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The symbolic approach to study the family of Appell-λ matrix polynomials

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Abstract. In this research article, we introduced certain hybrid and matrix special polynomial associated to λ polynomials and established their properties. Further, the monomiality principle and differential equations for these newly introduced hybrid special polynomials are obtained. Next, the determinantal representations of the special matrix polynomials and hybrid special polynomials associated with Appell-λ polynomials are obtained. Also, we derive several intrinsic outcomes for the special cases of these polynomials. The proposed approach in this article is symbolic. The Bernoulli-λ numbers and Euler-λ numbers are also obtained. The graphical representations are also given.

1. Introduction

The theory of special matrix functions can be considered as generalization of the special functions [24]. The special matrix polynomials represent the system of equations and hence, due to several applications of system of equations in certain areas like physics, mathematics and computers the importance of special matrix polynomials is realized. Matrix analogues of certain polynomials such as Hermite, Laguerre and Legendre polynomials and their corresponding differential equations are established in [11, 12]. The extensions and generalizations of the Hermite matrix polynomials, in which matrix is used as a parameter have been established and investigated for matrices in $\mathbb{C}^{m\times m}$ such that their all eigenvalues are located at the right open half-plane [11, 13].

Throughout this paper, we consider that M is a positive stable matrix in $\mathbb{C}^{m\times m}$, *i.e.*, M satisfies the following condition:

$$Re(\mu) > 0; \quad \forall \quad \mu \in \sigma(M), \tag{1}$$

where $\sigma(M)$ represent the set of all the eigenvalues of M. If D_0 is the complex plane, which is cut along its negative real axis and $log(z)$ represents the principal logaritheorem of z, then $z^{\frac{1}{2}}$ denotes $\exp(\frac{1}{2}log(z))$. If matrix $M \in \mathbb{C}^{m\times m}$ with $\sigma(M) \subset D_0$, then $M^{\frac{1}{2}} = \sqrt{M}$ denotes the image by $z^{\frac{1}{2}}$ of the matrix functional calculus

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[10] acting on the matrix M.

The class of Appell polynomials was introduced and characterized completely by Appell [2] in 1880. Further, Srivastava *et. al* [18, 19] studied this class of polynomials from different points of view. The Appell sets [2] can be defined by making use of the following conditions which are equivalent to each other [17, p.398]:
$A_n(x) \quad (n \in \mathbb{N}_0)$, is said to be an Appell polynomial (A_n being of degree atmost n), if either

(i) $\frac{d}{dx}A_n(x) = nA_{n-1}(x), \qquad (n \in \mathbb{N}_0);$ or

(ii) there exists a generating function of the following form

$$\sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!} = A(t)\exp(xt), \tag{2}$$

where $A(t)$ can be expressed as:

$$A(t) = \sum_{n=0}^{\infty} A_n\frac{t^n}{n!}, \quad A_0 \neq 0. \tag{3}$$

The function $A(t)$ is referred as determining function of the set $\{A_n(x)\}$, $n \in \mathbb{N}$. The Appell polynomials are quasi-monomials with their respective multiplicative and derivative operator. The Appell polynomials show different significance in the field of applied mathematics and pure mathematics as well. Recently Srivastava investigated certain properties satisfied by Appell and q-Appell polynomials[22]. Further, most of the researchers are working on the study of q-Appell polynomials due to its considerable popularity and importance to serve as a bridge between mathematics and physics [20, 25].

The study of special polynomials and special functions, becomes more simple and easy by the use of symbolic method in a unified manner [27, 28]. This method have been established recently to deal with Laguerre polynomials [3] and Bessel functions [7]. In fact, Bessel polynomials continue to serve as an important tool in numerical and approximation techniques to solve a wide variety of problems stemming from mathematical, physical, chemical, biological and engineering sciences [15]. Before going to the specific details of this paper, first, we present main formalism and idea, which we will be present here to establish new hybrid families of special polynomials. Dattoli *et al.* provided a link between trigonometric function and Laguerre polynomials. They made strategy to introduce a new family of polynomials, which works like a bridge between Laguerre and trigonometric functions and this family of polynomials called, λ polynomials [8]. They further generalized it to associated-λ polynomials by introducing a parameter β.

The symbolic definition of associated-λ polynomials is as follows [8]:

$$\lambda_n^{(\beta)}(x, y) = \hat{C}^{\beta}(y - \hat{C}x)^n\psi_0, \tag{4}$$

where $\hat{C}$ denotes a symbolic operator given by Dattoli *et al.* [5], which operates on the vacuum function ψ_0 as:

$$\hat{C}^r\psi_0 = \frac{\Gamma(r+1)}{\Gamma(2r+1)} \quad (r \in \mathbb{R}), \tag{5}$$

such that

$$\hat{C}^n\hat{C}^m = \hat{C}^{n+m}. \tag{6}$$

The generating relation and explicit form of associated-λ polynomials are [8]

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \lambda_n^{(\beta)}(x,y) = e^{yt} \cos(\sqrt{xt};\beta), \tag{7}$$

$$\lambda_n^{(\beta)}(x,y) = n! \sum_{r=0}^{n} (-1)^r \frac{\Gamma(\beta+r+1) y^{n-r} x^r}{r!(n-r)!\Gamma(2(\beta+r)+1)}. \tag{8}$$

The associated cosine function $\cos(x;\beta)$ is symbolically defined as [8]:

$$\cos(x;\beta) = \hat{C}^{\beta}\, e^{-\hat{C}x^2} \psi_0 = \sum_{r=0}^{\infty} \frac{(-1)^r (\beta+r)! x^{2r}}{r![2(\beta+r)]!} \qquad (\beta \in \mathbb{N}). \tag{9}$$

For $\beta = 0$, the associated cosine function reduces to ordinary cosine function and consequently, the associated-λ polynomials reduce to λ polynomials.

The ordinary generating relation for λ polynomials is given by[8]

$$\sum_{n=0}^{\infty} t^n \lambda_n(x,y) = \frac{1}{1-yt}\, e_0\left(\frac{xt}{1-yt}\right), \tag{10}$$

where $e_0(x)$ is the function defined as [8]:

$$e_0(x) = \sum_{r=0}^{\infty} (-1)^r \frac{r!}{(2r)!} x^r. \tag{11}$$

The symbolic definition for $e_0(x)$ is given by

$$e_0(x) = \frac{1}{1+\hat{C}x} \psi_0. \tag{12}$$

The abstraction of poweroid, given by Steffensen [26], gives the idea of the monomiality. This concept was reformulated and established by Dattoli [6]. Further, he has explored monomiality principles for classical special polynomials with other researchers [9].

The monomiality principle states that:
The term "quasi-monomial" refers to the polynomial sequence $\{S_n(x)\}_{n=0}^{\infty}$, which has two operators, namely multiplicative operator $\hat{M}$ and derivative operator $\hat{P}$ satisfying the following relations [6]:

$$\hat{M}\{S_n(x)\} = S_{n+1}(x) \tag{13}$$

and

$$\hat{P}\{S_n(x)\} = nS_{n-1}(x), \tag{14}$$

respectively.

The operators $\hat{M}$ and $\hat{P}$ satisfy the following commutation relation:

$$[\hat{P}, \hat{M}] = \hat{P}\hat{M} - \hat{M}\hat{P} = \hat{1}. \tag{15}$$

Thus, the operators $\hat{M}$ and $\hat{P}$ show a weyl group structure [6]. Using the operators $\hat{M}$ and $\hat{P}$, several characteristics of polynomial $S_n(x)$ can be obtained. If $\hat{M}$ and $\hat{P}$ have differential realizations, then the polynomials $S_n(x)$ satisfy the differential equation

$$\hat{M}\hat{P}\{S_n(x)\} = nS_n(x). \tag{16}$$

In this research paper, we instigate the λ matrix polynomials $\lambda_n^{(M)}(x,y)$ and then we make the convolution of Appell polynomials with λ matrix polynomials to introduce the family of Appell-λ matrix polynomials ${}_A\lambda_n^{(M)}(x,y)$ by using symbolic methods. Several characteristics of this family associated with λ matrix polynomials are established and some special cases of this family are discussed. The determinantal form of the Appell-λ associated families are obtained. Bernoulli-λ and Euler-λ numbers are also introduced.

2. Appell-λ matrix polynomials

In this section, first we define λ matrix polynomials then we introduce Appell-λ matrix polynomials and study their properties. We recall that the matrix cosine functions are defined, for all $X \in \mathbb{C}^{m\times m}$ as [1]:

$$\cos X = \sum_{r=0}^{\infty} \frac{(-1)^r X^{2r}}{(2r)!}. \tag{17}$$

In view of equations (9) and (17), the symbolic definition for cosine-matrix function is given by

$$\cos X = e^{-\hat{C}X^2}\psi_0, \tag{18}$$

where X is a positive stable matrix in $\mathbb{C}^{m\times m}$.

For introducing λ matrix polynomials, we define the matrix exponent of symbolic operator $\hat{C}$ as:

$$\hat{C}^M\psi_0 = \Gamma(M+I)(\Gamma(2M+I))^{-1}, \tag{19}$$

such that

$$\hat{C}^M\hat{C}^D = \hat{C}^{M+D}, \tag{20}$$

where M and D are positive stable matrices in $\mathbb{C}^{m\times m}$ and $I \in \mathbb{C}^{m\times m}$.

Now, in view of equations (12) and (9), we define associated matrix e_0 functions $e_0(x;M)$ and associated cosine matrix function by means of the following symbolic definitions:

$$e_0(x;M) = \hat{C}^M \frac{1}{1+\hat{C}x}\psi_0 \tag{21}$$

and

$$\cos(x;M) = \hat{C}^M\, e^{-\hat{C}x^2}\psi_0, \tag{22}$$

respectively.

Also, expanding the exponential of equation (22) and then using equations (19) and (20), the series expansion for the associated cosine matrix function can be obtained.

Differentiating equation (22) partially with respect to x, we find

$$\frac{d}{dx}\cos(x;M) = -2x\hat{C}^{M+I}\, e^{-\hat{C}x^2}\psi_0 = -\sin(x;M), \tag{23}$$

which for $M = 0 \in \mathbb{C}^{1\times 1}$, reduces to the sine function $\sin x$.

Now, we are able to introduce the symbolic definition for λ matrix polynomials in the form

$$\lambda_n^{(M)}(x,y) = \hat{C}^M(y-\hat{C}x)^n\psi_0. \tag{24}$$

We have the following theorem for generating functions and series definition for λ matrix polynomials:

Theorem 2.1. *The following ordinary and exponential generating functions and the series expansion hold true for the λ matrix polynomials*

$$\sum_{n=0}^{\infty} t^n \lambda_n^{(M)}(x,y) = \frac{1}{1-yt}\, e_0\Big(\frac{xt}{(1-yt)}; M\Big), \tag{25}$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \lambda_n^{(M)}(x,y) = e^{yt} \cos(\sqrt{xt}; M) \tag{26}$$

and

$$\lambda_n^{(M)}(x,y) = n! \sum_{r=0}^{n} \frac{(-1)^r \Gamma(M+(r+1)I)(\Gamma(2M+(2r+1)I))^{-1} x^r y^{n-r}}{r!(n-r)!}, \qquad n \geq 0, \tag{27}$$

respectively, where $M \in \mathbb{C}^{m\times m}$, $M + \left(n+\frac{1}{2}\right)I$ is invertible for every integer $n \geq 0$ and the generating functions (25), (26) *are defined for complex values of x, y and t with $|yt| < 1$.*

Proof. *From equation* (24)*, we have*

$$\sum_{n=0}^{\infty} t^n \lambda_n^{(M)}(x,y) = \hat{C}^M \frac{1}{(1-yt)\Big[1+\frac{\hat{C}xt}{(1-yt)}\Big]} \psi_0 \tag{28}$$

and

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \lambda_n^{(M)}(x,y) = \hat{C}^M e^{(y-\hat{C}x)t} \psi_0. \tag{29}$$

Using equation (21) *and* (22) *in the right hand sides of equations* (28) *and* (29) *yields assertions* (25) *and* (26), *respectively.*

Expanding the right hand side of any of equations (28) *or* (29) *and then comparing the equal powers of t, we get assertion* (27). □

Now, we introduce the symbolic definitions for Appell and Appell-λ matrix polynomials. We define a symbolic operator $\hat{a}$ which acts on vacuum ϕ_0 such that $A_n = \hat{a}^n \phi_0$. Thus from equation (3), we have

$$A(t) = e^{\hat{a}t} \phi_0. \tag{30}$$

Using above equation in equation (2) and then simplifying, we get the following symbolic definition for the Appell polynomials as:

$$A_n(x) = (x+\hat{a})^n \phi_0. \tag{31}$$

Now, we introduce Appell-λ matrix polynomials ${}_A\lambda_n^{(M)}(x,y)$ as:

$${}_A\lambda_n^{(M)}(x,y) := \lambda_n^{(M)}(x, y+\hat{a}) \phi_0. \tag{32}$$

Using equation (24) in equation (32), we get symbolic definition of Appell-λ matrix polynomials ${}_A\lambda_n^{(M)}(x,y)$ as:

$${}_A\lambda_n^{(M)}(x,y) = \hat{C}^M (y+\hat{a}-\hat{C}x)^n \phi_0 \psi_0, \tag{33}$$

where $\hat{C}$ operates on ψ_0 and $\hat{a}$ operates on ϕ_0.

We establish the following theorem for the generating function and series definition for Appell-λ matrix polynomials.

Theorem 2.2. *The following generating function and series definition hold true for the Appell-λ matrix polynomials*

$$\sum_{n=0}^{\infty} {}_A\lambda_n^{(M)}(x,y)\frac{t^n}{n!} = A(t)\, e^{yt}\cos(\sqrt{x}t; M) \tag{34}$$

and

$${}_A\lambda_n^{(M)}(x,y) = \sum_{r=0}^{n}\binom{n}{r}(-1)^r A_{n-r}(y)\Gamma(M+(r+1)I)(\Gamma(2M+(2r+1)I))^{-1}x^r, \tag{35}$$

respectively.

Proof. *From equation* (33), *we have*

$$\sum_{n=0}^{\infty} {}_A\lambda_n^{(M)}(x,y)\frac{t^n}{n!} = \hat{C}^M e^{(y+\hat{a}-\hat{C}x)t}\phi_0\psi_0, \tag{36}$$

which on using equations (22) *and* (30) *in the right hand side of above equation, gives assertion* (34).

Expanding the right hand side of equation (33) *and then using equations* (19), (20) *and* (31), *we get assertion* (35). □

Using equation (33), we can directly obtain the following theorem for the Appell-λ matrix polynomials:

Theorem 2.3. *The differential recurrence relations satisfied by the Appell-λ matrix polynomials are given by*

$$\frac{\partial}{\partial x}{}_A\lambda_n^{(M)}(x,y) = -\hat{C}\, n\, {}_A\lambda_{n-1}^{(M)}(x,y), \tag{37}$$

$$\frac{\partial}{\partial y}{}_A\lambda_n^{(M)}(x,y) = n\, {}_A\lambda_{n-1}^{(M)}(x,y) \tag{38}$$

and

$$\hat{\Delta}\,{}_A\lambda_n^{(M)}(x,y) = n\, {}_A\lambda_{n-1}^{(M)}(x,y), \tag{39}$$

where,

$$\hat{\Delta} := -4\sqrt{x}\frac{\partial}{\partial x}\sqrt{x}\frac{\partial}{\partial x}. \tag{40}$$

Using above theorem, we obtain the following theorem for symbolic differential equation of the Appell-λ matrix polynomials:

Theorem 2.4. *The Appell-λ matrix polynomials satisfy the following second order symbolic differential equation:*

$$\left(4x(y+\hat{a})\frac{\partial^2}{\partial x^2} + (2(y+\hat{a})-x)\frac{\partial}{\partial x} + (y+\hat{a}-\hat{C}x)\frac{\partial}{\partial y}\right){}_A\lambda_n^{(M)}(x,y) = 0. \tag{41}$$

Next, we obtain the following theorem for operational representation of Appell-λ matrix polynomials:

Theorem 2.5. *The operational representation satisfied by the Appell-λ matrix polynomials is given by*

$$ {}_A\lambda_n^{(M)}(x,y) = \hat{C}^M e^{y\hat{\triangle}}(\hat{a}-\hat{C}x)^n \phi_0\psi_0. \tag{42}$$

Proof. *From equations* (38) *and* (39), *we obtain the partial differential equation satisfied by the Appell-λ matrix polynomials, given by*

$$\frac{\partial}{\partial y} {}_A\lambda_n^{(M)}(x,y) = \hat{\triangle}_A\lambda_n^{(M)}(x,y) \tag{43}$$

and from equation (33), *we get the initial condition*

$$ {}_A\lambda_n^{(M)}(x,0) = \hat{C}^M(\hat{a}-\hat{C}x)^n \phi_0\psi_0. \tag{44}$$

Solving equation (43) *with the initial condition* (44), *we get the assertion* (42). □

The Appell-λ matrix polynomials are frame within the context of monomiality principle formalism in the next section. Also, we discuss some special cases of the Appell-λ matrix polynomials.

3. Monomiality property and Determinant form

This section deals with the quasi-monomiality property of the Appell-λ matrix polynomials and determinantal representation for the same family of polynomials. Also, we discuss few special cases of the established results obtained in Section 2.

We establish the following result to frame the Appell-λ matrix polynomials with respect to monomiality principle formalism:

Theorem 3.1. *The Appell-λ matrix polynomials are quasi-monomial with respect to the following multiplicative and derivative operators:*

$$M_{A\lambda} = (y + \hat{a} - \hat{C}x) \tag{45}$$

and

$$P_{A\lambda} = D_y, \tag{46}$$

or equivalently,

$$P_{A\lambda} = \hat{\triangle}, \tag{47}$$

respectively.

Proof. *Operating* $(y+\hat{a}-\hat{C}x)$ *on the both sides of equation* (33) *and then using equation* (33) *in right hand side of the resultant equation, we get*

$$(y+\hat{a}-\hat{C}x)_A\lambda_n^{(M)}(x,y) = {}_A\lambda_{n+1}^{(M)}(x,y), \tag{48}$$

which in view of equations (13) *and* (48) *gives assertion* (45).

Again, in view of equations (14), (38) *and* (39), *we get assertions* (46) *and* (47). □

Using equations (16), (45) and (46), we prove the following theorem for differential equation of Appell-λ matrix polynomials:

Theorem 3.2. *The differential equation satisfied by the Appell-λ matrix polynomials is given by*

$$\left(\left(y+\hat{a}-\hat{C}x\right)\frac{\partial}{\partial y}-n\right){}_A\lambda_n^{(M)}(x,y)=0. \tag{49}$$

Further, we discuss the determinant form of Appell-λ matrix polynomials.

Theorem 3.3. *The Appell-λ matrix polynomial of degree n is defined by*

$${}_A\lambda_0^{(M)}(x,y)=\frac{1}{\gamma_0}\Gamma(M+I)\Gamma(2M+I)^{-1}, \tag{50}$$

$${}_A\lambda_n^{(M)}(x,y)=\frac{(-1)^n}{(\gamma_0)^{n+1}}\begin{vmatrix}\Gamma(M+I)\Gamma(2M+I)^{-1} & \lambda_1^{(M)}(x,y) & \lambda_2^{(M)}(x,y) & \cdots & \lambda_{n-1}^{(M)}(x,y) & \lambda_n^{(M)}(x,y)\\ \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{n-1} & \gamma_n\\ 0 & \gamma_0 & \binom{2}{1}\gamma_1 & \cdots & \binom{n-1}{1}\gamma_{n-2} & \binom{n}{1}\gamma_{n-1}\\ 0 & 0 & \gamma_0 & \cdots & \binom{n-1}{2}\gamma_{n-3} & \binom{n}{2}\gamma_{n-2}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \gamma_0 & \binom{n}{n-1}\gamma_1\end{vmatrix}, \tag{51}$$

where

$$\gamma_0=\frac{1}{\beta_0},$$

$$\gamma_n=-\frac{1}{\beta_0}\Big(\sum_{k=1}^{n}\binom{n}{k}\beta_k\gamma_{n-k}\Big),\quad n=1,2,\ldots$$

Proof. *Let Appell-λ matrix polynomial, given by generating function* (34) *forms a sequence of polynomials and* $(\beta_n)_{n\in\mathbb{N}}$, $(\gamma_n)_{n\in\mathbb{N}}$ *be two numerical sequences given by*

$$A(t)=\beta_0+\frac{t}{1!}\beta_1+\frac{t^2}{2!}\beta_2+\ldots+\frac{t^n}{n!}\beta_n+\ldots,\quad n=0,1\ldots;\beta_0\neq 0, \tag{52}$$

$$\hat{A}(t)=\gamma_0+\frac{t}{1!}\gamma_1+\frac{t^2}{2!}\gamma_2+\ldots+\frac{t^n}{n!}\gamma_n+\ldots,\quad n=0,1\ldots;\gamma_0\neq 0, \tag{53}$$

satisfying

$$A(t)\hat{A}(t)=1. \tag{54}$$

Using the Cauchy-product rules in the above equation, we find

$$A(t)\hat{A}(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\beta_k\gamma_{n-k}\frac{t^n}{n!}, \tag{55}$$

where

$$\sum_{k=0}^{n}\binom{n}{k}\beta_k\gamma_{n-k}=\begin{cases}1 & for \quad n=0\\ 0 & for \quad n>0.\end{cases} \tag{56}$$

Hence,

$$\begin{cases} \gamma_0 = \frac{1}{\beta_0}, \\ \gamma_n = -\frac{1}{\beta_0}\left(\sum_{k=1}^{n} \binom{n}{k}\beta_k\gamma_{n-k}\right), \quad n = 1,2,\ldots \end{cases}$$

On multiplying both sides of equation (34) *by* $\hat{A}(t)$ *and then using equation* (54) *in the right hand side of resultant equation, we obtain*

$$\hat{A}(t)\sum_{n=0}^{\infty} {}_A\lambda_n^{(M)}(x,y)\frac{t^n}{n!} = e^{yt}\cos(\sqrt{xt};M). \tag{57}$$

Using equations (26), (53) *in the above equation and then comparing equal powers of t from both sides of above equation, we find the following series*

$$\sum_{k=0}^{n}\binom{n}{k}{}_A\lambda_{n-k}^{(M)}(x,y)\gamma_k = \lambda_n^{(M)}(x,y), \tag{58}$$

which on using equation (27), *gives the system of infinite equations in the unknown* ${}_A\lambda_n^{(\beta)}(x,y), \ \ n = 0,1,\ldots,$ *in the following form:*

$$\begin{cases} {}_A\lambda_0^{(M)}(x,y)\gamma_0 = \Gamma(M+I)\Gamma(2M+I)^{-1}, \\ {}_A\lambda_1^{(M)}(x,y)\gamma_0 + {}_A\lambda_0^{(M)}(x,y)\gamma_1 = \lambda_1^{(M)}(x,y), \\ {}_A\lambda_2^{(M)}(x,y)\gamma_0 + \binom{2}{1}{}_A\lambda_1^{(M)}(x,y)\gamma_1 + {}_A\lambda_0^{(M)}(x,y)\gamma_2 = \lambda_2^{(M)}(x,y), \\ \vdots \\ {}_A\lambda_n^{(M)}(x,y)\gamma_0 + \binom{n}{1}{}_A\lambda_1^{(M)}(x,y)\gamma_{n-1} + \ldots + {}_A\lambda_0^{(M)}(x,y)\gamma_n = \lambda_n^{(M)}(x,y) \\ \vdots \end{cases} \tag{59}$$

From first equation of system (59), *yields assertion* (50). *The lower triangular form of system* (59), *provides a way to find the unknown* ${}_A\lambda_n^{(M)}(x,y)$. *Simplifying the first* $n+1$ *equations by using Cramer's rule and then obtaining the determinant of the denominator and transposing the determinant in the numerator of the resultant equation, we find*

$${}_A\lambda_n^{(M)}(x,y) = \frac{1}{(\gamma_0)^{n+1}}\begin{vmatrix} \gamma_0 & \gamma_1 & \gamma_2 & \cdots & \gamma_{n-1} & \gamma_n \\ 0 & \gamma_0 & \binom{2}{1}\gamma_1 & \cdots & \binom{n-1}{1}\gamma_{n-2} & \binom{n}{1}\gamma_{n-1} \\ 0 & 0 & \gamma_0 & \cdots & \binom{n-1}{2}\gamma_3 & \binom{n}{2}\gamma_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \gamma_0 & \binom{n}{n-1}\gamma_1 \\ \Gamma(M+I)\Gamma(2M+I)^{-1} & \lambda_1^{(M)}(x,y) & \lambda_2^{(M)}(x,y) & \cdots & \lambda_{n-1}^{(M)}(x,y) & \lambda_n^{(M)}(x,y) \end{vmatrix}, \quad n = 1,2,3,\ldots, \tag{60}$$

which after circular row exchanging n times, i.e. *gives assertion* (51). □

Next, we give the following special cases of the Appell-λ matrix polynomials:

I. For the appropriate choices of function $A(t)$ the Appell-λ matrix polynomials reduce to the corresponding member of the family of Appell-λ matrix polynomials.
II. Since, in view of equations (4) and (24), for $M = \beta \in \mathbb{C}^{1\times 1}$, the λ matrix polynomials $\lambda_n^{(M)}(x, y)$ transform to the associated-λ polynomials $\lambda_n^{(\beta)}(x, y)$. Therefore, for the same choice of M, the Appell-λ matrix polynomials ${}_A\lambda_n^{(M)}(x, y)$ transform to the Appell-associated-λ polynomials ${}_A\lambda_n^{(\beta)}(x, y)$. Thus, for $M = \beta \in \mathbb{C}^{1\times 1}$, equations (33), (34), (35), (43), (41), (42), (45), (46), (49), (50) and (51) reduce to respective symbolic definition, generating function, series definition, partial differential equation, symbolic differential equation, operational representation, multiplicative operator, derivative operator, differential equation and determinantal representation for ${}_A\lambda_n^{(\beta)}(x, y)$.
III. Since, in view of equation (24), for $M = 0 \in \mathbb{C}^{1\times 1}$, the λ matrix polynomials $\lambda_n^{(M)}(x, y)$ transform to the λ polynomials $\lambda_n(x, y)$. Therefore, for the same choice of M, the Appell-λ matrix polynomials ${}_A\lambda_n^{(M)}(x, y)$ transform to the Appell-λ polynomials ${}_A\lambda_n(x, y)$. Thus, for $M = 0 \in \mathbb{C}^{1\times 1}$, equations (33), (34), (35), (43), (41), (42), (45), (46), (49), (50) and (51) reduce to the corresponding symbolic definition, generating function, series definition, partial differential equation, symbolic differential equation, operational representation, multiplicative operator, derivative operator, differential equation and determinantal representation for ${}_A\lambda_n(x, y)$.
IV. From equation (31) for $x = y$, we have ${}_A\lambda_n(0, y) = A_n(y)$ and from equation (4), we have $\lambda_n(0, y) = y^n$. Therefore, for $x = 0$, we get the determinant form for the Appell polynomials $A_n(y)$ [4, p.1533].

In continuation, the next section deals with graphical representations of certain members of the family of Appell-λ polynomials.

4. Graphical representation

First, we draw the surface plots of the Bernoulli-associated-λ polynomials ${}_B\lambda_n^{(\beta)}(x, y)$ and Bernoulli-λ polynomials ${}_B\lambda_n(x, y)$.

To draw the surface plots of the Bernoulli-associated-λ polynomials and Bernoulli-λ polynomials, we consider first six Bernoulli polynomials $B_n(x)$, given in the following Table:

Table 4.1. List of first six Bernoulli polynomials

n	0	1	2	3	4	5
$B_n(x)$.	1	$x - \frac{1}{2}$	$x^2 - x + \frac{1}{6}$	$x^3 - \frac{3}{2}x^2 + \frac{x}{2}$	$x^4 - 2x^3 + x^2 - \frac{1}{30}$	$x^5 - \frac{5}{2}x^4 + \frac{5}{3}x^3 - \frac{x}{6}$

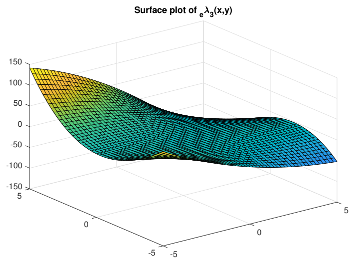
Using the appropriate expressions of $B_n(y)$ from Table 4.1, we get the following expression for the Bernoulli-associated-λ polynomials ${}_B\lambda_n^{(\beta)}(x, y)$ for $n = 3, \beta = \frac{1}{2}$:

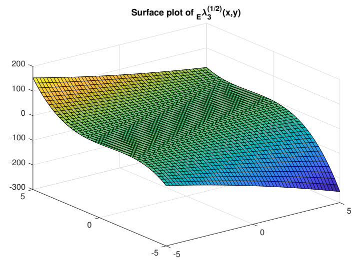
$$ {}_B\lambda_3^{(\frac{1}{2})}(x, y) = \frac{1.772}{2}y^3 - \frac{5.316}{4}y^2 + \frac{1.772}{4}y - \frac{5.316}{8}xy^2 + \frac{5.316}{8}xy - \frac{1.772}{16}x + \frac{5.316}{64}x^2y - \frac{5.316}{128}x^2 - \frac{1.772}{768}x^3 \quad (61) $$

and Bernoulli-λ polynomials ${}_B\lambda_n(x, y)$ for $n = 3$

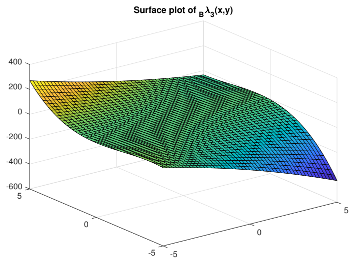
$$ {}_B\lambda_3(x, y) = y^3 - \frac{3}{2}y^2 + \frac{1}{2}y - \frac{3}{2}xy^2 + \frac{3}{2}xy - \frac{3}{12}x + \frac{1}{4}x^2y - \frac{1}{8}x^2 - \frac{1}{120}x^3, \quad (62) $$

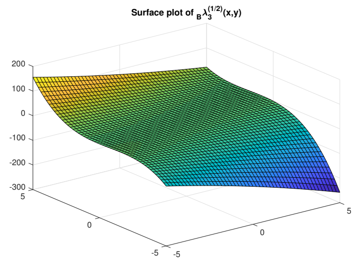
which are used to obtain the surface plots of Bernoulli-associated-λ polynomials ${}_B\lambda_n^{(\beta)}(x, y)$ and Bernoulli-λ polynomials ${}_B\lambda_n(x, y)$, respectively.



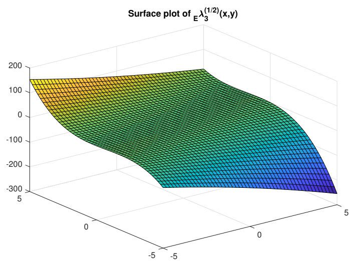


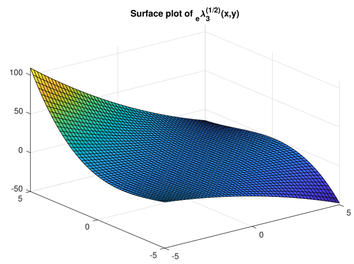
Similarly, we obtain the following surface plots of the Euler-associated-λ polynomials $_E\lambda_n^{(\beta)}(x,y)$ and Euler-λ polynomials $_E\lambda_n(x,y)$ by using the appropriate expressions of Euler polynomials $E_n(x)$.





In the similar way, we get the following surface plots of the truncated exponential-associated-λ polynomials $_e\lambda_n^{(\beta)}(x,y)$ and truncated exponential-λ polynomials $_e\lambda_n(x,y)$.





5. Concluding remarks

To conclude this paper, we introduce the 1-variable λ matrix polynomials $\lambda_n^{(M)}(x)$. To obtain the symbolic image of 1-variable λ matrix polynomials $\lambda_n^{(M)}(x)$, we interchange x and y in equation (24) and then substitute $y = 1$ in the resultant equation to get

$$\lambda_n^{(M)}(x) = \hat{C}^M (x - \hat{C})^n \psi_0. \tag{63}$$

Also, the generating function and series definition for 1-variable λ matrix polynomials can be obtained by using equation (63) as:

$$\sum_{n=0}^{\infty} \lambda_n^{(M)}(x) \frac{t^n}{n!} = e^{xt} \cos(\sqrt{t}; M) \tag{64}$$

and

$$\lambda_n^{(M)}(x) = n! \sum_{r=0}^{n} \frac{(-1)^r \Gamma(M + (r+1)I)(\Gamma(2M + (2r+1)I))^{-1} x^{n-r}}{r!(n-r)!}. \tag{65}$$

Now, we are at a position to introduce the 1-variable Appell-λ matrix polynomials ${}_A\lambda_n^{(M)}(x)$ by replacing x with the symbolic operator of Appell polynomials $A_n(x)$, given by equation (31) in equation (63) as:

$$ {}_A\lambda_n^{(M)}(x) = \hat{C}^M(x + \hat{a} - \hat{C})^n \phi_0\psi_0. \tag{66}$$

The generating function and series definition for ${}_A\lambda_n^{(M)}(x)$ can be obtained from equation (66) as:

$$\sum_{n=0}^{\infty} {}_A\lambda_n^{(M)}(x)\frac{t^n}{n!} = A(t)\, e^{xt} \cos(\sqrt{t}; M) \tag{67}$$

and

$$ {}_A\lambda_n^{(M)}(x) = n! \sum_{r=0}^{n} \frac{(-1)^r \Gamma(M + (r+1)I)(\Gamma(2M + (2r+1)I))^{-1} A_{n-r}(x)}{r!(n-r)!}. \tag{68}$$

For $M = 0$ equations (64) and (65), give the generating function and series definition of 1-variable λ polynomials $\lambda_n(x)$. Consequently, for the same choice of M equations (66), (67) and (68), give the symbolic image, generating function and series definition for 1-variable Appell-λ polynomials ${}_A\lambda_n(x)$.
We know that Taylor series expansions of the tangent and hyperbolic tangent functions give birth to the Bernoulli numbers B_n [14] whereas Taylor series expansions of the secant and hyperbolic secant functions is the origin of the Euler numbers E_n [14]. The Bernoulli and Euler numbers have occurrence in the area of combinatorics and have relations with number theory. These numbers can be viewed as special values of the Bernoulli and Euler polynomials. We have following relations for the Bernoulli numbers B_n and the Euler numbers E_n of order n:

$$B_n := B_n(0), \quad E_n := 2^n E_n\left(\frac{1}{2}\right), \quad (n \in \mathbb{N}_0). \tag{69}$$

Here, we introduce the numbers related to Bernoulli-λ and Euler-λ polynomials by using the Bernoulli and Euler numbers. We define the Bernoulli-λ number ${}_B\lambda_n$ and Euler-λ number ${}_E\lambda_n$ as:

$$ {}_B\lambda_n = {}_B\lambda_n(0) \tag{70}$$

and

$$ {}_E\lambda_n := {}_E\lambda_n\left(\frac{1}{2}\right). \tag{71}$$

Since, for $A(t) = \frac{t}{e^t-1}$, $A_n(x) := B_n(x)$ and for $A(t) = \frac{2}{e^t+1}$, $A_n(x) := E_n(x)$, therefore in view of equations (70) and (71), for $M = 0$ appropriate choices of $A(t)$ and x equation (67) gives the respective generating functions of Bernoulli-λ numbers and Euler-λ numbers.

$$\sum_{n=0}^{\infty} {}_B\lambda_n \frac{t^n}{n!} = \frac{t}{e^t - 1} \cos(\sqrt{t}) \tag{72}$$

and

$$\sum_{n=0}^{\infty} {}_E\lambda_n \frac{t^n}{n!} = \frac{2}{e^t + 1} e^{\frac{t}{2}} \cos(\sqrt{t}). \tag{73}$$

Here, we list first six members of Bernoulli-λ numbers and Euler-λ numbers.

Table 5.1. First six numbers ${}_B\lambda_n$ and ${}_E\lambda_n$

n	0	1	2	3	4	5
$B^{\lambda}n$	1	-1	$\frac{3}{4}$	$-\frac{23}{60}$	$\frac{113}{1680}$	$\frac{1027}{15120}$
$E^{\lambda}n$	1	$-\frac{1}{2}$	$-\frac{1}{6}$	$\frac{11}{30}$	$\frac{79}{420}$	$-\frac{5749}{7560}$

During the last two three decades, much research work has been done for orthogonal polynomials. By orthogonality property, we can explore other characteristics of these polynomials. The theory of orthogonal polynomials is wide and certainly provides an inexhaustible field of research. A large number of polynomials are recognized as belonging to orthogonal polynomials family (see [16, 21, 23]) may be worthy of consideration by the targeted readers.

References

[1] Alonso, P.; Peinado, J.; Ibáñez, J. J.; Martínez, J. S. A fast implementation of matrix trigonometric functions sine and cosine. *Proceeding of the 17th International Conference on Computational and Mathematical Methods in Science and Engineering, Costa Ballena Rota, Cádiz (Spain)*, 2017.

[2] Appell, P. Sur une classe de polynômes. (French) *Ann. Sci. École Norm. Sup.* **9** (1880), (2), 119-144.

[3] Babusci, D.; Dattoli, G.; Górska, K.; Penson, K. A. Lacunary generating functions for the Laguerre polynomials. *Sém. Lothar. Combin.* **76** (2017), Art. B76b, 19 pp.

[4] Costabile, F. A.; Longo, E. A determinantal approach to Appell polynomials. *J. Comput. Appl. Math.* **234** (2010), (5), 1528-1542.

[5] Dattoli, G.; Górska, K.; Horzela A.; Licciardi S.; Pidatella R. M. Comments on the properties of Mittag-Leffler function. *The European Physical Journal Special Topics*, **226** (2017), pp 3427-3443.

[6] Dattoli, G. Hermite-Bessel and Laguerre-Bessel functions: A by-product of the monomiality principle. *Advanced special functions and applications (Melfi, 1999)*, 147-164, *Proc. Melfi Sch. Adv. Top. Math. Phys.*, **1**, *Aracne, Rome*, 2000.

[7] Dattoli, G.; Licciardi, S. Operational, umbral methods, Borel transform and negative derivative operator techniques. *Integral Transforms Spec. Funct.* **31** (2020), no. 3, 192-220.

[8] Dattoli, G.; Licciardi, S.; Palma, E. D.; Sabia, E. From circular to Bessel functions: A transition through the umbral method. *Fractal Fract*, **9** (2017), **1** (1), 1-11.

[9] Dattoli, G.; Migliorati, M.; Srivastava, H. M. Sheffer polynomials, monomiality principle, algebraic methods and the theory of classical polynomials. *Math. Comput. Modelling.* **45** (2007), no. 9-10, 1033-1041.

[10] Dunford, N., Schwartz, J. *Linear Operators. Part I*, Interscience, New York, 1957.

[11] Jódar, L.; Company, R., Navarro, E. Laguerre matrix polynomials and systems of second order differential equations. *Appl. Numer. Math.* **15** (1994), 53-63.

[12] Jódar, L., Company, R., Ponsoda, E. Orthogonal matrix polynomials and systems of second order differential equations. *Differential Equations Dynam. Systems*. **3** (1995), 269-288 .

[13] Metwally, M. S.; Mohamed, M. T.; Shehata, A. Generalizations of two-index two-variable Hermite matrix polynomials. *Demonstratio Math.* **42** (2009), no. 4, 687-701.

[14] Rainville, E. D. *Special Functions*, Macmillan, New York, 1960, reprinted by Chelsea Publ. Co., Bronx, New York, 1971.

[15] Srivastava, H. M. An introductory overview of Bessel polynomials, the generalized Bessel polynomials and the q-Bessel polynomials. *Symmetry*. **15** (2023), 822, 1-28.

[16] Srivastava, H. M.; Ben Cheikh, Y. Orthogonality of some polynomial sets via quasi-monomiality. *Appl. Math. Comput.* **141** (2003), 415-425.

[17] Srivastava, H. M.; Manocha, H. L. *A Treatise on Generating Functions*. Ellis Horwood Limited, New York, 1984.

[18] Srivastava, H. M.; Masjed-Jamei, M.; Beyki, M. R. Some new generalizations and applications of the Apostol-Bernoulli, Apostol-Euler and Apostol-Genocchi polynomials. *Rocky Mountain J. Math.* **49** (2019), no. 2, 681-697.

[19] Srivastava, H. M.; Ricci, P. E.; Natalini, P. A family of complex Appell polynomial sets. *Rev. Real Acad. Cienc. Exactas, Fís. Natur. Ser. A Mat.* **113** (2019), 2359-2371.

[20] Srivastava, H. M.; Riyasat M. q-Difference equations for the 2-iterated q-Appell and mixed type q-Appell polynomials *Arabian J. Math.* **8** (2019), 63-77.

[21] Srivastava, H. M. Some applications of the Lagrange expansion theorem associated with general polynomial systems. *J. Nonlinear Convex Anal.* **24** (2023), no. 5, 1113-1127.

[22] Srivastava, H. M. Some characterizations of Appell and q-Appell polynomials. *Annali di Matematica Pura ed Applicata* **130** (1982), 321-329.

[23] Srivastava, H. M. Some families of generating functions associated with orthogonal polynomials and other higher transcendental functions. *Mathematics*. **10** (2022), 3730, 1-28.

[24] Srivastava, H. M.; Khan, W. A.; Haroon, H. Some expansions for a class of generalized Humbert matrix polynomials. *Rev. Real Acad. Cienc. Exactas, Fís. Natur. Ser. A Mat.* **11** (2019), 1307, 1-23.

[25] Srivastava, H. M.; Yasmin, G.; Muhyi A.; Araci, S. Certain results for the twice-iterated 2D q-Appell polynomials *Symmetry*. **11** (2019), no. 10, 1307.

[26] Steffensen, J. F. The Poweroid an extension of the mathematical notion of power. *Acta Math.* **73** (1941), 333-366.

[27] Raza, N.; Zainab, U.; Araci, S. Lacunary generating functions of hybrid type polynomials in view point of symbolic approach. *Comput. Model. Eng. and Sci.* **129**, (2021), 903-921.

[28] Raza, N.; Zainab, U. The Mittag-Leffler-Legendre polynomials and their Lie-algebraic relations. *Rep. Math. Phys.* **89** (2022), no. 1, 97-129.