Filomat 38:4 (2024), 1343–1355
https://doi.org/10.2298/FIL2404343S

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Geometric inequalities for CR δ-invariant on generic statistical submanifolds

Aliya Naaz Siddiqui[a,*], Ali Hussain Alkhaldi[b], Mohammad Hasan Shahid[c]

[a] *Division of Mathematics, School of Basic Sciences, Galgotias University, Greater Noida, Uttar Pradesh 203201, India*
[b] *Department of Mathematics, College of Science, King Khalid University, 9004 Abha, Saudi Arabia*
[c] *Department of Mathematics, Faculty of Natural Sciences, Jamia Millia Islamia, New Delhi - 110025, India*

Abstract. B.-Y. Chen [8] introduced the notion of CR δ-invariant on CR-submanifolds. Recently, F.R. Al-Solamy et al. [3, 4] and I. Mihai et al. [11], respectively, established optimal inequalities for this invariant on anti-holomorphic submanifolds in complex space forms and for generic submanifolds in Sasakian space forms. Furthermore, A.N. Siddiqui et al. [19] derived equivalent inequalities for the contact CR δ-invariant, but in the context of a generic submanifold within trans-Sasakian generalized Sasakian space forms. They also managed to identify a lower limit for the squared norm of the mean curvature. This was achieved by relating it to a CR δ-invariant and the Laplacian of the warping function. This was done in the case of CR-warped products existing within the same ambient space forms. In the present paper, we obtain two optimal inequalities involving the CR δ-invariant for a generic statistical submanifold in a holomorphic statistical manifold of constant holomorphic sectional curvature. Finally, we consider a generic statistical submersion from a holomorphic statistical manifold onto a statistical manifold.

1. Introduction

A statistical manifold of probability distributions is equipped with a Riemannian metric and a pair of conjugate affine connections [5]. A statistical structure can be considered as a generalization of a Riemannian metric and its Levi-Civita connection. The theory of statistical manifold and its submanifolds plays a crucial role in several fields of mathematics.

The study of Riemannian submersions between two Riemannian manifolds has been initiated by B. O'Neill [13, 14]. Recently, such submersions have been studied widely in differential geometry. In 2001, N. Abe et al. [1] have introduced the notion of statistical submersion between two statistical manifolds. In 2004, K. Takano [20] has defined Kähler-like statistical manifolds, by considering the notion of complex structure on statistical manifolds, and a Kähler-like statistical submersion. He has also introduced Sasaki-like statistical manifolds, that is, statistical manifolds endowed with contact structures in [21]. Recently, G.E. Vilcu et al. [22] have studied the concept of quaternionic Kähler-like statistical manifold and obtained

2020 *Mathematics Subject Classification*. Primary 53B25, 53C15, 53B35, 53C40.
Keywords. CR δ-invariant; Holomorphic statistical manifolds; Generic statistical submanifolds; Generic statistical submersion.
Received: 06 May 2023; Accepted: 10 August 2023
Communicated by Ljubica Velimirović
Research supported by the Deanship of Scientific Research at King Khalid University for funding this work through large group Research project under grant number RGP2/429/44.
* Corresponding author: Aliya Naaz Siddiqui
Email addresses: aliya.siddiqui@galgotiasuniversity.edu.in (Aliya Naaz Siddiqui), ahalkhaldi@kku.edu.sa (Ali Hussain Alkhaldi), mshahid@jmi.ac.in (Mohammad Hasan Shahid)

several geometric properties of quaternionic Kähler-like statistical submersions. In [9], H. Furuhata et al. have considered another notion of complex structure on statistical manifolds and defined holomorphic statistical manifold. By keeping this idea, in [10], the statistical counterpart of a Sasakian manifold, that is, Sasakian statistical manifold has been investigated. Several results have been derived by distinguished geometers in the area of statistical manifolds (see [6, 7, 12, 15–18]).

In the early 1990s, B.-Y. Chen introduced new types of Riemannian invariants, called δ-invariants (or Chen invariants) on Riemannian manifolds. The δ-invariants are not similar in nature to the classical scalar and Ricci curvatures because both of them are total sum of sectional curvatures on Riemannian manifolds. In contrast, all of the non-trivial δ-invariants are derived from the scalar curvature by removing a definite amount of sectional curvatures. He considered the concept of δ-invariants in order to find new necessary conditions for the existence of minimal immersions into a Euclidean space of an arbitrary dimension and to obtain applications of the celebrated Nash embedding theorem.

The *CR* δ-invariant $\delta(\mathbf{D})$ on a *CR*-submanifold N in a Kähler manifold $\overline{N}$ is defined by [8]

$$\delta(\mathbf{D})(p) = \tau(p) - \tau(\mathbf{D}_p), \tag{1}$$

where τ denotes the scalar curvature of N and $\tau(\mathbf{D})$ denotes the scalar curvature of the holomorphic distribution $\mathbf{D}$ of N. He [8] also established an inequality for anti-holomorphic warped product submanifold $N = N^T \times_f N^\perp$ in a complex space form $\overline{N}(4\bar{c})$ involving the *CR* δ-invariant $\delta(\mathbf{D})$. F.R. Al-Solamy et al. [3, 4] proved an optimal inequality for this CR δ-invariant on an anti-holomorphic submanifold N in $\overline{N}(4\bar{c})$. Recently, Mihai et al. [11] studied Chen's *CR* δ-invarinat on an odd dimensional contact *CR*-submanifold, called contact *CR* δ-invariant, and obtained an optimal estimate for a generic submanifold N in a Sasakian space form $\overline{N}(\bar{c})$ of constant ϕ-sectional curvature $\bar{c}$.

The purpose of this paper is to define the concept of *CR* δ-invariant on a *CR*-statistical submanifold in a holomorphic statistical manifold and to prove two optimal inequalities for newly defined *CR* δ-invariant on a generic statistical submanifold in a holomorphic statistical manifold of constant holomorphic sectional curvature. Also, we define the concept of generic statistical submersion and investigate the integrability of the distributions which arise from the definition of a generic statistical submersion from a holomorphic statistical manifold onto a statistical manifold.

2. Statistical Manifolds and their Submanifolds

Definition 2.1. *[5] A Riemannian manifold $(\overline{N}, \bar{g})$ endowed with a pair of torsion-free affine connections $\overline{\nabla}$ and $\overline{\nabla}^*$ satisfying*

$$Z\bar{g}(X, Y) = \bar{g}(\overline{\nabla}_Z X, Y) + \bar{g}(X, \overline{\nabla}^*_Z Y),$$

for all $X, Y, Z \in \Gamma(T\overline{N})$, and $\overline{\nabla}\bar{g}$ is symmetric, called statistical manifold. Here $\bar{g}$ is a Riemannian metric, and the connections $\overline{\nabla}$ and $\overline{\nabla}^*$ are called dual connections on $\overline{N}$; they satisfy $(\overline{\nabla}^*)^* = \overline{\nabla}$.

Remark that if $(\overline{\nabla}, \bar{g})$ is a statistical structure on $\overline{N}$, then $(\overline{\nabla}^*, \bar{g})$ is also a statistical structure.

Definition 2.2. *[5] Let $(\overline{N}, \overline{\nabla}, \bar{g})$ be a statistical manifold and N be a submanifold of $\overline{N}$. Then (N, ∇, g) is also a statistical manifold with the induced statistical structure (∇, g) on N from $(\overline{\nabla}, \bar{g})$ and we call (N, ∇, g) a statistical submanifold in $(\overline{N}, \overline{\nabla}, \bar{g})$.*

The fundamental equations in the geometry of Riemannian submanifolds (see [24]) are the Gauss and Weingarten formulas and the Gauss equation. In our setting, the Gauss and Weingarten formulas are, respectively, given by [23]

$$\begin{aligned} &\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \qquad \overline{\nabla}^*_X Y = \nabla^*_X Y + h^*(X, Y), \\ &\overline{\nabla}_X U = -A_U(X) + \nabla^\perp_X U, \qquad \overline{\nabla}^*_X U = -A^*_U(X) + \nabla^{\perp *}_X U, \end{aligned} \tag{2}$$

for all $X, Y \in \Gamma(TN)$ and $U \in \Gamma(T^{\perp}N)$, where $\overline{\nabla}$ and $\overline{\nabla}^*$ (resp., ∇ and ∇^*) are the dual connections on $\overline{N}$ (resp., on N). The symmetric and bilinear imbedding curvature tensor of N in $\overline{N}$ for $\overline{\nabla}$ and $\overline{\nabla}^*$ are denoted by h and h^*, respectively. The relations between h (resp., h^*) and A (resp., A^*) are given by [23]

$$\overline{g}(h(X,Y),U) = g(A^*_U X, Y), \qquad \overline{g}(h^*(X,Y),U) = g(A_U X, Y). \tag{3}$$

Let $\overline{R}$ and R be the curvature tensor fields of $\overline{\nabla}$ and ∇, respectively. Then the corresponding Gauss equation is given by [23]

$$\overline{R}(X,Y,Z,W) = R(X,Y,Z,W) + \overline{g}(h(X,Z),h^*(Y,W)) - \overline{g}(h^*(X,W),h(Y,Z)), \tag{4}$$

for all $X, Y, Z, W \in \Gamma(TN)$. Similarly, if $\overline{R}^*$ and R^* are the curvature tensor fields of $\overline{\nabla}^*$ and ∇^*, respectively, the Gauss equation for $\overline{\nabla}^*$ is similar to (4).

$$\begin{aligned}\overline{R}^*(X,Y,Z,W) &= R^*(X,Y,Z,W) + \overline{g}(h^*(X,Z),h(Y,W)) \\ &\quad - \overline{g}(h(X,W),h^*(Y,Z)).\end{aligned} \tag{5}$$

The curvature tensor fields of $\overline{N}$ and N are defined as

$$\overline{\mathbf{S}} = \frac{1}{2}(\overline{R} + \overline{R}^*), \quad \mathbf{S} = \frac{1}{2}(R + R^*). \tag{6}$$

Thus, the sectional curvature $\mathbf{K}^{\nabla,\nabla^*}$ on N is given by [15, 16]

$$\begin{aligned}\mathbf{K}^{\nabla,\nabla^*}(X \wedge Y) &= g(\mathbf{S}(X,Y)Y,X) \\ &= \frac{1}{2}(g(R(X,Y)Y,X) + g(R^*(X,Y)Y,X)),\end{aligned} \tag{7}$$

for all orthonormal vectors $X, Y \in T_pN$, $p \in N$.

Suppose that $\dim(N) = n$ and $\dim(\overline{N}) = m$. Let $\{v_1, \ldots, v_n\}$ and $\{v_{n+1}, \ldots, v_m\}$ be respectively orthonormal bases of T_pN and $T^{\perp}_pN$ at $p \in N$. Then the scalar curvature τ^{∇,∇^*} of N is given by

$$\tau^{\nabla,\nabla^*} = \sum_{1 \le i < j \le n} \mathbf{K}^{\nabla,\nabla^*}(v_i \wedge v_j). \tag{8}$$

Definition 2.3. *[9] Let $(\overline{N}, \mathbf{J}, \overline{g})$ be a Kähler manifold and $\overline{\nabla}$ be an affine connection on $\overline{N}$. Then $(\overline{N}, \overline{\nabla}, \overline{g}, \mathbf{J})$ is said to be a holomorphic statistical manifold if*

(a) $(\overline{N}, \overline{\nabla}, \overline{g})$ is a statistical manifold, and

(b) the 2-form ω on $\overline{N}$, given by

$$\omega(X,Y) = \overline{g}(X, \mathbf{J}Y),$$

for all $X, Y \in \Gamma(T\overline{N})$, is $\overline{\nabla}$-parallel.

For a holomorphic statistical manifold $(\overline{N}, \overline{\nabla}, \overline{g}, \mathbf{J})$, we have the following relation [9]:

$$\overline{\nabla}_X(\mathbf{J}Y) = \mathbf{J}\overline{\nabla}^*_X Y, \tag{9}$$

for all $X, Y \in \Gamma(T\overline{N})$, where $\overline{\nabla}^*$ is the dual connection of $\overline{\nabla}$ with respect to $\overline{g}$.

Definition 2.4. *[9] A holomorphic statistical manifold $(\overline{N}, \overline{\nabla}, \overline{g}, \mathbf{J})$ is said to be of constant holomorphic curvature $\overline{c} \in \mathbb{R}$ if the following curvature equation holds*

$$\begin{aligned}\overline{S}(X,Y)Z &= \frac{\overline{c}}{4}(\overline{g}(Y,Z)X - \overline{g}(X,Z)Y + \overline{g}(\mathbf{J}Y,Z)\mathbf{J}X - \overline{g}(\mathbf{J}X,Z)\mathbf{J}Y \\ &\quad + 2\overline{g}(X,\mathbf{J}Y)\mathbf{J}Z),\end{aligned} \tag{10}$$

for all $X, Y, Z \in \Gamma(T\overline{N})$. It is denoted by $(\overline{N}(c), \overline{\nabla}, \overline{g}, \mathbf{J})$.

Let (N, g) be a statistical submanifold in $(\overline{N}, \overline{\nabla}, \overline{g}, \mathbf{J})$. Then for all $X \in \Gamma(TN)$, we put [9, 24]

$$\mathbf{J}X = \mathbf{P}X + \mathbf{F}X,$$

where $\mathbf{P}X$ and $\mathbf{F}X$ are the tangential and normal components of $\mathbf{J}X$, respectively. Then $\mathbf{P}$ is an endomorphism of the tangent bundle TN, and $\mathbf{F}X$ is a normal-bundle-valued 1-form on the tangent bundle TN. In the similar way, for all $V \in \Gamma(T^{\perp}N)$, we put [9, 24]

$$\mathbf{J}V = tV + fV,$$

where tV and fV are the tangential and normal components of $\mathbf{J}V$, respectively. Then f is an endomorphism of the normal bundle $T^{\perp}N$, and t is a tangent bundle-valued 1-form on the normal bundle $T^{\perp}N$.

The statistical version of the definition of a CR-submanifold is the following:

Definition 2.5. *[9] A statistical submanifold N in a holomorphic statistical manifold $(\overline{N}, \overline{\nabla}, \overline{g}, \mathbf{J})$ of dimension $2m \geq 4$ is called a CR-statistical submanifold if N is CR-submanifold in $\overline{N}$, that is, there exists a differentiable distribution $\mathbf{D} : p \to \mathbf{D}_p \subseteq T_pN$ on N satisfying the following conditions:*

(a) $\mathbf{D}$ is holomorphic, that is, $\mathbf{J}\mathbf{D}_p = \mathbf{D}_p \subset T_pN$ for each $p \in N$, and

(b) the complementary orthogonal distribution $\mathbf{D}^{\perp} : p \to \mathbf{D}_p^{\perp} \subseteq T_pN$ is totally real, that is, $\mathbf{J}\mathbf{D}_p^{\perp} \subset T_p^{\perp}N$ for each $p \in N$.

Remark 2.6. *[9] CR-statistical submanifolds are characterized by the condition* $\mathbf{FP} = 0$.

Definition 2.7. *[9] Let N be a CR-statistical submanifold of a holomorphic statistical manifold $(\overline{N}, \overline{\nabla}, \overline{g}, \mathbf{J})$. Then N is said to be a*

(a) mixed totally geodesic with respect to $\overline{\nabla}$ if $h(X, Y) = 0$, for all $X \in \Gamma(\mathbf{D})$ and $Y \in \Gamma(\mathbf{D}^{\perp})$.

(a) mixed totally geodesic with respect to $\overline{\nabla}^*$ if $h^*(X, Y) = 0$, for all $X \in \Gamma(\mathbf{D})$ and $Y \in \Gamma(\mathbf{D}^{\perp})$.*

Definition 2.8. *[9] Let N be a CR-statistical submanifold of a holomorphic statistical manifold $(\overline{N}, \overline{\nabla}, \overline{g}, \mathbf{J})$. If $\mathbf{J}\mathbf{D}^{\perp} = T^{\perp}N$ and $\mathbf{D} \neq 0$, then N is called a generic statistical submanifold $(f = 0)$.*

Let μ be the the orthogonal complementary subbundle of $\mathbf{J}\mathbf{D}^{\perp}$ within $T^{\perp}N$, then

$$T^{\perp}N = \mathbf{F}\mathbf{D}^{\perp} \oplus \mu.$$

For a generic statistical submanifold N, we have the following [9]:

$$\mathbf{P}\mathbf{D} = \mathbf{D}, \quad \mathbf{P}\mathbf{D}^{\perp} = 0, \quad \mathbf{F}\mathbf{D}^{\perp} = T^{\perp}N, \quad t(T^{\perp}N) = t(\mathbf{F}\mathbf{D}^{\perp}) = \mathbf{D}^{\perp}. \tag{11}$$

3. Generic Statistical Submanifolds with Canonical Structures

In this section, we give some results based on the distributions of generic statistical submanifolds in holomorphic statistical manifolds.

Proposition 3.1. *Let N be a generic submanifold of a holomorphic statistical manifold $(\overline{N}, \overline{\nabla}, \overline{g}, \mathbf{J})$. Then we have*

(a) the holomorphic distribution $\mathbf{D}$ is integrable if and only if $\mathbf{F}\nabla^*_X Y = \nabla^{\perp}_X \mathbf{F}Y$, for all $X, Y \in \Gamma(TN)$, holds.

(b) $\mathbf{P}\nabla^*_X Y - \nabla_X \mathbf{P}Y \in \Gamma(\mathbf{D}^{\perp})$ for all $X \in \Gamma(TN)$ and $Y \in \Gamma(\mathbf{D})$.

Proof. Following [12], we have

$$A^*_U\mathbf{P}Y = -A_{fU}Y, \tag{12}$$

for all $Y \in \Gamma(TN)$ and $U \in \Gamma(T^\perp N)$.

On the other hand, we derive

$$\nabla_X\mathbf{P}Y + h(X,\mathbf{P}Y) = \mathbf{P}\nabla^*_XY + th^*(X,Y) + \mathbf{F}\nabla^*_XY + fh^*(X,Y), \tag{13}$$

for all $X \in \Gamma(TN)$ and $Y \in \Gamma(\mathbf{D})$, which yields

$$g(A^*_U\mathbf{P}Y,X) = -g(\nabla^*_XY,tU) - g(A_{fU}Y,X). \tag{14}$$

On combining (12) and (14), we arrive at $g(\nabla^*_XY,tU) = 0$, which implies $\nabla^*_XY \in \Gamma(\mathbf{D})$ as $tU \in \Gamma(\mathbf{D}^\perp)$. Thus, $\mathbf{D}$ is integrable. Conversely is trivial. Hence, the assertion (a) follows.

From (13), we get $\mathbf{P}\nabla^*_XY - \nabla_X\mathbf{P}Y = th^*(X,Y)$, which gives the assertion (b). □

Proposition 3.2. *Let N be a generic submanifold of a holomorphic statistical manifold $(\overline{N},\overline{\nabla},\overline{g},\mathbf{J})$. If N is totally umbilical with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$, then the purely real distribution $\mathbf{D}^\perp$ is integrable.*

Proof. Following [9], we have

$$\begin{aligned}\nabla_X(tU) &= -A_{fU}X - \mathbf{P}A^*_UX + t\nabla^{\perp *}_XU\\ &= -g(U,H)X - g(U,H^*)\mathbf{P}X + t\nabla^{\perp *}_XU,\end{aligned} \tag{15}$$

for all $X \in \Gamma(TN)$ and $U \in \Gamma(T^\perp N)$, where we have used $A_UX = g(U,H)X$ and $A^*_UX = g(U,H^*)X$. Since, $\mathbf{P}\mathbf{D}^\perp = 0$ and $t(T^\perp N) = t(\mathbf{F}\mathbf{D}^\perp) = \mathbf{D}^\perp$. Thus, (15) implies that $\nabla_ZW \in \Gamma(\mathbf{D}^\perp)$ for all $Z,W \in \Gamma(\mathbf{D}^\perp)$. This shows that $\mathbf{D}^\perp$ is integrable. □

4. Two Optimal Estimates for Generic Statistical Submanifolds

By analogy with the Chen's *CR* δ-invariant, we define a *CR* δ-invariant on a *CR*-statistical submanifold N in a holomorphic statistical manifold $\overline{N}$ by

$$\delta^{\nabla,\nabla^*}(\mathbf{D})(p) = \tau^{\nabla,\nabla^*}(p) - \tau^{\nabla,\nabla^*}(\mathbf{D}_p),$$

where τ^{∇,∇^*} signifies the scalar curvature of N with respect to ∇ and ∇^* and $\tau^{\nabla,\nabla^*}(\mathbf{D})$ represents the scalar curvature of the holomorphic distribution $\mathbf{D}$ of N with respect to ∇ and ∇^*.

We choose a local orthonormal frame $\{v_1,\dots,v_{2m_1+m_2}\}$ on N such that $v_1,\dots,v_{m_1},v_{m_1+1},\dots,v_{2m_1}$ are tangents to $\mathbf{D}$ and $v_{2m_1+1},\dots,v_{2m_1+m_2}$ are tangents to $\mathbf{D}^\perp$, where $v_{m_1+1} = \mathbf{J}v_1,\dots,v_{2m_1} = \mathbf{J}v_{m_1}$. Then two partial mean curvature vectors denoted by $\overrightarrow{H}_{\mathbf{D}}$ (resp. $\overrightarrow{H}^*_{\mathbf{D}}$) and $\overrightarrow{H}_{\mathbf{D}^\perp}$ (resp. $\overrightarrow{H}^*_{\mathbf{D}^\perp}$) of N are

$$\overrightarrow{H}_{\mathbf{D}} = \frac{1}{2m_1}\sum_{i=1}^{2m_1} h(v_i,v_i), \quad \text{(resp. } \overrightarrow{H}^*_{\mathbf{D}} = \frac{1}{2m_1}\sum_{i=1}^{2m_1} h^*(v_i,v_i)),$$

and

$$\overrightarrow{H}_{\mathbf{D}^\perp} = \frac{1}{m_2}\sum_{r=1}^{m_2} h(v_{2m_1+r},v_{2m_1+r}), \quad \text{(resp. } \overrightarrow{H}^*_{\mathbf{D}^\perp} = \frac{1}{m_2}\sum_{r=1}^{m_2} h^*(v_{2m_1+r},v_{2m_1+r})).$$

Also, the two partial mean curvature vectors $\overrightarrow{H}^0_{\mathbf{D}}$ and $\overrightarrow{H}^0_{\mathbf{D}^\perp}$ with respect to the Levi-Civita connection $\overline{\nabla}^0$ of N are expressed as

$$\overrightarrow{H}^0_{\mathbf{D}} = \frac{1}{2m_1}\sum_{i=1}^{2m_1} h^0(v_i,v_i), \qquad \overrightarrow{H}^0_{\mathbf{D}^\perp} = \frac{1}{m_2}\sum_{r=1}^{m_2} h^0(v_{2m_1+r},v_{2m_1+r}),$$

where h^0 is the second fundamental form of N in $\overline{N}$ with respect to the Levi-Civita connection $\overline{N}^0$.

We use some more notations for later use.

$$h(v_r, v_s) = h_{\mathbf{D}^\perp}(v_r, v_s), \quad h^*(v_r, v_s) = h^*_{\mathbf{D}^\perp}(v_r, v_s), \quad h^0(v_r, v_s) = h^0_{\mathbf{D}^\perp}(v_r, v_s), \tag{16}$$

for $r, s \in \{2m_1 + 1, \dots, 2m_1 + m_2\}$.

Since $2h^0 = h + h^*$, then one has

$$4h^{02} = h^2 + h^{*2} + 2g(h, h^*). \tag{17}$$

For our convenience, we denote the squared norm of $\mathbf{A}$ by $\mathbf{A}^2$ and denote the square of $\mathbf{A}$ by $(\mathbf{A})^2$.

In this section, we prove an optimal estimate of the *CR* δ-invariant of a generic statistical submanifold in a holomorphic statistical manifold of constant holomorphic sectional curvature $\bar{c}$.

Theorem 4.1. *Let N be a generic statistical submanifold in a holomorphic statistical manifold $(\overline{N}, \overline{\nabla}, \overline{g}, \mathbf{J})$ of constant holomorphic sectional curvature $\bar{c}$. Then, we have*

$$\begin{aligned}\delta^{\nabla,\nabla^*}(\mathbf{D}) \quad &\geq \quad 2\delta^0(\mathbf{D}) + \frac{3}{4}\frac{m_2^2}{m_2+2}[\overrightarrow{H}^2_{\mathbf{D}^\perp} + \overrightarrow{H}^{*2}_{\mathbf{D}^\perp}] - \frac{(2m_1+m_2)^2}{4}(\overrightarrow{H}^2 + \overrightarrow{H}^{*2}) \\ &\quad - \frac{\bar{c}m_2}{8}(4m_1 + m_2 - 1),\end{aligned} \tag{18}$$

where $\delta^0(\mathbf{D})$ denotes the CR δ-invariant of N with respect to ∇^0.

Proof. Let us choose a local orthonormal frame $\{v_1, \dots, v_{2m_1+m_2}\}$ on N as above. Then the scalar curvature $\tau^{\nabla,\nabla^*}(p)$ of N and $\tau(\mathbf{D}_p)^{\nabla,\nabla^*}$ of $\mathbf{D}$, $p \in N$, are given by

$$\begin{aligned}2\tau^{\nabla,\nabla^*}(p) \quad &= \sum_{1\leq A\neq B\leq 2m_1+m_2} g(\mathbf{S}(v_A, v_B)v_B, v_A) \\ &= \sum_{1\leq i\neq j\leq 2m_1} g(\mathbf{S}(v_i, v_j)v_j, v_i) + 2\sum_{i=1}^{2m_1}\sum_{r=2m_1+1}^{m_2} g(\mathbf{S}(v_i, v_r)v_r, v_i) \\ &\quad + \sum_{2m_1+1\leq r\neq s\leq 2m_1+m_2} g(\mathbf{S}(v_r, v_s)v_s, v_r),\end{aligned} \tag{19}$$

and

$$2\tau^{\nabla,\nabla^*}(\mathbf{D}_p) = \sum_{1\leq i\neq j\leq 2m_1} g(\mathbf{S}(v_i, v_j)v_j, v_i). \tag{20}$$

By the definition of the *CR* δ-invariant and the relations (19) and (20), we have

$$\begin{aligned}\delta^{\nabla,\nabla^*}(\mathbf{D})(p) \quad &= \quad \tau^{\nabla,\nabla^*}(p) - \tau^{\nabla,\nabla^*}(\mathbf{D}_p) \\ &= \sum_{i=1}^{2m_1}\sum_{r=2m_1+1}^{m_2} g(\mathbf{S}(v_i, v_r)v_r, v_i) + \frac{1}{2}\sum_{2m_1+1\leq r\neq s\leq 2m_1+m_2} g(\mathbf{S}(v_r, v_s)v_s, v_r).\end{aligned} \tag{21}$$

By straightforward calculations, the first term of (21) becomes

$$\begin{aligned}\sum_{i=1}^{2m_1}\sum_{r=2m_1+1}^{m_2} g(\mathbf{S}(v_i, v_r)v_r, v_i) \quad &= \quad \sum_{1=i}^{2m_1}\sum_{r=2m_1+1}^{m_2} g(\overline{\mathbf{S}}(v_i, v_r)v_r, v_i) \\ &= \quad 2m_1m_2\frac{\bar{c}}{4} + \frac{1}{2}\sum_{i=1}^{2m_1}\sum_{r=2m_1+1}^{m_2}[g(h^*(v_i, v_i), h(v_r, v_r)) \\ &\quad + g(h(v_i, v_i), h^*(v_r, v_r)) - 2g(h(v_i, v_r), h^*(v_i, v_r))],\end{aligned} \tag{22}$$

and the second term of (21)

$$\begin{aligned}
\frac{1}{2} & \sum_{2m_1+1\le r\neq s\le 2m_1+m_2} g(\mathbf{S}(v_r,v_s)v_s,v_r) \\
&= \frac{m_2(m_2-1)}{2}\frac{\bar{c}}{4} \\
&\quad -\frac{1}{2}\sum_{2m_1+1\le r\neq s\le 2m_1+m_2} g(h^*(v_s,v_r),h(v_s,v_r)) \\
&\quad +\frac{1}{4}\sum_{2m_1+1\le r\neq s\le 2m_1+m_2} [g(h(v_r,v_r),h^*(v_s,v_s))+g(h(v_s,v_s),h^*(v_r,v_r))]. \qquad (23)
\end{aligned}$$

On the other hand, we derive

$$\begin{aligned}
& \frac{1}{2}\sum_{i=1}^{2m_1}\sum_{r=2m_1+1}^{m_2}\{g(h^*(v_i,v_i),h(v_r,v_r))+g(h(v_i,v_i),h^*(v_r,v_r))\} \\
& +\frac{1}{4}\sum_{2m_1+1\le r\neq s\le 2m_1+m_2}\{g(h(v_r,v_r),h^*(v_s,v_s))+g(h(v_s,v_s),h^*(v_r,v_r))\} \\
= & \ \frac{1}{2}\{(2m_1+m_2)^2 g(\overrightarrow{H},\overrightarrow{H}^*)-(2m_1)^2 g(\overrightarrow{H}_{\mathbf{D}},\overrightarrow{H}^*_{\mathbf{D}})\}. \qquad (24)
\end{aligned}$$

By combining the above equations (21-24), we arrive at

$$\begin{aligned}
\delta^{\nabla,\nabla^*}(\mathbf{D})(p) &= \frac{m_2}{8}(4m_1+m_2-1)\bar{c} \\
&\quad +\frac{1}{2}[(2m_1+m_2)^2 g(\overrightarrow{H},\overrightarrow{H}^*)-(2m_1)^2 g(\overrightarrow{H}_{\mathbf{D}},\overrightarrow{H}^*_{\mathbf{D}})] \\
&\quad -\frac{1}{2}\sum_{2m_1+1\le r\neq s\le 2m_1+m_2} g(h^*(v_s,v_r),h(v_s,v_r)) \\
&\quad -\sum_{i=1}^{2m_1}\sum_{r=2m_1+1}^{m_2} g(h(v_i,v_r),h^*(v_i,v_r)). \qquad (25)
\end{aligned}$$

From relations (16) and (17), we have

$$-\frac{1}{2}\sum_{2m_1+1\le r\neq s\le 2m_1+m_2} g(h^*(v_s,v_r),h(v_s,v_r)) = \frac{1}{4}(h^{*2}_{\mathbf{D}^\perp}+h^2_{\mathbf{D}^\perp})-h^{02}_{\mathbf{D}^\perp}.$$

Thus, (25) gives

$$\begin{aligned}
\delta^{\nabla,\nabla^*}(\mathbf{D})(p) &= \frac{m_2}{8}(4m_1+m_2-1)\bar{c}-\sum_{i=1}^{2m_1}\sum_{r=2m_1+1}^{m_2} g(h(v_i,v_r),h^*(v_i,v_r)) \\
&\quad +\frac{1}{2}[(2m_1+m_2)^2 g(\overrightarrow{H},\overrightarrow{H}^*)-(2m_1)^2 g(\overrightarrow{H}_{\mathbf{D}},\overrightarrow{H}^*_{\mathbf{D}})] \\
&\quad +\frac{1}{4}(h^{*2}_{\mathbf{D}^\perp}+h^2_{\mathbf{D}^\perp})-h^{02}_{\mathbf{D}^\perp}. \qquad (26)
\end{aligned}$$

By analogy with [3, 11], we have the following relations:

$$h^2_{\mathbf{D}^\perp} \ge \frac{3m_2^2}{m_2+2}\overrightarrow{H}^2_{\mathbf{D}}, \qquad h^{*2}_{\mathbf{D}^\perp} \ge \frac{3m_2^2}{m_2+2}\overrightarrow{H}^{*2}_{\mathbf{D}}. \qquad (27)$$

Substituting (27) into (26), we get

$$\begin{aligned}\delta^{\nabla,\nabla^*}(\mathbf{D})(p) \geq\ & \frac{m_2}{8}(4m_1+m_2-1)\bar{c} - \sum_{i=1}^{2m_1}\sum_{r=2m_1+1}^{m_2} g(h(v_i,v_r),h^*(v_i,v_r)) \\ & +\frac{1}{2}[(2m_1+m_2)^2 g(\overrightarrow{H},\overrightarrow{H}^*) - (2m_1)^2 g(\overrightarrow{H}_{\mathbf{D}},\overrightarrow{H}^*_{\mathbf{D}})] \\ & +\frac{1}{4}\frac{3m_2^2}{m_2+2}(\overrightarrow{H}^{*2}_{\mathbf{D}}+\overrightarrow{H}^2_{\mathbf{D}}) - h^{02}_{\mathbf{D}^\perp}. \end{aligned} \tag{28}$$

Also, from $2\overrightarrow{H}^0_{\mathbf{D}} = \overrightarrow{H}_{\mathbf{D}} + \overrightarrow{H}^*_{\mathbf{D}}$, we derive the following

$$g(\overrightarrow{H}_{\mathbf{D}},\overrightarrow{H}^*_{\mathbf{D}}) \leq \overrightarrow{H}^{02}_{\mathbf{D}}.$$

Thus, equation (28) takes the following form:

$$\begin{aligned}\delta^{\nabla,\nabla^*}(\mathbf{D})(p) \geq\ & \frac{m_2}{8}\bar{c}(4m_1+m_2-1) - \sum_{i=1}^{2m_1}\sum_{r=2m_1+1}^{m_2} g(h(v_i,v_r),h^*(v_i,v_r)) \\ & +\frac{1}{2}\{(2m_1+m_2)^2 g(\overrightarrow{H},\overrightarrow{H}^*) - (2m_1)^2 \overrightarrow{H}^{02}_{\mathbf{D}} \\ & +\frac{1}{4}\frac{3m_2^2}{m_2+2}(\overrightarrow{H}^{*2}_{\mathbf{D}}+\overrightarrow{H}^2_{\mathbf{D}}) - h^{02}_{\mathbf{D}^\perp}. \end{aligned} \tag{29}$$

On the other hand, the CR $\delta-$invariant $\delta^0(\mathbf{D})$ of N with respect to ∇^0 can be obtained as

$$\begin{aligned}\delta^0(\mathbf{D})(p) =\ & \frac{m_2}{8}\bar{c}(4m_1+m_2-1) - \sum_{i=1}^{2m_1}\sum_{r=2m_1+1}^{m_2} h^{02}(v_i,v_r) + \frac{(2m_1+m_2)^2}{2}\overrightarrow{H}^{02} - 2m_1^2\overrightarrow{H}^{02}_{\mathbf{D}} \\ & +\frac{1}{4}\frac{3m_2^2}{m_2+2}(\overrightarrow{H}^{*2}_{\mathbf{D}}+\overrightarrow{H}^2_{\mathbf{D}}) - h^{02}_{\mathbf{D}^\perp}. \end{aligned} \tag{30}$$

Substituting (30) into (29), we find

$$\begin{aligned}\delta^{\nabla,\nabla^*}(\mathbf{D}) \geq\ & 2\delta^0(\mathbf{D}) + \frac{3}{4}\frac{m_2^2}{m_2+2}(\overrightarrow{H}^2_{\mathbf{D}^\perp}+\overrightarrow{H}^{*2}_{\mathbf{D}^\perp}) - \frac{(2m_1+m_2)^2}{4}(\overrightarrow{H}^2+\overrightarrow{H}^{*2}) \\ & -\frac{m_2}{8}\bar{c}(4m_1+m_2-1) + \frac{1}{2}\left[\sum_{i=1}^{2m_1}\sum_{r=2m_1+1}^{m_2} h^2(v_i,v_r) + h^{*2}(v_i,v_r)\right], \end{aligned}$$

where we have used two relations

$$\begin{aligned}\frac{1}{2}g(\overrightarrow{H},\overrightarrow{H}^*) - \overrightarrow{H}^{02} &= -\frac{1}{4}(\overrightarrow{H}^2+\overrightarrow{H}^{*2}), \\ \sum_{i=1}^{2m_1}\sum_{r=2m_1+1}^{m_2} g(h(v_i,v_r),h^*(v_i,v_r)) &= \sum_{i=1}^{2m_1}\sum_{r=2m_1+1}^{m_2}[-2g(h^0(v_i,v_r),h^0(v_i,v_r)) \\ &\quad +\frac{1}{2}h(h(v_i,v_r),h(v_i,v_r)) + g(h^*(v_i,v_r),h^*(v_i,v_r))]. \end{aligned}$$

Hence our assertion follows. □

Theorem 4.2. *Let N be a generic statistical submanifold in a holomorphic statistical manifold $(\overline{N},\overline{\nabla},\overline{g},\mathbf{J})$ of constant holomorphic sectional curvature $\bar{c}$. Then the equality case of the inequality (18) holds if and only if*

- *(a) N is $\mathbf{D}$-minimal with respect to ∇ and ∇^*,*
- *(b) N is mixed totally geodesic with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$,*

(c) there exists an orthonormal basis $\{v_{2m_1+1}, \ldots, v_{2m_1+m_2}\}$ *of* $\mathbf{D}^\perp$ *such that*

(i) $h^r_{rr} = 3h^r_{ss}, \quad h^{*r}_{rr} = 3h^{*r}_{ss}, \forall 2m_1 + 1 \leq r \neq s \leq 2m_1 + m_2,$

(ii) $h^r_{st} = 0, \quad h^{*r}_{st} = 0, \forall r, s, t \in \{2m_1 + 1, \ldots, 2m_1 + m_2\}, r \neq s \neq t.$

Note that generic submanifold with $\dim(\mathbf{D}^\perp) = 1$ is nothing but a real hypersurface. Thus, in the following, we give another optimal estimate of the *CR* δ-invariant ($\delta^{\nabla,\nabla^*}(\mathbf{D}) = Ric^{\nabla,\nabla^*}(\mathbf{JN}, \mathbf{JN})$) of a real statistical hypersurface in $\overline{N}(\overline{c})$.

Theorem 4.3. *Let N be a real statistical hypersurface in a holomorphic statistical manifold* $(\overline{N}, \overline{\nabla}, \overline{g}, \mathbf{J})$ *of constant holomorphic sectional curvature* $\overline{c}$ *such that* $\dim(\mathbf{D}) = 2m_1$ *and* $\dim(\mathbf{D}^\perp) = 1$. *Then, we have*

$$Ric^{\nabla,\nabla^*}(\mathbf{JN}, \mathbf{JN}) \quad \geq \quad 2Ric^0(\mathbf{JN}, \mathbf{JN}) - \frac{\overline{c}m_1}{2} - \frac{(2m_1+1)^2}{4}(\overrightarrow{H}^2 + \overrightarrow{H}^{*2}),$$

where Ric^0 *denotes the Ricci tensor of N with respect to* ∇^0.

Proof. For a real statistical hypersurface of $(\overline{N}(\overline{c}), \overline{\nabla}, \overline{g}, \mathbf{J})$, the definition of $\delta^{\nabla,\nabla^*}(\mathbf{D})$ and $\delta^0(\mathbf{D})$ gives

$$\delta^{\nabla,\nabla^*}(\mathbf{D})(p) = Ric^{\nabla,\nabla^*}(\mathbf{JN}, \mathbf{JN}), \tag{31}$$

and

$$\delta^0(\mathbf{D})(p) = Ric^0(\mathbf{JN}, \mathbf{JN}), \tag{32}$$

respectively.

Now, let $\{v_1, \ldots, v_{m_1}, v_{m_1+1}, \ldots, v_{2m_1}\}$ be an orthonormal frame on the holomorphic distribution $\mathbf{D}$, where $v_{m_1+1} = \mathbf{J}v_1, \ldots, v_{2m_1} = \mathbf{J}v_{m_1}$. We assume that $v_{2m_1+1} = e$ be a unit vector field in the complementary orthogonal distribution $\mathbf{D}^\perp$. Then

$$\begin{aligned}\delta^{\nabla,\nabla^*}(\mathbf{D})(p) &= \sum_{i=1}^{2m_1} g(\mathbf{S}(v_i, e)e, v_i) \\ &= \frac{\overline{c}m_1}{2} + \frac{1}{2}\sum_{i=1}^{2m_1}[g(h(e,e), h^*(v_i, v_i)) \\ &\quad + g(h^*(e,e), h(v_i, v_i)) - 2g(h(v_i, e), h^*(v_i, e))].\end{aligned} \tag{33}$$

On the other hand, we derive

$$\begin{aligned}&\frac{1}{2}\sum_{i=1}^{2m_1}[g(h(e,e), h^*(v_i, v_i)) + g(h^*(e,e), h(v_i, v_i))] \\ &= \frac{(2m_1+1)^2}{2}g(\overrightarrow{H}, \overrightarrow{H}^*) - \frac{(2m_1)^2}{2}g(\overrightarrow{H}_{\mathbf{D}}, \overrightarrow{H}^*_{\mathbf{D}}) - \frac{1}{2}g(h(e,e), h^*(e,e)).\end{aligned} \tag{34}$$

From (33) and (34), we get

$$\begin{aligned}\delta^{\nabla,\nabla^*}(\mathbf{D})(p) &= \frac{\overline{c}m_1}{2} + \frac{(2m_1+1)^2}{2}g(\overrightarrow{H}, \overrightarrow{H}^*) - \frac{(2m_1)^2}{2}g(\overrightarrow{H}_{\mathbf{D}}, \overrightarrow{H}^*_{\mathbf{D}}) \\ &\quad - \frac{1}{2}g(h(e,e), h^*(e,e)) - 2\sum_{i=1}^{2m_1} g(h(v_i, e), h^*(v_i, e)).\end{aligned}$$

By a similar argument as in Theorem 4.1, we derive the CR $\delta-$invariant $\delta^0(\mathbf{D})$ of N with respect to ∇^0 as

$$2\delta^0(\mathbf{D})(p) \quad = \quad \overline{c}m_2 + (2m_1+1)^2\overrightarrow{H}^{02} - (2m_1)^2\overrightarrow{H}^{02}_{\mathbf{D}} - h^{02}_{\mathbf{D}^\perp} - 2\sum_{i=1}^{2m_1} g(h^0(v_i, e), h^0(v_i, e)).$$

By combining the last two relations, we find that

$$\begin{aligned}\delta^{\nabla,\nabla^*}(\mathbf{D})(p) &= 2\delta^0(\mathbf{D})(p) - \frac{(2m_1+1)^2}{4}(\overrightarrow{H}^2 + \overrightarrow{H}^{*2}) - \frac{\bar{c}m_1}{2} \\ &+\frac{1}{4}[g(h(e,e),h(e,e)) + g(h^*(e,e),h^*(e,e))] \\ &+\frac{1}{2}\sum_{i=1}^{2m_1}[g(h(v_i,e),h(v_i,e)) + g(h^*(v_i,e),h^*(v_i,e))],\end{aligned}$$

or

$$\delta^{\nabla,\nabla^*}(\mathbf{D})(p) \geq 2\delta^0(\mathbf{D})(p) - \frac{(2m_1+1)^2}{4}(\overrightarrow{H}^2 + \overrightarrow{H}^{*2}) - \frac{\bar{c}m_1}{2}. \tag{35}$$

Our assertion follows from (31), (32) and (35). □

Theorem 4.4. *Let N be a real statistical hypersurface in a holomorphic statistical manifold $(\overline{N},\overline{\nabla},\overline{g},\mathbf{J})$ of constant holomorphic sectional curvature $\bar{c}$ such that $\dim(\mathbf{D}) = 2m_1$ and $\dim(\mathbf{D}^\perp) = 1$. Then the equality case of the inequality (31) holds if and only if*

(a) N is $\mathbf{D}$-minimal with respect to ∇ and ∇^*,
(b) there exists an orthonormal frame $\{v_1,\ldots,v_{2m_1}\}$ of $\mathbf{D}$ such that
  (i) $h(v_i,e) = 0$, $h^*(v_i,e) = 0$, $\forall i = 1,\ldots,2m_1$,
  (ii) $h(e,e) = 0$, $h^*(e,e) = 0$.

5. Generic Statistical Submersion

For the theory of statistical submersion, we follow [1, 2, 13, 14, 20–22]:

Definition 5.1. *[20] Let $(\overline{N},\overline{g},\overline{\nabla})$ and (N',g',∇') be two statistical manifolds. Then a Riemannian submersion $\phi:\overline{N}\to N'$ is said to be a statistical submersion if*

$$\phi_*(\overline{\nabla}_X Y)_p = (\nabla'_{X'}Y')_{\phi(p)},$$

for all basic vector fields X,Y on $\overline{N}$ $\phi-$related to X' and Y' on N', and $p\in\overline{N}$.

The vector fields in $ker\phi_*$ are tangent to the fibres $\phi^{-1}(p)$, $p\in N'$ and are called vertical vector fields. The vectors which are orthogonal to the vertical distribution (or orthogonal to fibers), denoted by $(ker\phi_*)^\perp$, are said to be horizontal. We set the projection mappings on the distributions $ker\phi_*$ and $(ker\phi_*)^\perp$ by $\mathbb{V}$ and $\mathbb{H}$, respectively. The tensor fields of type (1,2) are denoted by $\mathbb{T}$ (resp. $\mathbb{T}^*$), and $\mathbb{A}$ (resp. $\mathbb{A}^*$) and defined by [20]

$$\begin{aligned}\mathbb{T}_X Y &= \mathbb{H}\overline{\nabla}_{\mathbb{V}X}\mathbb{V}Y + \mathbb{V}\overline{\nabla}_{\mathbb{V}X}\mathbb{H}Y, \\ \mathbb{T}^*_X Y &= \mathbb{H}\overline{\nabla}^*_{\mathbb{V}X}\mathbb{V}Y + \mathbb{V}\overline{\nabla}^*_{\mathbb{V}X}\mathbb{H}Y, \\ \mathbb{A}_X Y &= \mathbb{H}\overline{\nabla}_{\mathbb{H}X}\mathbb{V}Y + \mathbb{V}\overline{\nabla}_{\mathbb{H}X}\mathbb{H}Y, \\ \mathbb{A}^*_X Y &= \mathbb{H}\overline{\nabla}^*_{\mathbb{H}X}\mathbb{V}Y + \mathbb{V}\overline{\nabla}^*_{\mathbb{H}X}\mathbb{H}Y,\end{aligned}$$

for all vector fields X and Y on $\overline{N}$. For the geometric properties of $\mathbb{T}$ (resp. $\mathbb{T}^*$), and $\mathbb{A}$ (resp. $\mathbb{A}^*$), see [20].

Lemma 5.2. *[20] For all $X,Y\in\Gamma(ker\phi_*)$ and $U,V\in\Gamma((ker\phi_*)^\perp)$, we have*

$$\begin{aligned}&\overline{\nabla}_U V = \mathbb{T}_U V + \mathbb{V}\overline{\nabla}_U V, \quad \overline{\nabla}^*_U V = \mathbb{T}^*_U V + \mathbb{V}\overline{\nabla}^*_U V, \\ &\overline{\nabla}_U X = \mathbb{H}\overline{\nabla}_U X + \mathbb{T}_U X, \quad \overline{\nabla}^*_U X = \mathbb{H}\overline{\nabla}^*_U X + \mathbb{T}^*_U X, \\ &\overline{\nabla}_X U = \mathbb{A}_X U + \mathbb{V}\overline{\nabla}_X U, \quad \overline{\nabla}^*_X U = \mathbb{A}^*_X U + \mathbb{V}\overline{\nabla}^*_X U, \\ &\overline{\nabla}_X Y = \mathbb{H}\overline{\nabla}_X Y + \mathbb{A}_X Y, \quad \overline{\nabla}^*_X Y = \mathbb{H}\overline{\nabla}^*_X Y + \mathbb{A}^*_X Y.\end{aligned}$$

Moreover, if X is basic, then $\mathbb{H}\overline{\nabla}_U X = \mathbb{A}_X U$ and $\mathbb{H}\overline{\nabla}^*_U X = \mathbb{A}^*_X U$.

Definition 5.3. *Let $\overline{N}$ be a holomorphic statistical manifold with Riemannian metric $\overline{g}$ and an almost complex structure $\mathbf{J}$ and N' be a holomorphic statistical manifold with Riemannian metric g' and an almost complex structure $\mathbf{J}'$. A statistical submersion $\phi : \overline{N} \to N'$ is said to be a holomorphic statistical submersion if ϕ is a holomorphic map, that is, $\phi_* \circ \mathbf{J} = \mathbf{J}' \circ \phi_*$.*

We give the statistical version of the definition of a generic Riemannian submersion.

Definition 5.4. *Let $\overline{N}$ be a holomorphic statistical manifold with Riemannian metric $\overline{g}$ and an almost complex structure $\mathbf{J}$ and N' be a statistical manifold with Riemannian metric g'. A statistical submersion $\phi : \overline{N} \to N'$ is called a generic statistical submersion if there exists a differentiable distribution $\mathbf{D}_1 \subset \ker\phi_*$ such that*

$$\ker\phi_* = \mathbf{D}_1 \oplus \mathbf{D}_2, \quad \mathbf{J}\mathbf{D}_1 = \mathbf{D}_1,$$

where $\mathbf{D}_2$ is the orthogonal complement of $\mathbf{D}_1$, and is purely real distribution on the fibres of the submersion ϕ.

Note that a statistical submersion $\phi : \overline{N} \to N'$ is said to be a generic statistical submersion from a holomorphic statistical manifold onto a statistical manifold if ϕ is a generic Riemannian submersion.

For all $X \in \Gamma(\ker \phi_*)$, we put

$$\mathbf{J}X = PX + FX,$$

where $PX \in \Gamma(\mathbf{D}_1)$ and $FX \in \Gamma((\mathrm{Ker}\ \phi_*)^\perp)$. So, the decomposition of $(\ker \phi_*)^\perp$ is as follows:

$$(\ker\phi_*)^\perp = F\mathbf{D}_2 \oplus \nu,$$

where ν is invariant under $\mathbf{J}$. For all $V \in \Gamma((\ker \phi_*)^\perp)$, we have

$$\mathbf{J}V = BV + CV,$$

where $BV \in \Gamma(\mathbf{D}_2)$ and $CV \in \Gamma(\nu)$.

Here, we investigate the integrability of the distributions $\mathbf{D}_1$ and $\mathbf{D}_2$, which arise from the definition of a generic statistical submersion from a holomorphic statistical manifold onto a statistical manifold.

Theorem 5.5. *If $\phi : \overline{N} \to N'$ is a generic statistical submersion from a holomorphic statistical manifold $(\overline{N}, \overline{g}, \mathbf{J}, \overline{\nabla})$ onto a statistical manifold (N', g', ∇'), then the following conditions are equivalent:*

- *(a)* *$\mathbf{D}_1$ is integrable,*
- *(b)* *$\overline{g}(\mathbb{T}_X\mathbf{J}Y, FZ) = \overline{g}(\mathbb{T}_Y\mathbf{J}X, FZ)$,*
- *(b)** *$\overline{g}(\mathbb{T}^*_X\mathbf{J}Y, FZ) = \overline{g}(\mathbb{T}^*_Y\mathbf{J}X, FZ)$,*

for all $X, Y \in \Gamma(\mathbf{D}_1)$ and $Z \in \Gamma(\mathbf{D}_2)$.

Proof. For all $X, Y \in \Gamma(\mathbf{D}_1)$, we have

$$\begin{aligned} \mathbf{J}[X, Y] &= \mathbf{J}(\overline{\nabla}^*_X Y - \overline{\nabla}^*_Y X) = \overline{\nabla}_X\mathbf{J}Y - \overline{\nabla}_Y\mathbf{J}X \\ &= \mathbb{T}_X\mathbf{J}Y + \mathbb{V}\overline{\nabla}_X\mathbf{J}Y - \mathbb{T}_Y\mathbf{J}X - \mathbb{V}\overline{\nabla}_Y\mathbf{J}X, \end{aligned} \tag{36}$$

where we have used Lemma 5.2. By the hypothesis of the theorem, we get

$$\overline{g}(\mathbf{J}[X, Y], FZ) = \overline{g}(\mathbb{V}\overline{\nabla}_X\mathbf{J}Y - \mathbb{V}\overline{\nabla}_Y\mathbf{J}X, FZ),$$

for all $Z \in \Gamma(\mathbf{D}_2)$. Since, $\mathbb{V}\overline{\nabla}_X\mathbf{J}Y - \mathbb{V}\overline{\nabla}_Y\mathbf{J}X \in \Gamma(\ker \phi_*)$ and $FZ \in \Gamma((\ker \phi_*)^\perp)$, which imply that $\overline{g}([X, Y], Z) = 0$. From this, we conclude that $[X, Y] \in \Gamma(\mathbf{D}_1)$. Thus, $\mathbf{D}_1$ is integrable.

On the other hand, we assume $\mathbf{D}_1$ is integrable. By comparing the vertical and horizontal components of relation (36), we get our assertion (b). This shows that (a) and (b) are equivalent. Similarly, we can show that (a) and (b)* are equivalent. □

Theorem 5.6. *If $\phi : \overline{N} \to N'$ is a generic statistical submersion from a holomorphic statistical manifold $\overline{N}$ onto a statistical manifold N', then the following conditions are equivalent:*

(a) $\mathbf{D}_2$ *is integrable,*

(b) $\mathbb{V}\overline{\nabla}_V PU - \mathbb{V}\overline{\nabla}_U PV + \mathbb{T}_V FU - \mathbb{T}_U FV \in \Gamma(\mathbf{D}_2)$,

*(b)** $\mathbb{V}\overline{\nabla}^*_V PU - \mathbb{V}\overline{\nabla}^*_U PV + \mathbb{T}^*_V FU - \mathbb{T}^*_U FV \in \Gamma(\mathbf{D}_2)$,

for all $U, V \in \Gamma(\mathbf{D}_2)$.

Proof. Since, ker ϕ_* is integrable, then $[U, V] \in \Gamma(\ker \phi_*)$, for all $U, V \in \Gamma(\mathbf{D}_2)$. Thus, for all $Z \in \Gamma(\mathbf{D}_1)$, we have

$$
\begin{aligned}
\overline{g}([U, V], \mathbf{J}Z) &= -g(\mathbf{J}^2[U, V], \mathbf{J}Z) \\
&= -g(\mathbf{J}(\mathbf{J}\overline{\nabla}UV - \mathbf{J}\overline{\nabla}VU), \mathbf{J}Z) \\
&= -g(\mathbf{J}(\overline{\nabla}^* U\mathbf{J}V - \overline{\nabla}^* V\mathbf{J}U), \mathbf{J}Z) \\
&= g(\mathbf{J}(\overline{\nabla}^* V\mathbf{J}U), \mathbf{J}Z) - g(\mathbf{J}(\overline{\nabla}^* U\mathbf{J}V), \mathbf{J}Z) \\
&= g(B(\mathbb{T}^* VPU - \mathbb{T}^* UPV + \mathbb{A}^* FUV - \mathbb{A}^* FVU), \mathbf{J}Z) \\
&\quad + g(P(V\overline{\nabla}^* VPU - V\overline{\nabla}^* UPV + \mathbb{T}^* VFU - \mathbb{T}^* UFV), \mathbf{J}Z).
\end{aligned}
$$

Hence, $\mathbf{D}_2$ is integrable if and only if (b) holds. Similarly, we can show that (a) and (b)* are equivalent. □

Remark 5.7. *Theorems 4.1 and 4.3 can be studied for the generic statistical submersions.*

Acknowledgements: The authors are highly thankful to the referee for the valuable suggestions. The second author extends his appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through large group Research project under grant number RGP2/429/44.

References

[1] N. Abe, K. Hasegawa, *An affine submersion with horizontal distribution and its applications*, Diff. Geom. Appl. **14** (2001), 235-250.
[2] S. Ali, T. Fatima, *Generic Riemannian Submersions*, Tamkang J. Math. **44(4)** (2013), 395-409.
[3] F.R. Al-Solamy, B.-Y. Chen, S. Deshmukh, *Two optimal inequalities for anti-holomorphic submanifolds and their applications*, Taiwanese J. Math. **18** (2014), 199-217.
[4] F. Al-Solamy, B.-Y. Chen, S. Deshmukh, *Erratum to: Two optimal inequalities for anti-holomorphic submanifolds and their applications*, Taiwanese J. Math. **22** (2018), 615-616.
[5] S. Amari, *Differential-Geometrical Methods in Statistics*, (Volume 28) Lecture Notes in Statistics, Springer-Verlag: New York, USA, 1985.
[6] M.E. Aydin, A. Mihai, I. Mhai, *Some Inequalities on Submanifolds in Statistical Manifolds of Constant Curvature* Filomat **29(3)** (2015), 465-477.
[7] M.E. Aydin, A. Mihai, I. Mhai, *Generalized Wintgen inequality for statistical submanifolds in statistical manifolds of constant curvature*, Bull. Math. Sci. **7** (2017), 155-166.
[8] B.-Y. Chen, *An optimal inequality for CR-warped products in complex space forms involving CR δ-invariant*, Int. J. Math. **23(3)** (2012), 1250045.
[9] H. Furuhata, I. Hasegawa, *Submanifold theory in holomorphic statistical manifolds*, In: Dragomir, S., Shahid, M.H., Al-Solamy, F.R. (eds.) Geometry of Cauchy-Riemann Submanifolds, Springer, Singapore, 2016, 179–215.
[10] H. Furuhata, I. Hasegawa, Y. Okuyama, K. Sato, M.H. Shahid, *Sasakian statistical manifolds*, J. Geom. Phys. **117** (2017), 179-186.
[11] I. Mihai, I. Presură, *An inequality for contact CR-submanifolds in Sasakian space forms*, J. Geom. **109(34)** (2018).
[12] M.N. Boyom, A.N. Siddiqui, W.A. Mior Othman, M.H. Shahid, *Classification of totally umbilical CR-statistical submanifolds in holomorphic statistical manifolds with constant holomorphic curvature*, In: Nielsen F., Barbaresco F. (eds) Geometric Science of Information. GSI 2017. Lecture Notes in Computer Science, **10589**. Springer, Cham, 2017.
[13] B. O'Neill, *The fundamental equations of a submersion*, Mich. Math. J. **13** (1966), 459-469.
[14] B. O'Neill, *Submersions and geodesics*, Duke Math. J. **34** (1967), 363-373.
[15] B. Opozda, *Bochner's technique for statistical structures*, Ann. Glob. Anal. Geom. **48(4)** (2015), 357-395.
[16] B. Opozda, *A sectional curvature for statistical structures*, Linear Algebra Appl. **497** (2016), 134-161.
[17] A.N. Siddiqui, F.R. Al-Solamy, M.H. Shahid, I. Mihai, *On CR-statistical submanifolds of holomorphic statistical manifolds* Filomat **35 (11)** (2021), 3571-3584.

[18] A.N. Siddiqui, M.H. Shahid, *On totally real statistical submanifolds*, Filomat **32(13)** (2018), 4473-4483.
[19] A.N. Siddiqui, A. Ali, A.H. Alkhaldi, *Chen optimal inequalities of CR-warped products of generalized Sasakian space form*, J. Taibah Univ. Sci., **14 (1)** (2020), 322-330, DOI: 10.1080/16583655.2020.1738704.
[20] K. Takano, *Statistical manifolds with almost complex structures and its statistical submersions*, Tensor N.S. **65** (2004), 128-142.
[21] K. Takano, *Statistical manifolds with almost contact structures and its statistical submersions*, J. Geom. **85** (2006), 171-187.
[22] A.D. Vilcu, G.E. Vilcu, *Statistical Manifolds with almost Quaternionic Structures and Quaternionic Kähler-like Statistical Submersions*, Entropy **17** (2015), 6213-6228.
[23] P.W. Vos, *Fundamental equations for statistical submanifolds with applications to the Bartlett correction*, Ann. Inst. Stat. Math. **41(3)** (1989), 429-450.
[24] K. Yano, M. Kon, *Structures on manifolds*, (Volume 3) Worlds Scientific, Singapore, 1985.