Filomat 38:4 (2024), 1305–1315
https://doi.org/10.2298/FIL2404305S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

New criteria for blow up of fractional differential equations

Yeqing Shan[a], Guangying Lv[a,*]

[a] *School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China*

Abstract. In this paper, some new blow-up criteria are given for a single equation, and the blow-up problem of the solution of a single nonlocal equation is solved by changing the equation into a system of equations by introducing an auxiliary function. In addition, the theory of ordinary differential equation is extended to partial differential equation by using the first eigenvalue theory. The results show that the blow-up criteria of the Liouville-Caputo and the Caputo-Hadamard fractional differential equations are different.

1. Introduction

Increasingly, it has been realized that properties of many phenomena occurring in real life problems are not adequately described by evolution equations of integer order in time, such as memory effects, material science, anomalous diffusion, rheology, fractals and control theory. The references [9, 12, 16] covered several of these phenomena and demonstrated the importance of the fractional modeling. A large number of investigations of basic calculus have been reported by recent survey-cum-expository review papers [29–31]. Additionally, [29, 30] also provided meaningful fractional calculus operators related to higher transcendental functions and their applications. Furthermore, the well-posedness of fractional differential equations (FDEs) has been well studied by many research groups [7, 16, 20, 24]. It is generally accepted that the blow-up problem is an important issue in the well-posedness, which is a common phenomenon of the FDEs [1]. Therefore, we intend in this work to focus on the finite time blow-up solutions. In this paper, we are concerned with the blow-up phenomenon of FDEs. For FDEs, there are different definitions about the fractional derivative. We only focus on two types: the Liouville-Caputo and the Caputo-Hadamard fractional differential equations. We will present different blow-up criteria to the following Cauchy problems for the time fractional diffusion equation

$$\begin{cases} {}_0^C\mathbb{D}_t^\alpha f(t) = f^p(t) \quad \textit{and} \quad {}_0^H\mathbb{D}_t^\alpha f(t) = f^p(t), \quad t > 0, \\ f(0) = f_0, \end{cases} \tag{1}$$

2020 *Mathematics Subject Classification*. Primary 34A08, 35B44; Secondary 35B51.
Keywords. Time-fractional; Caputo-Hadamard; Comparison principle; Blow-up.
Received: 04 April 2023; Accepted: 16 August 2023
Communicated by Hari M. Srivastava
Research supported by NSF of China grants 11771123, 12171247, Jiangsu Provincial Double-Innovation Doctor Program JSSCBS20210466 and the Qing Lan Project of Jiangsu Province and Postgraduate Research and Practice Innovation Program of Jiangsu Province (No. KYCX21_0932).
* Corresponding author: Guangying Lv
Email addresses: shan_yeqing@163.com (Yeqing Shan), gylvmaths@nuist.edu.cn (Guangying Lv)

where $p > 1$, $f \in C([0,T], \mathbb{R})$ and ${}_0^c\mathbb{D}_t^\alpha$ is the Liouville-Caputo fractional derivative of order $\alpha \in (0,1)$,

$$ {}_0^c\mathbb{D}_t^\alpha u = \frac{\partial}{\partial t} {}_0\mathbb{I}_t^{1-\alpha}(u(t,x) - u_0(x)), $$

and referred to [29, 30], ${}_0\mathbb{I}_t^{1-\alpha}$ denotes the left Riemann-Liouville fractional integrals of order $1-\alpha$, which is defined by

$$ {}_0\mathbb{I}_t^{1-\alpha} u = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u(s) ds. $$

The Hadamard fractional derivative (Hadamard 1892) of a given function $f(t)$ with order $\alpha > 0$ is defined by

$$ {}_0^H\mathbb{D}_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} f(\tau) \frac{d\tau}{\tau}, \quad t > a > 0, $$

where $\Gamma(\cdot)$ is the Gamma function. The existence and uniqueness of fractional differential equations have been studied by many researchers. Lakshmikantham & Vatsala [20] studied the basic properties of fractional differential equations, also see [19, 22]. Bai et al. [2] used the monotone iterative method to studied the fractional differential equations. Boulfoul et al. [5] obtained the existence and uniqueness of nonlinear fractional integro-differential equation on an unbounded domain in a weighted Banach space. Srivastava et al. [32] verified the existence of nondecreasing integrable solution of a nonlinear hybrid implicit functional differential inclusion of arbitrary fractional order with complex initial condition, and then provided some examples to prove the results. Li & Liu [26] studied a generalized definition of Caputo derivatives and its application to fractional ODEs. Izadi and Srivastava [14] investigated a novel set of basis functions to treat a class of multi-order fractional pantograph differential equations computationally, whose feasibility was further proved via different evidences. Li and Sarwar [23] considered the Caputo type fractional differential equations and obtained the existence and continuation of solutions of FDEs. Recently, Li & Li [25] obtained the stability and logarithmic decay of the solution to Hadamard-type fractional differential equation. Develi & Duman [8] obtained the existence and stability analysis of solution for fractional delay differential equations. Srivastava et al. [34] systematically investigated a certain UFLCDE, and found its fractional sum equation. Baleanu et al. [3] solved a system of fractional differential equations and obtained the necessary operational matrix of fractional integral with the help of the Clenshaw-Curtis formula. Yang and Srivastava [36] first proposed a non-differentiable asymptotic approximation theory along with revealing the fractal behavior of a linear oscillator in fractal medium, and then proved the effectiveness through graphical results. Srivastava et al. [33] revealed the existence of a unique solution for a coupled system of higher-order fractional differential equations with multi-point boundary conditions. Herzallah and Radwan [13] established the existence and uniqueness of solutions to some classes of nonlocal semilinear conformable fractional differential. There are a lot of important results about the well-posedness of FDEs. The above-mentioned literature demonstrated important results for the well-posedness of the FDEs, all of which provide meaningful ideas for addressing the finite time blow-up problem in this paper.

There are a lot of results on solutions of fractional differential equations. In this paper, we will give some new blow-up criteria for (1). Moreover, we find the blow-up criteria for the Liouville-Caputo and the Caputo-Hadamard fractional differential equations are different, see Section 2. Based on the book [17], *we first give some new blow-up criteria for the Liouville-Caputo derivative and the Caputo-Hadamard derivative. Then we generalize the blow-up results to time-fractional partial differential equations.* Laskri and Tatar [21] considered the following FDEs

$$ \begin{cases} \mathbb{D}_+^\alpha u(t) \geq t^\beta |u|^m, & t > 0, m > 1, 0 < \alpha < 1, \\ \mathbb{D}_+^{\alpha-1} u(0^+) = b, \end{cases} \tag{2} $$

where $\mathbb{D}_+^\alpha$ is the left-sided Riemann-Liouville fractional derivative. Under the assumptions that $\beta > -\alpha$ and $1 < m \leq \frac{1+\beta}{1-\alpha}$, problem (2) does not admit global nontrivial solutions when $b \geq 0$. Obviously, equations

(1) and (2) are different. Furthermore, our method is very easy to understand. We remark that the Volterra equations and FDEs have some common properties. Mydlarczyk et al. [28] obatined the blow-up solutions to a system of nonlinear Volterra equations, also see [18]. Kassim et al. [15] obtained the non-existence for fractionally damped fractional differential problems and our blow-up criteria are different from theirs. Notably, [15, 18, 28] contains blow-up results for FDEs, but there is no result to describe the relation between p and α.

For the finite time blow-up solutions of fractional partial differential equations (FPDEs), there are abundant results. Firstly, there are many results about the Cauchy problem of FPDEs. Zhang and Li [38, 39] studied the time-fractional superdiffusion equation with or without nonlinear memory, also see [6]. Because we only consider the FPDEs on bounded domain, we will not recall other results of FPDEs in the whole space. Alsaedi et al. [1] studied the following problem

$$\begin{cases} \mathbb{D}_{+}^{\alpha} u(t) + (-\Delta)_{\Omega}^{s} u = -u(1-u), & x \in \Omega, t > 0, 0 < \alpha < 1, \\ u = 0, & x \in \mathbb{R}^n \setminus \Omega, t > 0, \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{3}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $(-\Delta)_{\Omega}^{s}$ is the regional fractional Laplacian with $s \in (0,1)$. They obtained the global solution of (3) under the assumption that $0 < u_0 < 1$ and finite time blow-up solution of (3) under the assumptions that the initial data is suitable large. There are rich results about the blow-up solution of SPDEs in the whole space. In [24], Li and Li considered the Caputo-Hadamard fractional partial differential equations and obtained the existence of finite time blow-up solutions. In [27], Manimaran and Shangerganesh considered the blow-up phenomenon of FPDEs with variable exponents on bounded domain. Until now there are few results about Capito-Hadamard fractional partial differential equations on bounded domain. In this paper, we give some positive answers.

The novelties of this paper are as follows: (1) some new blow-up criteria for the Liouville-Caputo derivative and the Caputo-Hadamard derivative are given; (2) the difference between the Liouville-Caputo derivative and the Caputo-Hadamard derivative is considered; (3) a new blow-up criterion of nonlocal FDEs is given by transforming the single equation into a system; (4) some new blow-up criteria for partial differential equations with the Liouville-Caputo derivative and the Caputo-Hadamard derivative are given.

This paper is arranged as follows. In Section 2, we study the blow-up phenomenon of FDEs, and Section 3 is devoted to FPDEs on a bounded domain.

2. Blow-up results of FDEs

Consider the initial value problem (IVP) for fractional differential equations

$$\begin{cases} {}_0^C\mathbb{D}_t^{\alpha} f(t) = f^p(t), & t > 0, \\ f(0) = f_0, \end{cases} \tag{4}$$

where $0 < \alpha < 1$ and $f \in C([0,T],\mathbb{R})$. Since f is assumed continuous, it is easy to see that the solution of initial value problem (4) is equivalent to the solution of the following integral equation

$$f(t) = f(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f^p(s) ds, \quad 0 \le t \le T. \tag{5}$$

Here Γ denotes the Gamma function.

And we quote the following fundamental result relative to the fractional integral.

Lemma 2.1. *[20] (Comparison principle) Let $v, w \in C([0,T],\mathbb{R})$, $f \in C([0,T] \times \mathbb{R}, \mathbb{R})$ and*

(1) $v(t) \le v(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f^p(s, v(s)) ds$,

(2) $w(t) \ge w(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f^p(s, w(s)) ds, 0 \le t \le T$.

One of the foregoing inequalities is strict. Furthermore, $f^p(t,x)$ is nondecreasing in x for each t and $v(0) < w(0)$, then we have $v(t) < w(t)$.

2.1.1 ***The Liouville-Caputo fractional differential equation***

In this subsection, we will give our blow-up criteria for the Liouville-Caputo fractional differential equation

Theorem 2.2. *Assume that f_0 is positive, let $0 < \alpha < 1$ and $p > 1$, then for IVP (4), we have the following results.*

(1) If $0 < p(1-\alpha) \leq 1$, then the solution of problem (4) with $f_0 \geq 0$ blows up in a finite time;

(2) If $p(1-\alpha) > 1$ and $f_0 > (\frac{p-1}{\Gamma(\alpha)[p(1-\alpha)-1]})^{p-1}$, then there exists a constant T, such that the solution of the problem (4) blows up in a finite time.

Proof Note that $0 < t-s < t < t+1$, we have

$$\begin{aligned} f(t) &= f(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f^p(s)ds \\ &\geq f(0) + \frac{1}{\Gamma(\alpha)(t+1)^{1-\alpha}}\int_0^t f^p(s)ds, \end{aligned}$$

By the Lemma 2.1, using comparison principle, we know that $f(s)$ is an upper solution of

$$\begin{cases} u'(t) = \frac{u^p(t)}{\Gamma(\alpha)(t+1)^{p(1-\alpha)}}, \\ u(0) = f(0), \end{cases}$$

where $u(t) = (1+t)^{1-\alpha} f(t)$. Set $u(t) = (1+t)^{1-\alpha} f(t)$, then we get

$$\begin{aligned} u(t) &\geq (1+t)^{1-\alpha} f(0) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{u^p(s)}{(1+s)^{p(1-\alpha)}}ds \\ &\geq f(0) + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{u^p(s)}{(1+s)^{p(1-\alpha)}}ds. \end{aligned}$$

Next, we consider

$$\begin{cases} u'(t) = \frac{u^p(t)}{\Gamma(\alpha)(t+1)^{p(1-\alpha)}}, \\ u(0) = f(0). \end{cases} \tag{6}$$

It is easy to show that

$$u^{1-p}(t) = f^{1-p}(0) + \frac{p-1}{\Gamma(\alpha)(1-p(1-\alpha))} - \frac{(p-1)(1+t)^{1-p(1-\alpha)}}{\Gamma(\alpha)(1-p(1-\alpha))}. \tag{7}$$

Proof of (1): Note that $0 < p(1-\alpha) < 1$, we have the solution $u(t)$ will blow-up at time T, where

$$T = \Big(1 + \frac{f^{1-p}(0)\Gamma(\alpha)(1-p(1-\alpha))}{p-1}\Big)^{\frac{1}{1-p(1-\alpha)}} - 1.$$

The function $f(t)$ is an upper solution of (4) by comparison principle.

If $p(1-\alpha) = 1$, (6) is transformed as follows

$$u'(t) = \frac{u^p(t)}{\Gamma(\alpha)(t+1)}, \quad u(0) = f(0).$$

Solving this, we get

$$u^{1-p}(t) = u^{1-p}(0) - \frac{(p-1)\ln(1+t)}{\Gamma(\alpha)}, \quad u(0) = f(0).$$

Hence, there exists a constant T^*, we know that the solution $u(t)$ blows up at time $t > T^*$ for any nonnegative initial value.

Proof of (2): If $p(1-\alpha) > 1$, then we have the following results. We first transform equation (7) into

$$u(t) = \left(\frac{1}{f^{1-p}(0) + \frac{(p-1)(1+t)^{1-p(1-\alpha)}}{\Gamma(\alpha)[p(1-\alpha)-1]} - \frac{p-1}{\Gamma(\alpha)[p(1-\alpha)-1]}} \right)^{\frac{1}{p-1}}.$$

When $f^{1-p}(0) < \frac{p-1}{\Gamma(\alpha)[p(1-\alpha)-1]}$, there exists a constant $T^* > 0$, such that the solution $u(t)$ blows up. The proof is complete. □

Next, we consider another case, if the bound of blow-up time T is fixed, can we give an initial data such that the solution will blow-up before time T? We now give a positive answer.

Theorem 2.3. *Let $f(t)$ be the solution of the following ODE:*

$$\begin{cases} {}_0^C\mathbb{D}_t^\alpha f(t) = f^p(t), \quad 0 < t \le T, \\ f(0) = f_0, \end{cases}$$

where $T > 0$ is a fixed constant. Then for any $T^* \in (0, T]$, there exists a positive constant C_1 such that for any $f_0 > C_1$, the solution $f(t)$ will be infinite with initial data f_0 whenever $t > T^*$.

The proof is easier than that of Theorem 2.2. In order to read easily, we also give a proof to briefly explain it.

Proof Similar to the proof of Theorem 2.2, noting that $0 < t \le T$, we have

$$f(t) \ge f(0) + \frac{1}{\Gamma(\alpha)T^{1-\alpha}} \int_0^t f^p(s)ds.$$

Based on the comparison principle, we consider

$$\begin{cases} f'(t) = \frac{1}{\Gamma(\alpha)T^{1-\alpha}} f^p(t), \\ f(0) = f_0. \end{cases}$$

whose solution is given by

$$f(t) = \Big(\frac{1}{f_0^{1-p} - \frac{t(p-1)}{\Gamma(\alpha)T^{1-\alpha}}} \Big)^{\frac{1}{p-1}}, \quad for \ \ t \le T.$$

Thus the blow-up occurs at $t = \frac{\Gamma(\alpha)T^{1-\alpha}}{f_0^{p-1}(p-1)}$. We can choose $f_0 \ge (\frac{T^{1-\alpha}\Gamma(\alpha)}{T^*(p-1)})^{\frac{1}{p-1}}$, for any $T^* < T$, then $f(t)$ blows up before time T^*. The proof is complete. □

As a special example, we consider the linear perturbation

$$\begin{cases} f'(t) = \lambda f(t) + f^p(t), \quad t > 0, \\ f(0) = f_0. \end{cases}$$

Like equation (4), Theorem 2.2 and Theorem 2.3 also hold for (6), the reason is that the linear term has no effect on blow-up.

2.1.2 *Nonlocal FDEs*.

We shall be concerned with the following nonlocal differential equation of time-fractional order:

$$\begin{cases} \mathbb{D}_t^\alpha (f - f_0) = [\int_0^t (t-s)^{\gamma-1} f^p(s)ds]^q, \quad t > 0, \\ f|_{t=0} = f_0. \end{cases} \tag{8}$$

Let $g = \int_0^t (t-s)^{\gamma-1} f^p(s)ds$, then

$$\begin{cases} \mathbb{D}_t^{\gamma}(g) = f^p(t), \quad t > 0, \\ g|_{t=0} = 0. \end{cases} \tag{9}$$

Hence

$$\begin{cases} \mathbb{D}_t^{\alpha}(f - f_0) = g^q, \quad t > 0, \\ \mathbb{D}_t^{\gamma} g = f^p, \quad t > 0, \\ f|_{t=0} = f_0, \quad g|_{t=0} = 0. \end{cases} \tag{10}$$

Here $\mathbb{D}_{a+}^{\alpha} f(x)$ is the Riemann-Liouville fractional derivative of order $\alpha \in C$. When $0 < \alpha < 1$, $\mathbb{D}_{a+}^{\alpha} f(x)$ has the following relationship with the Liouville-Caputo fractional derivative ${}^{C}\mathbb{D}_{a+}^{\alpha} f(x)$:

$$({}^{C}\mathbb{D}_{a^+}^{\alpha} f)(x) = (\mathbb{D}_{a^+}^{\alpha} [f(t) - f(a)])(x),$$

and

$$\mathbb{D}_{a^+}^{\alpha} f = \frac{d^m}{dx^m}(J^{m-\alpha} f(x)) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m}\Big[\int_0^x (x-t)^{m-\alpha-1} f(t)dt\Big],$$

where $\mathbb{D}_{a+}^{\alpha}$ is the Riemann-Liouville fractional derivative operator, and J^{α} is the Riemann-Liouville fractional integral operator of order $\alpha \in \mathbb{C}$. In this section, we note the Liouville-Caputo fractional derivative ${}^{C}\mathbb{D}_{a^+}^{\alpha}$ instead of ${}_0^C\mathbb{D}_t^{\alpha}$ as ${}^{C}\mathbb{D}_{a^+}^{\alpha}$, which means the fractional derivative from the initial time a. We have the following results.

Theorem 2.4. *Let $f_0 > 0$, $p > 1$, $q > 1$, if $\alpha > 1 - \frac{1}{p}$, the solution of the equation (10) blows up in finite time.*

Proof We assume that (f, g) is the global solution of equation (10), set

$$\varphi(t) = \begin{cases} \frac{(T-t)^{\lambda}}{T^{\lambda}} \quad , \quad t \in [0, T], \\ \quad 0 \quad\quad , \quad t > T. \end{cases}$$

satisfies

$$\int_0^T \mathbb{D}_{T-}^{\alpha} \varphi(t)dt = C_{\alpha,\lambda} T^{1-\alpha}, \quad C_{\alpha,\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+2-\alpha)},$$

$$\int_0^T |\mathbb{D}_{T-}^{\alpha} \varphi(t)|^p \varphi^{1-p}(t)dt = C_{p,\alpha} T^{1-\alpha p}, \quad C_{p,\alpha} = \frac{1}{\lambda - p\alpha + 1}\Big[\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)}\Big]^p,$$

and $\lambda > \max\{\alpha p - 1, \gamma p - 1\}$. Next, we will use the following properties of Riemann-Liouville fractional derivative operator.

$$\int_0^T f(t)\mathbb{D}_{0^+}^{\alpha} g(t)dt = \int_0^T g(t)\mathbb{D}_{T-}^{\alpha} f(t)dt, \quad 0 < \alpha < 1.$$

Multiply both sides of equation system (10) by φ, and then integrate t on $[0, T]$, there is

$$\begin{cases} \int_0^T f \cdot \mathbb{D}_{T-}^{\alpha} \varphi dt = \int_0^T f_0 \cdot \mathbb{D}_{T-}^{\alpha} \varphi dt + \int_0^T |g(t)|^q \varphi dt, \\ \int_0^T g \cdot \mathbb{D}_{T-}^{\gamma} \varphi dt = \int_0^T |f(t)|^p \varphi dt. \end{cases} \tag{11}$$

Estimate the above formula by using Hölder inequality

$$\int_0^T f\cdot \mathbb{D}^{\alpha}_{T-}\varphi dt \le \Big[\int_0^T |f|^p\varphi(t)dt\Big]^{\frac{1}{p}}\Big[\int_0^T |\mathbb{D}^{\alpha}_{T-}\varphi|^{p'}\varphi^{-\frac{p'}{p}}dt\Big]^{\frac{1}{p'}},$$
$$\int_0^T g\cdot \mathbb{D}^{\gamma}_{T-}\varphi dt \le \Big[\int_0^T |g|^q\varphi(t)dt\Big]^{\frac{1}{q}}\Big[\int_0^T |\mathbb{D}^{\gamma}_{T-}\varphi|^{q'}\varphi^{-\frac{q'}{q}}dt\Big]^{\frac{1}{q'}},$$

where $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{q}+\frac{1}{q'}=1$. Let

$$\int_0^T |g|^q\varphi(t)dt = A,\quad \int_0^T |f|^p\varphi(t)dt = B,$$
$$\int_0^T |\mathbb{D}^{\alpha}_{T-}\varphi|^{p'}\varphi^{-\frac{p'}{p}}dt = E,\quad \int_0^T |\mathbb{D}^{\gamma}_{T-}\varphi|^{q'}\varphi^{-\frac{q'}{q}}dt = F.$$

Since $f_0>0$, $\int_0^T \mathbb{D}^{\alpha}_{T-}\varphi dt>0$, and

$$\int_0^t |g(t)|^q\varphi(t)dt = \int_0^T f\cdot\mathbb{D}^{\alpha}_{T-}\varphi dt - \int_0^T f_0\cdot\mathbb{D}^{\alpha}_{T-}\varphi dt$$
$$\Rightarrow \int_0^t |g(t)|^q\varphi(t)dt \le \int_0^T f\cdot\mathbb{D}^{\alpha}_{T-}\varphi dt \le B^{\frac{1}{p}}\cdot E^{\frac{1}{p'}}$$
$$\Rightarrow A\le B^{\frac{1}{p}}\cdot E^{\frac{1}{p'}} \quad and \quad B = A^{\frac{1}{q}}\cdot F^{\frac{1}{q'}}.$$

Arrange the above two equations, we obtain

$$A^{1-\frac{1}{pq}} \le (F^{\frac{1}{q'}})^{\frac{1}{p}}\cdot E^{\frac{1}{p'}},$$
$$B^{1-\frac{1}{pq}} \le (E^{\frac{1}{p'}})^{\frac{1}{q}}\cdot F^{\frac{1}{q'}},$$

and because of $\lambda>\max\{\alpha p-1,\gamma p-1\}$, then we get $E=C_{p',\alpha}T^{1-\alpha p'}$, $F=C_{q',\gamma}T^{1-\gamma q'}$, we get

$$\begin{aligned} f_0\int_0^T \mathbb{D}^{\alpha}_{T-}\varphi dt &= \int_0^T f\cdot\mathbb{D}^{\alpha}_{T-}\varphi dt - \int_0^T |g(t)|^q\varphi dt\\ &\le B^{\frac{1}{p}}\cdot E^{\frac{1}{p'}} = B^{\frac{1}{p}}\cdot[C_{p',\alpha}T^{1-\alpha p'}]^{\frac{1}{p'}}\\ &= B^{\frac{1}{p}}\cdot[C_{p',\alpha}^{\frac{1}{p'}}T^{\frac{1}{p'}-\alpha}],\end{aligned}$$

when $\frac{1}{p'}-\alpha<0$, as $\alpha>\frac{1}{p'}$, if $T\to\infty$, then we can get $f_0\le 0$, which contradicts the initial conditions $f_0>0$, so equation (10) blows up at a finite time. The proof is complete. □

2.2.1 ***The Caputo-Hadamard fractional differential equation***

In this subsection, we will give a new blow-up criterion for the Caputo-Hadamard fractional differential equation. Consider the initial value problem (IVP) for the Caputo-Hadamard fractional differential equations given by

$$\begin{cases} {}^H_0\mathbb{D}^{\alpha}_{a+}f(t) = f^p(t), & t>a,\\ {}^H_0\mathbb{D}^{\alpha}_{a+}f(a+) = b_0, & f(a+)=b_0,\end{cases} \tag{12}$$

where $0<\alpha<1$ and $f\in C([0,T],\mathbb{R})$. Since f is assumed continuous, it is easy to see that the solution of initial value problem (12) can be written as the following fractional equation

$$f(t) = \frac{b_0}{\Gamma(\alpha)}(\log\frac{t}{a})^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_a^t (\log\frac{\tau}{a})^{\alpha-1}f^p(\tau)\frac{d\tau}{\tau}. \tag{13}$$

Here Γ denotes the Gamma function. We noted that $\log\frac{\tau}{a} < \log\frac{t}{a}$ for $t > \tau > a$, so that

$$\left(\log\frac{\tau}{a}\right)^{\alpha-1} > \left(\log\frac{t}{a}\right)^{\alpha-1}, \quad 0 < \alpha < 1.$$

Consequently,

$$f(t) \geq \frac{b_0}{\Gamma(\alpha)}(\log\frac{t}{a})^{\alpha-1} + \frac{1}{\Gamma(\alpha)}(\log\frac{t}{a})^{\alpha-1}\int_0^t \frac{f^p(\tau)}{\tau}d\tau. \tag{14}$$

Note that $\log\frac{\tau}{a} < \frac{\tau}{a}$ if $\tau > a$, then we get $\frac{1}{\tau} < \frac{1}{a}\times(\log\frac{\tau}{a})^{-1}$, which yields

$$f(t) \geq \frac{b_0}{\Gamma(\alpha)}(\log\frac{t}{a})^{\alpha-1} + \frac{1}{a\Gamma(\alpha)}\int_0^t \frac{f^p(\tau)}{(\log\frac{\tau}{a})^{p(1-\alpha)}}d\tau. \tag{15}$$

Theorem 2.5. *Let $0 < \alpha < 1$ and $p > 1$, then the solution of problem (12) blows up for arbitrary initial value.*

Proof By Lemma 2.1 and comparison principle, we know that $f(s)$ is an upper solution of the following equation. Next, we consider

$$\begin{cases} u'(t) = \frac{u^p(t)}{a\Gamma(\alpha)(\log\frac{t}{a})^{p(1-\alpha)}}, \\ u(0) = u(a), \end{cases} \tag{16}$$

and $u(a) = \frac{b_0}{\Gamma(\alpha)}(\log\frac{t}{a})^{\alpha-1}$, it is easy to show that

$$u(t) = \left(\frac{1}{u^{1-p}(a) - \frac{p-1}{\Gamma(\alpha)}\int_0^{\log\frac{t}{a}} s^{p(\alpha-1)}e^s ds}\right)^{\frac{1}{p-1}}. \tag{17}$$

If $0 < p(1-\alpha) < 1$, and t is large enough, then we have

$$\begin{aligned} \int_0^{\log\frac{t}{a}} s^{p(\alpha-1)}e^s ds &= \Big(\int_0^1 + \int_1^{\log\frac{t}{a}}\Big)s^{p(\alpha-1)}e^s ds \\ &\geq \int_1^{\log\frac{t}{a}} s^{p(\alpha-1)}e^s ds \\ &\geq (\log\frac{t}{a})^{p(\alpha-1)}(\frac{t}{a} - e) \to \infty, \quad \textit{as } t \to \infty. \end{aligned}$$

Thus, the solution $u(t)$ will blows up at $t \geq T^*$.

If $p(1-\alpha) = 1$, We first transform equation (16) into

$$u'(t) = \frac{u^p}{a\Gamma(\alpha)\log(\frac{t}{a})}, \quad u(0) = u(a),$$

as well as equation (17), we can get

$$u(t) = \left(\frac{1}{u^{1-p}(a) - \frac{p-1}{\Gamma(\alpha)}\int_0^{\log\frac{t}{a}} se^s ds}\right)^{\frac{1}{p-1}}.$$

Hence, there exists a constant T^*, we know that the solution $u(t)$ will blows up at time $t > T^*$ for any nonnegative initial value.

When $p(1-\alpha) > 1$, we split the interval, along with

$$\int_0^1 s^{p(\alpha-1)}e^s ds \geq \int_0^1 s^{p(\alpha-1)} ds = \infty.$$

Hence, the solution $u(t)$ will blows up at time $t > T^*$ for arbitrarily initial value. The proof is complete. □

Remark 2.6. *Comparing with Theorems 2.4 and 2.1, we can find that there is no relationship between* α *and* p*, which is different from the Liouville-Caputo derivative. The reason is that* $\int_0^{\log\frac{t}{a}} s^{p(\alpha-1)}e^s ds$ *has two singularities* $s = 0$ *and* $s = +\infty$.

3. Blow-up results of FPDEs

In this section, we generalize the blow-up results of FDEs to FPDEs. Consider the following FPDEs

$$\begin{cases} \mathbb{D}_+^\alpha u(t) - \Delta u = f(u), & x \in \Omega, t > 0, 0 < \alpha < 1, \\ u = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{18}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. We consider the eigenvalue problem for the elliptic equation

$$\begin{cases} -\Delta\phi = \lambda\phi, & \text{in}\Omega, \\ \phi = 0, & \text{on } \partial\Omega. \end{cases} \tag{19}$$

Then, all the eigenvalues are strictly positive, increasing and the eigenfunction ϕ_1 corresponding to the smallest eigenvalue λ_1 does not change sign in domain Ω, as shown in [11]. Therefore, we normalize it as

$$\phi_1(x) \geq 0, \quad \int_\Omega \phi_1(x)dx = 1.$$

Multiplying ϕ_1 on both sides of the first equation of (18) and integrating over Ω, we have

$$(\mathbb{D}_+^\alpha u(t), \phi_1) - (\Delta u, \phi_1) = (f(u), \phi_1).$$

If we let $v(t) = (u(t,\cdot), \phi_1(\cdot))$, then f satisfies that

$$\begin{cases} \mathbb{D}_+^\alpha v(t) = \lambda_1 v(t) + (f(u), \phi_1), & t > 0, \\ v(0) = v_0 = (u_0, \phi_1). \end{cases}$$

This is the classical first eigenvalue method, the most important is the solution keeps nonnegative. It is easy to see that the solution will keep nonnegative if the initial value is nonnegative. Consequently, we have $v(t) > 0$. In particular, if $f(u) = u^p$, then by using Jensen's inequality, we have

$$\begin{cases} \mathbb{D}_+^\alpha v(t) \geq \lambda_1 v(t) + v^p(t) \geq v^p(t), & t > 0, \\ v(0) = v_0 = (u_0, \phi_1). \end{cases}$$

In other words, v is an upper solution of the following equation

$$\begin{cases} \mathbb{D}_+^\alpha v(t) = v^p(t), & t > 0, \\ v(0) = v_0 = (u_0, \phi_1). \end{cases}$$

It follows from the results of Section 2, we have the following result.

Theorem 3.1. *Let* $0 < \alpha < 1$ *and* $p > 1$*, then for problem (18), we have the following results.*

(1) If $0 < p(1-\alpha) \leq 1$*, then the solution of problem (18) with* $u_0 \geq 0$ *blows up in a finite time;*

(2) If $p(1-\alpha) > 1$ *and* $(u_0, \phi_1) > (\frac{p-1}{\Gamma(\alpha)[p(1-\alpha)-1]})^{p-1}$*, then exists a constant* T*, such that the solution of problem (18) blows up in a finite time.*

Similarly, one can consider the following reaction-diffusion equation

$$\begin{cases} {}^{H}_{0}\mathbb{D}^{\alpha}_{a+}u(t) - \Delta u = f(u), & x \in \Omega, t > 0, 0 < \alpha < 1, \\ u = 0, & x \in \partial\Omega, t > 0, \\ {}^{H}_{0}\mathbb{D}_{a+}u(x, a) = u_a(x), & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain. Furthermore, we can also deduce that

$$\begin{cases} \mathbb{D}^{\alpha}_{+}u(t) - \Delta u = [\int_0^t (t-s)^{\gamma-1} u^p(s) ds]^q, & x \in \Omega, t > 0, 0 < \alpha < 1, \\ u = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $p, q > 1$. It is easy to see that the results of Section 2 can be generalized by using the first eigenvalue method.

Acknowledgment This work was supported in part by NSF of China grants 11771123, 12171247, Jiangsu Provincial Double-Innovation Doctor Program JSSCBS20210466 and the Qing Lan Project of Jiangsu Province and Postgraduate Research and Practice Innovation Program of Jiangsu Province (No. KYCX21_0932).

References

[1] A. Alsaedi, M. Kirane and B. Torebek, *Global existence and blow-up for a space and time nonlocal reaction-diffusion equation*, Quaest. Math. **44** (2021), 747–753.

[2] Z. Bai, S. Zhang, S. Sun and C. Yin, *Monotone iterative method for fractional differential equations*, Electron. J. Differential Equations **2016** (2016), 1–8.

[3] D. Baleanu, B. Shiri, H. M. Srivastava, M. Al Qurashi, *A Chebyshev spectral method based on operational matrix for fractional differential equations involving non-singular Mittag-Leffler kernel*, Adv. Difference Equ. **2018** (2018), 1–23.

[4] E.G. Bazhlekova, *Subordination principle for fractional evolution equations*, Fract. Calc. Appl. Anal. **3** (2000), 213–230.

[5] A. Boulfoul, B. Tellab, N. Abdellouahab and Kh. Zennir, *Existence and uniqueness results for initial value problem of nonlinear fractional integro-differential equation on an unbounded domain in a weighted Banach space*, Math. Methods Appl. Sci. **44** (2021), 3509–3520.

[6] N. Cui and H. Sun, *Existence and multiplicity results for the fractional Schrödinger equations with indefinite potentials*, Appl. Anal. **100** (2021), 1198–1212.

[7] R. Dahal, C. Goodrich and B. Lyons, *Monotonicity results for sequential fractional differences of mixed orders with negative lower bound*, J. Difference Equ. Appl. **27** (2021), 1574–1593.

[8] F. Develi and O. Duman, *Existence and stability analysis of solution for fractional delay differential equations*, Filomat **37** (2023), 1869–1878.

[9] S.D. Eidelman and A.N. Kochubei, *Cauchy problem for fractional diffusion equations*, J. Differential Equations, **199** (2004), 211–255.

[10] H. Fujita, *On the blowing up of solutions of the Cauchy problem for* $u^t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo **13** (1966), 109–124.

[11] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, (2nd edition), Springer-Verlag, New York, 1983.

[12] R. Gorenflo and F. Mainardi, *Fractional Calculus: Integral and Differential Equations of Fractional Order, A. Carpinteri and F. Mainardi (Editors): Fractals and Fractional Calculus in Continuum Mechanics*, Springer Verlag, Wien and New York 1997, 223-276.

[13] M. Herzallah and A. Radwan, *Existence and uniqueness of solutions to some classes of nonlocal semilinear conformable fractional differential or integrodifferential equations*, Filomat **36** (2022), 2717–2728.

[14] M. Izadi, H.M. Srivastava, *A novel matrix technique for multi-order pantograph differential equations of fractional order*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **477** (2021).

[15] M. Kassim, K. Furati and N. Tatar, *Non-existence for fractionally damped fractional differential problems*, Acta Math. Sci. Ser. B (Engl. Ed.) **37** (2017), 119–130.

[16] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol 204. Elsevier Science B.V., Amsterdam, 2006.

[17] M. Kirane, Y. Laskri and N.E. Tatar, *Critical exponents of Fujita type for certain evolution equations and systems with spatio-temporal fractional derivatives*, J. Math. Anal. Appl. **312** (2005), 488–501.

[18] C. Kirk, W. Olmstead and C. Roberts, *A system of nonlinear Volterra equations with blow-up solutions*, J. Integral Equations Appl. **25** (2013), 377–393.

[19] Z. Laadjal, *Sharp estimates for the unique solution for a class of fractional differential equations*, Filomat **37** (2023), 435–441.

[20] V. Lakshmikantham and A. S. Vatsala, *Basic theory of fractional differential equations*, Nonlinear Anal. **69** (2008), 2677–2682.

[21] Y. Laskri and N. Tatar, *The critical exponent for an ordinary fractional differential problem*, Comput. Math. Appl. **59** (2010), 1266–1270.

[22] C. Li, Y. Chen and J. Kurths, *Fractional calculus and its applications*, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **371** (2013), 20130037.

[23] C. Li and S. Sarwar, *Existence and continuation of solutions for Caputo type fractional differential equations*, Electron. J. Differential Equations **2016**, (2016), 1-14

[24] C. Li and Z. Li, *The blow-up and global existence of solution to Caputo-Hadamard fractional partial differential equation with fractional laplacian*, J. Nonlinear Sci. **31** (2021), no. 5, Paper No. 80, 35 pp.
[25] C. Li and Z. Li, *Stability and logarithmic decay of the solution to Hadamard-type fractional differential equation*, J. Nonlinear Sci. **31** (2021), no. 2, Paper No. 31, 60 pp.
[26] L. Li and J. Liu, *A generalized definition of Caputo derivatives and its application to fractional ODEs*, SIAM J. Math. Anal. **50** (2018), 2867–2900.
[27] J. Manimaran and L, Shangerganesh, *Blow-up solutions of a time-fractional diffusion equation with variable exponents*, Tbilisi Math. J. **12** (2019), 149–157.
[28] W. Mydlarczyk, W. Okrasiski and C. Roberts, *Blow-up solutions to a system of nonlinear Volterra equations*, J. Math. Anal. Appl. **301** (2005), 208–218.
[29] H.M. Srivastava, *A Survey of Some Recent Developments on Higher Transcendental Functions of Analytic Number Theory and Applied Mathematics*, Symmetry. **13** (2021), 2294.
[30] H.M. Srivastava, *An Introductory Overview of Fractional-Calculus Operators Based Upon the Fox-Wright and Related Higher Transcendental Functions*. J. Adv. Engrg. Comput. **5** (2021), 135–166.
[31] H.M. Srivastava, *Some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformations*, J. Nonlinear Convex Anal. **22** (2021), 1501–1520.
[32] H.M. Srivastava, A.M.A. El-Sayed, H.H.G. Hashem, Sh.M. Al-Issa, *Analytical investigation of nonlinear hybrid implicit functional differential inclusions of arbitrary fractional orders*, Rev. Real Acad. Cienc. Exactas Fs. Nat. Ser. A Mat. (RACSAM) **116** (2022), 26.
[33] H.M. Srivastava, A. Shehata, S.I. Moustafa, *Some Fixed Point Theorems for $F(\psi,\varphi)$-Contractions and Their Application to Fractional Differential Equations*, Russian. J. Math. Phys. **27** (2020), 385–398.
[34] H.M. Srivastava, P.O. Mohammed, C.S. Ryoo, Y.S. Hamed, *Existence and uniqueness of a class of uncertain Liouville-Caputo fractional difference equations*, J.King Saud Univ. Sci. **33** (2021) 101497.
[35] N.E. Tatar, *Nonexistence results for a fractional problem arising in thermal diffusion in fractal media*, Chaos Solitons Fractals, **36** (2008), 1205–1214.
[36] X.J. Yang, H.M. Srivastava, *An asymptotic perturbation solution for a linear oscillator of free damped vibrations in fractal medium described by local fractional derivatives*, Commun. Nonlinear Sci. Numer. Simul. **29** (2015), 499–504.
[37] Q. Zhang and H. Sun, *The blow-up and global existence of solutons of Cauchy problem for a time fractional diffusion equation*, Topol. Methods Nonlinear Anal. **46** (2015), 69–92.
[38] Q. Zhang and Y. Li, *The critical exponent for a time fractional diffusion equation with nonlinear memory*, Math. Methods Appl. Sci. **41** (2018), 6443–6456.
[39] Q. Zhang and Y. Li, *Global well-posedness and blow-up solutions of the Cauchy problem for a time-fractional superdiffusion equation*, J. Evol. Equ. **19** (2019), 271–303.