the chapter, Also using Lawson-Simons's result, the results for the homology groups on the 4-dimensional CR-submanifolds of the 6-dimensional sphere are presented. We first prove the below lemma which will be crucial for our theorems.

Lemma 3.1. *Let M be a* 4− *dimensional compact CR-warped product submanifold in the form $M_{\top} \times_f M_{\perp}$ of S^6. Then we get*

$$\|h\|^2 = 2 - 2\tau_{D^{\perp}} + 6\|\nabla ln f\|^2 + \sum_{\alpha,\beta}^{2} g(h(v_\alpha, v_\alpha), h(v_\beta, v_\beta)$$

where $\tau_{D^{\perp}}$ denotes the scalar curvature of the leaf $M_{\perp}$ of $D^{\perp}$.

Proof. Since M is a CR-warped product submanifold, we can take orthonormal frames $\{v_1, v_2 = Jv_1\}$ and $\{v_3, v_4\}$ of $D^{\top}$ and $D^{\perp}$, respectively. Then we have $\{v_1, Jv_1, v_3, v_4, Jv_3, Jv_4\}$ as an orthonormal frame of $\mathcal{S}^6$. Thus we obtain

$$\|h\|^2 = \sum_{i,j=1,\alpha=1}^{2} g(h(v_i, v_j), Jv_\alpha)^2 + 2\sum_{i,\alpha,\beta=1}^{2} g(h(v_i, v_\alpha), Jv_\beta)^2 + \sum_{\alpha,\beta,\gamma=1}^{2} g(h(v_\alpha, v_\beta), v_\gamma)^2. \tag{6}$$

On the other hand, from [25, Lemma 4.3] (see also: [1] and [16]) we get $g(h(\xi_1, \xi_2), J\xi_3) = 0$ for $\xi_1, \xi_2 \in \Gamma(D^{\top})$ and $\xi_3 \in \Gamma(D^{\perp})$. Using this in (6) we get

$$\|h\|^2 = 2\sum_{i,\alpha,\beta=1}^{2} g(h(v_i, v_\alpha), Jv_\beta)^2 + \sum_{\alpha,\beta,\gamma=1}^{2} g(h(v_\alpha, v_\beta), v_\gamma)^2. \tag{7}$$

Moreover, from [16, Lemma 4.1] we get $g(J\xi_1, \xi_3), J\xi_4) = (\xi_1(\ln f)g(\xi_3, \xi_4)$ for $\xi_1 \in \Gamma(D^{\top})$ and $\xi_3, \xi_4 \in \Gamma(D^{\perp})$. Putting this in (7) we derive

$$\|h\|^2 = 4\|\overset{T}{\nabla}\ln f\|^2 + \sum_{\alpha,\beta,\gamma=1}^{2} g(h(v_\alpha, v_\beta), v_\gamma)^2, \tag{8}$$

where $\overset{T}{\nabla}\ln f$ is the $D-$ component of $\nabla \ln f$. On the other hand, taking $\xi_1 = \xi_4 = v_\alpha, \xi_2 = \xi_3 = v_\beta$ in (5), we have

$$\sum_{\alpha,\beta=1}^{2} g(\bar{R}(v_\alpha, v_\beta)v_\beta, v_\alpha) = \sum_{\alpha,\beta=1}^{2} g(R(v_\alpha, v_\beta)v_\beta, v_\alpha) - g(h(v_\alpha, v_\alpha), h(v_\beta, v_\beta)) + \|h(v_\alpha, v_\beta)\|^2. \tag{9}$$

Since the ambient space is the unit sphere, we obtain

$$\sum_{\alpha,\beta=1}^{2} \|h(v_\alpha, v_\beta)\|^2 = 2 - \sum_{\alpha,\beta=1}^{2} g(R(v_\alpha, v_\beta)v_\beta, v_\alpha) - g(h(v_\alpha, v_\alpha), h(v_\beta, v_\beta)). \tag{10}$$

On the other hand, from (2) we have

$$\sum_{\alpha,\beta=1}^{2} g(R(v_\alpha, v_\beta)v_\beta, v_\alpha) = \sum_{\alpha,\beta=1}^{2} g(\overset{M_2}{R}(v_\alpha, v_\beta)v_\beta, v_\alpha) - 2\|\overset{T}{\nabla} \ln f\|^2. \tag{11}$$

Using (11) in (10) we get

$$\sum_{\alpha,\beta=1}^{2} \|h(v_\alpha, v_\beta)\|^2 = 2 - \sum_{\alpha,\beta=1}^{2} g(\overset{M_2}{R}(v_\alpha, v_\beta)v_\beta, v_\alpha) - g(h(v_\alpha, v_\alpha), h(v_\beta, v_\beta)) + 2\|\overset{T}{\nabla} \ln f\|^2. \tag{12}$$

Putting (12) in (8) we derive

$$\|h\|^2 = 6\|\overset{T}{\nabla} \ln f\|^2 + 2 - \sum_{\alpha,\beta=1}^{2} g(\overset{M_2}{R}(v_\alpha, v_\beta)v_\beta, v_\alpha) + g(h(v_\alpha, v_\alpha), h(v_\beta, v_\beta)) \tag{13}$$

which completes proof. □

From Lemma 3.1 and Yau's result [33, Theorem 8] we get the below theorem.

Theorem 3.2. *Let M be a $4-$ dimensional compact CR-warped product submanifold in the form $M_\top \times_f M_\perp$ of S^6 with parallel mean curvature. If*

$$\sum_{\alpha,\beta=1}^{2} g(h(v_\alpha, v_\alpha), h(v_\beta, v_\beta)) \leq 2\tau_{D^\perp} - 6\|\overset{T}{\nabla} \ln f\|^2 - 1,$$

then M lies in a totally geodesic $\mathcal{S}^5$.

From Lawson-Simons's result [18, Corollary 2] and Lemma 3.1, we obtain the below theorem

Theorem 3.3. *Let M be a $4-$ dimensional oriented compact CR-warped product submanifold of S^6 with the form $M_\top \times_f M_\perp$. If*

$$\sum_{\alpha,\beta} g(h(v_\alpha, v_\alpha), h(v_\beta, v_\beta)) < 1 + 2\tau_{D^\perp} - 6\|\nabla ln f\|^2$$

then M is a homotopy sphere.

From [30, Theorem 1.1] and Lemma 3.1, we get the below differentiable sphere theorem.

Theorem 3.4. *Let M be a $4-$ dimensional oriented complete CR-warped product submanifold of S^6 with the form $M_\top \times_f M_\perp$. If*

$$\sup_M (2 - 2\tau_{D^\perp} + 6\|\nabla ln f\|^2 + \sum_{\alpha,\beta}^{2} g(h(v_\alpha, v_\alpha), h(v_\beta, v_\beta))) < 2\sqrt{3},$$

then M is diffeomorphic to the standard unit $4-$ sphere $\mathcal{S}^4$.

If M is a minimal submanifold, from [19, Main Theorem] and Lemma 3.1 we get the below topological sphere theorem.

Theorem 3.5. *Let M be a $4-$ dimensional compact minimal CR-warped product submanifold in the form $M_\top \times_f M_\perp$ of S^6. If*

$$\|\overset{T}{\nabla} \ln f\|^2 < \frac{1}{3}(1 + \tau_{D^\perp}),$$

then M is homeomorphic to a sphere.

Proof. Since M is minimal, we get

$$H = \frac{1}{4}[h(v_1, v_1) + h(Jv_1, Jv_1) + h(v_3, v_3) + h(v_4, v_4)] = 0.$$

On the other hand, since M is a warped product submanifold in the form $M_\top \times_f M_\perp$ of of a nearly Kaehler manifold S^6, $D^\top$ is integrable. Then from [3, Theorem 2.4] we know that $h(\xi_1, J\xi_2) = h(J\xi_1, \xi_2)$ for $\xi_1, \xi_2 \in \Gamma(D^\top)$. Hence it follows that $h(v_1, v_1) + h(Jv_1, Jv_1) = 0$. Thus we get $h(v_3, v_3) + h(v_4, v_4) = 0$ which completes proof. □

We also get the below thereom due to Lemma 3.1 and [21, Theorem 3].

Theorem 3.6. *Let M be a $4-$ dimensional compact minimal CR-warped product submanifold in the form $M_\top \times_f M_\perp$ of S^6. If*

$$\frac{1}{3}(\frac{1}{3} + \tau_{D^\perp}) \geq \|\overset{T}{\nabla} \ln f\|^2,$$

then M is totally geodesic submanifold.

From [29, Main Theorem] and Lemma 3.1, we get the below result.

Theorem 3.7. *Let M be a $4-$dimensional compact oriented and connected CR-warped product submanifold of S^6 with the form $M_\top \times_f M_\perp$ with scalar curvature $12r$and nowhere-zero mean curvature. If*

$$r \geq \frac{2}{3}$$

and

$$6\|\nabla ln f\|^2 + \sum_{\alpha,\beta}^{2} g(h(v_\alpha, v_\alpha), h(v_\beta, v_\beta)) \leq 6r - 5 + \frac{1}{2r-1} + 2\tau_{D^\perp},$$

then either

(i) the fundamental group of M is finite and M is homeomorphic to a sphere;

(ii) M is isometric to the Riemannian product $\mathcal{S}^1(\sqrt{1-c^2}) \times \mathcal{S}^3(c)$ with $c^2 = \frac{1}{2r}$.

We now examine relations between the homology group of the CR-submanifold and our main lemma. Let $\tilde{M}$ be an m-dimensional compact Riemannian manifold with Riemannian metric $\tilde{g}$ and the Levi-Civita connection $\tilde{\nabla}$. Denote by (S, ϱ) the oriented, p-rectifiable set in $\tilde{M}$. The set of rectifiable p-currents is

$$\mathcal{R}_p(\tilde{M}) = \{\mathcal{S} : \sum_{n=1}^{\infty} n\mathcal{S}_n; \mathcal{S}_n = (S_n, \varrho_n), \tilde{M}(\mathcal{S}) = \sum_{n=1}^{\infty} n\mathcal{H}^p(S_n) < \infty\}.$$

$\mathcal{S} \in \mathcal{R}_p(\tilde{M})$ is named an integral p-current if $\mathcal{S}$ and $\partial\mathcal{S}$ are both rectifiable currents, for details see:[12]. It is known that any non-trivial integral homology class in $H_p(M, \mathbb{Z})$ corresponds to a stable current [13]. By using this correspondence, Lawson and Simons obtained vanishing theorems concerning $H_p(M, \mathbb{Z})$. Applying their theorem, Leung proved a vanishing theorem for closed oriented minimal submanifold M^n of a sphere $\mathcal{S}^{n+p}$ for $n \geq 4$. Using [18, Theorem 4], [19, Theorem 2] and Lemma 3.1, we get the below theorem about homology groups of proper CR-warped product submanifolds.

Theorem 3.8. *Let M be a $4-$ dimensional closed oriented minimal proper CR-warped product submanifold of unit sphere S^6 with the form $M_\top \times_f M_\perp$. If*

$$3\|\nabla lnf\|^2 < 1 + \tau_{D^\perp}$$

then $H_2(M, \mathbb{Z}) = 0$ where $H_i(M, \mathbb{Z})$ is the $i-$th homology group of M with integer coefficients.

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References

[1] Al-Luhaibi, N. S., Al-Solamy, F. R., Khan, V.A., *CR-warped product submanifolds of nearly Kaehler manifolds*, J. Korean Math. Soc., **46(5)** (2009), 979–995.
[2] Anarella, M., *A survey on submanifolds of nearly Kaehler spaces*, arXiv:2401.05175, 2024.
[3] Bejancu, A., *Geometry of CR-Submanifolds* , Kluwer Academic Publishers, Dortrecht, 1986.
[4] Bishop, R. L., O'Neill, B., *Manifolds of negative curvature* , Trans. Amer. Math. Soc., **145**(1969), 1–49.
[5] Chen, B. Y., *Differential Geometry of Warped Product Manifolds and Submanifolds* , World Scientific, 2017.
[6] Chen, B. Y., *Geometry of warped product CR-submanifolds in Kaehler manifold* , Monatsh. Math., **133**(2001), 177–195.
[7] Chen, B. Y., *Geometry of warped product CR-submanifolds in Kaehler manifolds II* , Monatsh. Math., **134**(2001), 103–119.
[8] Chen, B. Y., *CR-submanifolds of a Kaehler manifold I*, J. Differential Geometry, **16(2)**, (1981), 305–322
[9] Chen, B. Y., *CR-submanifolds of a Kaehler manifold II*, J. Differential Geometry, **16(3)**, (1981), 493–509.
[10] Deshmukh, S., Shahid, M. H., Shahid, A., *CR-submanifolds of a nearly Kaehler manifold II*, Tamkang J. Math., **17** (1986), no. 4, 17–27.
[11] Deshmukh, S., *Cohomology of CR-submanifolds of a nearly Kaehler manifold*, Math. Chronicle, **16** (1987), 47–52.
[12] Federer, H., *Geometric Measure Theory* , Springer, 1996.
[13] Federer, H.,Fleming, W., *Normal and integral currents* , Ann. of Math., **72**(1960), 458–520.
[14] Husain, S. I., Deshmukh, S., *CR-submanifolds of a nearly Kaehler manifold*, Indian J. Pure Appl. Math.,**18** (1987), no. 11, 979–990.
[15] Khan, V. A., Jamal, N., Al-Kathiri, A. A., *A note on warped products submanifolds of nearly Kaehler manifolds*, Toyama Math. J.,**32** (2009), 59–73.
[16] Khan, V. A., Khan, K. A., Uddin, S., *Warped product CR-submanifolds in nearly Kaehler manifolds*, SUT J. Math.,**43** (2007), no. 2, 201–213.
[17] Kon, S. H., Tan, S. L., *Totally umbilical CR-submanifolds of a nearly Kaehler manifold*, Tamkang J. Math., **27** (1996), no. 2, 145–149.
[18] Lawson, H. B., Simons, J., *On stable currents and their application to global problems in real and complex geometry*, Ann. of Math., **98**(1973), 427–450.
[19] Leung P. F., *Minimal submanifolds in a sphere*, Math. Z., **183**(1983),75–86.
[20] Ogiue, K., *Differential geometry of Kaehler submanifolds*, Advances in Mathematics, **13(1)**, (1974), 73–114.
[21] Min, L. A., Jimin, L., *An intrinsic rigidity theorem for minimal submanifolds in a sphere*, Arch. Math.,**58** (1992), 582–594.
[22] Perelman, G., *The entropy formula for the Ricci flow and its geometric applications*, arXiv:math/0211159.
[23] Perelman, G., *Ricci flow with surgery on three-manifolds*, preprint, math.DG/0303109.
[24] Perelman, G., *Finite extinction time for the solutions to the Ricci flow on certain threemanifolds*, preprint, math.DG/0307245.
[25] Sahin, B., Gunes, R., *CR-Warped Product Submanifolds of Nearly Kaehler Manifolds*, Contributions to Algebra and Geometry, **49(2)** (2008), 383–397.
[26] Sasahara, T., *Three-dimensional CR-submanifolds in the nearly Kaehler six-sphere satisfying B. Y. Chen's basic equality*, Tamkang J. Math.,**31** (2000), no. 4, 289–296.
[27] Shahid, A., Husain, S. I., *Submersions of CR-submanifolds of a nearly Kaehler manifold I*, Rad. Mat., **7** (1991), no. 2, 197–205.
[28] Shahid, A., *Submersions of CR-submanifolds of a nearly Kaehler manifold II*, Rad. Mat., **8** (1992/98), no. 2, 281–289.
[29] Shu, S. C. Liu, S. Y., *Curvature and topology of compact submanifolds in the unit sphere*, Pacific Journal of Mathematics, **228(2)** (2006), 371–386.
[30] Xu, H. W., Zhao, E. T., *Topological and differentiable sphere theorems for complete submanifolds*, Commun. Analysis and Geometry, **17(3)**(2009),565–585.
[31] Yano, K., Kon, M., *Anti-invariant submanifolds*, Marcel Dekker, 1976.
[32] Yano, K. Kon, M., *Structures on manifolds*, World Scientific, 1984.
[33] Yau, S. T., *Submanifolds with constant mean curvature II*, Amer. J. Math., **97(1)** (1975), 76–100.