Filomat 38:4 (2024), 1367–1374
https://doi.org/10.2298/FIL2404367G

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On convergence of certain Hermite-type operators

Vijay Gupta[a], Deepak Malik[a]

[a]*Department of Mathematics, Netaji Subhas University of Technology, Sector 3 Dwarka, New Delhi 110078, India*

Abstract. In this paper we study approximation properties of a discrete operators based on modified Hermite polynomials. We find moments using the concept of moment generating function and estimate convergence results for such operators.

1. Introduction

Krech [13] introduced a generalization of Szász-Mirakyan operators by means of Hermite polynomials, for $f \in C[0,\infty)$ as follows

$$(G_n^\alpha f)(x) \quad = \quad e^{-(nx+\alpha x^2)} \sum_{k=0}^{\infty} \frac{x^k}{k!} H_k(n,\alpha) f\left(\frac{k}{n}\right), \qquad \alpha, x \geq 0, n \in N, \tag{1}$$

where $H_k(n,\alpha)$ are the modified Hermite polynomials of two variables defined in [13] as

$$H_k(n,\alpha) := k! \sum_{m=0}^{[\frac{k}{2}]} \frac{\alpha^m}{m!} \frac{n^{k-2m}}{(k-2m)!}, n, k \in N.$$

As a special case when $\alpha = 0$, then $H_k(n,0) = n^k$ and we get the Szász-Mirakyan operators defined by

$$(S_n f)(x) := (G_n^0 f)(x) \quad = \quad \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} f\left(\frac{k}{n}\right), \qquad x \geq 0, n \in N, \tag{2}$$

Also, we have the connection for negative α viz. $H_k(2n,-1) = H_k(n)$, where $H_k(n) := k! \sum_{s=0}^{[\frac{k}{2}]} \frac{(-1)^s}{s!} \frac{(2n)^{k-2s}}{(k-2s)!}$. In this form the operator (3) takes the form:

$$(G_n^{-1} f)(x) \quad = \quad e^{-(2nx-x^2)} \sum_{k=0}^{\infty} \frac{x^k}{k!} H_k(n) f\left(\frac{k}{2n}\right), \qquad x \geq 0, n \in N. \tag{3}$$

This form corresponding to $\alpha = -1$ provides an operator, which is not positive but linear operator.

The present paper is an extension of the work of [13], we deal with some other approximation properties of the operators $G_n^\alpha, \alpha \geq 0$. Firstly, we estimate moment producing function and using it we find moments. We estimate some direct results including pointwise convergence and quantitative asymptotic formulae in terms of different moduli of continuity.

2020 *Mathematics Subject Classification*. 41A10; 41A25; 41A30.
Keywords. Hermite polynomials; Generating function; Exponential preservation.
Received: 27 March 2023; Accepted: 11 August 2023
Communicated by Snežana Č. Živkovic-Zlatanović
Email addresses: vijaygupta2001@hotmail.com (Vijay Gupta), malikdeepak1510@gmail.com (Deepak Malik)

2. Moment Estimation

In [9] authors estimated the moments of some other operators, using different methods. For the operators G_n^α, we provide in this section the moments.

Lemma 2.1. *For A is a real number, $n \in N$ and $\alpha \geq 0$, the moment producing function of the operators G_n^α is given by*

$$(G_n^\alpha e^{At})(x) = e^{(e^{A/n}-1)[nx+\alpha x^2(e^{A/n}+1)]}.$$

In particular, we have

$$\begin{aligned}
(G_n^\alpha e_0)(x) &= 1, \\
(G_n^\alpha e_1)(x) &= x + \frac{2\alpha x^2}{n}, \\
(G_n^\alpha e_2)(x) &= x^2 + \frac{4\alpha^2 x^3 + 4\alpha n x^3}{n^2} + \frac{(nx + 4\alpha x^2)}{n^2}, \\
(G_n^\alpha e_3)(x) &= x^3 + \frac{(nx + 8\alpha x^2 + 3n^2x^2 + 18\alpha n x^3 + 24\alpha^2 x^4 + 6\alpha n^2 x^4 + 12\alpha^2 n x^5 + 8\alpha^3 x^6)}{n^3}, \\
(G_n^\alpha e_4)(x) &= \frac{1}{n^4}\Big[nx + 16\alpha x^2 + 7n^2x^2 + 64\alpha n x^3 + 6n^3x^3 + 112\alpha^2 x^4 + 48\alpha n^2 x^4 + n^4x^4 \\
&\quad + 120\alpha^2 n x^5 + 8\alpha n^3 x^5 + 96\alpha^3 x^6 + 24\alpha^2 n^2 x^6 + 32\alpha^3 n x^7 + 16\alpha^4 x^8\Big].
\end{aligned}$$

Proof. By generating function of two variable Hermite polynomial

$$e^{nx+\alpha x^2} = \sum_{k=0}^{\infty} \frac{x^k}{k!} H_k(n, \alpha)$$

and by definition (3), we have

$$\begin{aligned}
(G_n^\alpha e^{At})(x) &= e^{-(nx+\alpha x^2)} \sum_{k=0}^{\infty} \frac{x^k}{k!} H_k(n, \alpha) e^{Ak/n} \\
&= e^{(e^{A/n}-1)[nx+\alpha x^2(e^{A/n}+1)]}.
\end{aligned}$$

The moments follow by using the relation

$$(G_n^\alpha e_r)(x) = \left[\frac{\partial^r}{\partial A^r} e^{(e^{A/n}-1)[nx+\alpha x^2(e^{A/n}+1)]}\right]_{A=0}, \quad r = 0, 1, 2, \ldots$$

□

Lemma 2.2. *If we denote $\mu_{n,m}^\alpha(x) := (G_n^\alpha(e_1 - x e_0)^m)(x), m \in N \cup \{0\}$, then*

$$\mu_{n,m}^\alpha(x) = \left[\frac{\partial^m}{\partial A^m} e^{(e^{A/n}-1)[nx+\alpha x^2(e^{A/n}+1)]-Ax}\right]_{A=0}.$$

In particular, we get

$$\begin{aligned}
\mu^{\alpha}_{n,0}(x) &= 1,\\
\mu^{\alpha}_{n,1}(x) &= \frac{2\alpha x^2}{n},\\
\mu^{\alpha}_{n,2}(x) &= \frac{nx+4\alpha x^2+4\alpha^2x^4}{n^2},\\
\mu^{\alpha}_{n,3}(x) &= \frac{nx+8\alpha x^2+6\alpha n x^3+24\alpha^2x^4+8\alpha^3x^6}{n^3},\\
\mu^{\alpha}_{n,4}(x) &= \frac{nx+16\alpha x^2+3n^2x^2+32\alpha n x^3+112\alpha^2x^4+24\alpha^2nx^5+96\alpha^3x^6+16\alpha^4x^8}{n^4}.
\end{aligned}$$

In general we have $\mu^{\alpha}_{n,m}(x) = O(n^{-[(m+1)/2]})$.

The central moments easily follow by the property of the moment generating function from Lemma 2.1 by using change of scale law.

Lemma 2.3. *For* $A > 0$ *real,* $n \in N$ *and* $\alpha \geq 0$*, we have*

$$(G^{\alpha}_n(t-x)^2e^{At})(x) \leq C(A,\alpha,x)\mu^{\alpha}_{n,2}(x),$$

where $C(A,\alpha,x) = e^{x(e^A-1)+\alpha x^2(e^{2A}-1)}\left[e^{4A}+x^2(e^A-1)^2+4\alpha x^3e^{2A}(e^A-1)\right]$.

Proof. By Lemma 2.2, differentiating moment producing function both sides with respect to A, we get

$$\begin{aligned}
(G^{\alpha}_n te^{At})(x) &= e^{(e^{A/n}-1)[nx+\alpha x^2(e^{A/n}+1)]}\left[xe^{A/n}+\frac{2\alpha x^2e^{2A/n}}{n}\right]\\
(G^{\alpha}_n t^2e^{At})(x) &= e^{(e^{A/n}-1)[nx+\alpha x^2(e^{A/n}+1)]}\left[\left(xe^{A/n}+\frac{2\alpha x^2e^{2A/n}}{n}\right)^2+\left(\frac{xe^{A/n}}{n}+\frac{4\alpha x^2e^{2A/n}}{n^2}\right)\right].
\end{aligned}$$

Finally by linearity property using the fact $n(e^{A/n}-1) < e^A-1$, we get

$$\begin{aligned}
(G^{\alpha}_n(t-x)^2e^{At})(x) &= e^{(e^{A/n}-1)[nx+\alpha x^2(e^{A/n}+1)]}\Big[x^2(e^{A/n}-1)^2+\frac{4\alpha x^2e^{2A/n}}{n^2}(1+\alpha x^2e^{2A/n})\\
&\quad +\frac{xe^{A/n}}{n}+\frac{4\alpha x^3e^{3A/n}}{n}-\frac{4\alpha x^3e^{2A/n}}{n}\Big]\\
&\leq C(A,\alpha,x)\mu^{\alpha}_{n,2}(x),
\end{aligned}$$

where $C(A,\alpha,x) = e^{x(e^A-1)+\alpha x^2(e^{2A}-1)}\left[e^{4A}+x^2(e^A-1)^2+4\alpha x^3e^{2A}(e^A-1)\right]$. □

3. Convergence

This section consists of some direct results, we first find in the following result the point-wise convergence of the operator (3):

Theorem 3.1. *If* $f \in C^B[0,\infty)$ *(class of bounded continuous functions on* $[0,\infty)$*), then*

$$\lim_{n\to\infty}(G^{\alpha}_n f(t))(x) = f(x),$$

and

$$\lim_{n\to\infty}(G^{\alpha}_{mn} f(nt))\left(\frac{x}{n}\right) = (S_m f)(x),$$

where $(S_m f)(x)$ *denotes the Szász-Mirakyan operator given by (2).*

Proof. For the operators G_n^α defined in (3), we have

$$\begin{aligned}\lim_{n\to\infty}(G_n^\alpha e^{ist})(x) &= \lim_{n\to\infty} e^{(e^{is/n}-1)[nx+\alpha x^2(e^{is/n}+1)]}\\ &= e^{isx} = (Id \;\; e^{ist})(x).\end{aligned}$$

Next, we have

$$\begin{aligned}\lim_{n\to\infty}(G_{mn}^\alpha e^{isnt})\left(\frac{x}{n}\right) &= \lim_{n\to\infty} e^{(e^{is/m}-1)\left[mx+\frac{\alpha x^2}{n^2}(e^{is/m}+1)\right]}\\ &= e^{mx(e^{is/m}-1)}\\ &= e^{-mx}\sum_{k=0}^{\infty}\frac{(mx)^k}{k!}e^{isk/m} = (S_m e^{ist})(x),\end{aligned}$$

where S_m is the Szász-Mrakyan operators defined in (2). Thus by [1, Th. 2.1] (also see [2, Th. 1]), we get the desired result. □

Let B_2 be the set of all functions satisfying $|f(x)| \leq c(1+x^2)$, for certain constant c depending on f. Also suppose C_2^k denotes the subspace of all functions $f \in B_2$ satisfying $\lim_{|x|\to\infty}|f(x)|/(1+x^2) = k$, for certain constant k, then following [5], [12], the moduli is defined by

$$\Omega(g,\delta) = \sup_{\substack{|e|\leq\delta\\ x\in R}}\frac{\left|g(x+e)-g(x)\right|}{(1+e^2)(1+x^2)}, \quad g \in C_2^k.$$

Theorem 3.2. *Let $G_n^\alpha : \widetilde{E} \to C[0,\infty)$, where $\widetilde{E}$ represents the space of the functions f having polynomial growth. If f and $f'' \in C_2^k \cap \widetilde{E}$, then*

$$\begin{aligned}&\left|(G_n^\alpha f)(x) - f(x) - \frac{2\alpha x^2}{n}f'(x) - \frac{nx+4\alpha x^2+4\alpha^2x^4}{2n^2}f''(x)\right|\\ \leq\;& 8\left(1+x^2\right)\left(\frac{nx+4\alpha x^2+4\alpha^2x^4}{n^2}\right)\Omega\left(f'', \left(\frac{\mu_{n,6}^\alpha(x)}{\mu_{n,2}^\alpha(x)}\right)^{1/4}\right).\end{aligned}$$

Proof. Using

$$f(u) = f(x) + (u-x)f'(x) + \frac{(u-x)^2}{2!}f''(x) + \frac{(u-x)^2}{2!}\varepsilon(u,x),$$

$\varepsilon(u,x) = f''(\xi) - f''(x), x < \xi < u$ and $\varepsilon(u,x) \to 0$ as $u \to x$ and applying Lemma 2.3, we have

$$\begin{aligned}&\left|(G_n^\alpha f)(x) - f(x) - \frac{2\alpha x^2}{n}f'(x) - \frac{nx+4\alpha x^2+4\alpha^2x^4}{n^2}f''(x)\right|\\ \leq\;& \frac{1}{2}\left(G_n^\alpha|\varepsilon(u,x)|(u-x)^2\right)(x). \qquad (4)\end{aligned}$$

Following [3], for any $\delta > 0$ the following inequality holds:

$$|\varepsilon(u,x)| = \left|f''(\xi) - f''(x)\right| \leq 4\left(1+\frac{(t-x)^4}{\delta^4}\right)\left(1+\delta^2\right)^2\left(1+x^2\right)\Omega(f,\delta).$$

Next, in the factor $\left(1+\delta^2\right)^2$ taking $\delta \leq 1$, we get

$$\left(G_n^\alpha|\varepsilon(u,x)|(u-x)^2\right)(x) \leq 16\mu_{n,2}^\alpha(x)\left(1+x^2\right)\left(1+\frac{\mu_{n,6}^\alpha(x)}{\delta^4\mu_{n,2}^\alpha(x)}\right)\Omega(f'',\delta).$$

Selecting $\delta^4 = \mu^{\alpha}_{n,6}(x)/\mu^{\alpha}_{n,2}(x) \leq 1$, where we can apply Lemma 2.2. Therefore by (4), the required inequality is proved. □

We consider $C^*[0,\infty)$, as subspace of $C[0,\infty)$ consisting of real-valued continuous functions defined in $[0,\infty)$ and $\lim_{x\to\infty} f(x)$ exists and is finite also it is endowed with uniform norm $\|.\|_\infty$.

Theorem 3.3. *If $f \in C^*[0,\infty)$ then the following inequality holds:*

$$\|(G^{\alpha}_n f) - f\|_\infty \leq 2\omega^*\left(f, \sqrt{2\alpha_1(n) + \alpha_2(n)}\right),$$

where $\alpha_s(n)$, $s = 1,2$ approaches to zero for a large enough n and

$$\omega^*(h,\delta) = \sup_{\substack{|e^{-t}-e^{-x}|\leq\delta \\ x,t\geq 0}} |h(t) - h(x)|.$$

Proof. Following [11], if we denote

$$\|(G^{\alpha}_n e^{-st}) - e^{-sx}\|_\infty = \alpha_s(n), \quad s = 0,1,2,$$

then for $f \in C^*[0,\infty)$, we have

$$\|(G^{\alpha}_n f) - f\|_\infty \leq \|f\|_\infty \alpha_0(n) + (2 + \alpha_0(n)) \cdot \omega^*\left(f, \sqrt{\alpha_0(n) + 2\alpha_1(n) + \alpha_2(n)}\right).$$

Here we have $\alpha_0(n) = 0$, because G^{α}_n preserve constant function. Next for $s = 1,2$, we can write

$$\begin{aligned}(G^{\alpha}_n e^{-st})(x) &= e^{(e^{-s/n}-1)[nx+\alpha x^2(e^{-s/n}+1)]} \\ &= \exp\left((e^{-s/n} - 1)[nx + \alpha x^2(e^{-s/n} + 1)]\right) \\ &= \exp\left(\sum_{r=1}^{\infty} \frac{(-s)^r}{r!n^{r-1}} x + \sum_{r=1}^{\infty} \frac{(-s)^r}{r!n^r} \alpha x^2(e^{-s/n} + 1)\right),\end{aligned}$$

implying

$$\alpha_s(n) = \| \exp\left(\sum_{r=2}^{\infty} \frac{(-s)^r}{r!n^{r-1}} x + \sum_{r=1}^{\infty} \frac{(-s)^r}{r!n^r} \alpha x^2(e^{-s/n} + 1)\right)\|_\infty.$$

the right hand side approaches to zero for sufficiently large n.
This leads to the proof of result. □

Theorem 3.4. *If f, f'' belongs to $C^*[0,\infty)$, then for $x \in [0,\infty)$, the following inequality exists:*

$$\begin{aligned}&\left| n\left[(G^{\alpha}_n f)(x) - f(x)\right] - 2\alpha x^2 f'(x) - \frac{nx + 4\alpha x^2 + 4\alpha^2 x^4}{2n} f''(x) \right| \\ \leq\ & \frac{1}{2}\omega^*\left(f'', \frac{1}{\sqrt{n}}\right)\left[\frac{nx + 4\alpha x^2 + 4\alpha^2 x^4}{n}\right. \\ &+ \left(\frac{[nx + 16\alpha x^2 + 3n^2x^2 + 32\alpha n x^3 + 112\alpha^2 x^4 + 24\alpha^2 n x^5 + 96\alpha^3 x^6 + 16\alpha^4 x^8]^{1/2}}{n}\right) \\ &\left.\left[n^2\left(G^{\alpha}_n\left(\frac{1}{e^x} - \frac{1}{e^t}\right)^4\right)(x)\right]^{1/2}\right].\end{aligned}$$

Proof. By Taylor's expansion

$$f(t) = \sum_{k=0}^{2} \frac{(t-x)^k}{k!} f^{(k)}(x) + \eta(t,x)(e_1 - xe_0)^2,$$

where

$$\eta(t,x) := \frac{f''(\delta) - f''(x)}{2}.$$

Using Lemma 2.2, we have

$$\left| n\left[(G_n^\alpha f)(x) - f(x)\right] - 2\alpha x^2 f'(x) - \frac{nx + 4\alpha x^2 + 4\alpha^2 x^4}{2n} f''(x) \right| \leq n\left(G_n^\alpha \left|\eta(t,x)\right| (e_1 - e_0 x)^2\right)(x).$$

Using the property of $\omega^*(f,\delta)$ given by

$$|f(t) - f(x)| \leq \left(1 + \frac{1}{\delta^2}\left(\frac{1}{e^x} - \frac{1}{e^t}\right)^2\right)\omega^*(f,\delta),$$

we obtain

$$\left|\eta(t,x)\right| \leq \frac{1}{2}\left(1 + \frac{1}{\delta^2}\left(\frac{1}{e^x} - \frac{1}{e^t}\right)^2\right)\omega^*\left(f'',\delta\right).$$

Applying Cauchy-Schwarz inequality and choosing $\delta = \frac{1}{\sqrt{n}}$, we get

$$n\left(G_n^\alpha \left|\eta(t,x)\right| (e_1 - xe_0)^2\right)(x) \leq \frac{1}{2}\omega^*\left(f'', n^{-1/2}\right)\left[n\mu_{n,2}^\alpha(x) + \sqrt{n^2\left(G_n^\alpha\left(e^{-x} - e^{-t}\right)^4\right)(x)}\sqrt{n^2\mu_{n,4}^\alpha(x)}\right].$$

Finally, we obtain the required outcome by using Lemma 2.2. □

Remark 3.5. *The convergence of the operator G_n^α in the preceding theorem takes place for n large enough. By using Lemma 2.2, we have*

$$\begin{aligned}
&\lim_{n\to\infty} n^2\left(G_n^\alpha\left(e^{-x} - e^{-t}\right)^4\right)(x) \\
&= \lim_{n\to\infty} n^2\Big[e^{-4x} - 4e^{-3x}e^{(e^{-1/n}-1)[nx+\alpha x^2(e^{-1/n}+1)]} + 6e^{-2x}e^{(e^{-2/n}-1)[nx+\alpha x^2(e^{-2/n}+1)]} \\
&\quad -4e^{-x}e^{(e^{-3/n}-1)[nx+\alpha x^2(e^{-3/n}+1)]} + e^{(e^{-4/n}-1)[nx+\alpha x^2(e^{-4/n}+1)]}\Big] \\
&= 3e^{-4x}x^2.
\end{aligned}$$

For $f \in U^*[0,\infty)$ and $A > 0$ be real, let us consider

$$\omega_1(f,\delta,A) = \sup_{u\geq 0, h\leq\delta} |f(u) - f(u+h)|e^{-Au}, \qquad \delta > 0,$$

as the modulus of continuity of first order defined by Ditzian [4] and

$$U^*[0,\infty) := \{f \in C[0,\infty) : \|f\|_A = \sup_{u\geq 0}|f(u)e^{-Au}| < \infty\}.$$

Further, for $\beta \in (0,1]$, consider the following Lipschitz space:

$$Lip_\beta(A) = \{f \in U^*[0,\infty) : \omega_1(f,\delta,A) \leq C\delta^\beta, \ \forall\delta < 1\}.$$

Theorem 3.6. *Let $f \in U^*[0,\infty) \cap C^2[0,\infty)$, $f'' \in Lip_\beta(A)$, $\beta \in (0,1]$, then for $x \in [0,\infty)$, we have*

$$\begin{aligned}&\left|(G_n^\alpha f)(x) - f(x) - \frac{2\alpha x^2}{n} f'(x) - \frac{nx + 4\alpha x^2 + 4\alpha^2 x^4}{2n^2} f''(x)\right| \\ \leq\ & \omega_1(f'', h, A) \cdot \left[2e^{2A,\alpha,x} + C(A,\alpha,x) + \sqrt{C(2A,\alpha,x)}\right] \cdot \left(\frac{nx + 4\alpha x^2 + 4\alpha^2 x^4}{2n^2}\right),\end{aligned}$$

where

$$h = \frac{1}{n}\left(\frac{nx + 16\alpha x^2 + 3n^2x^2 + 32\alpha n x^3 + 112\alpha^2 x^4 + 24\alpha^2 n x^5 + 96\alpha^3 x^6 + 16\alpha^4 x^8}{nx + 4\alpha x^2 + 4\alpha^2 x^4}\right)^{1/2}.$$

Proof. According to Taylor's expansion, η exists between x and t such that

$$f(t) = \sum_{k=0}^{2} \frac{(t-x)^k}{k!} f^{(k)}(x) + \epsilon(t,x)(e_1 - xe_0)^2, \tag{5}$$

where

$$\epsilon(t,x) := \frac{f''(\eta) - f''(x)}{2}.$$

On applying the operator G_n^α to (5) and also using Lemma 2.2, we get

$$\begin{aligned}&\left|(G_n^\alpha f)(x) - f(x) - \frac{2\alpha x^2}{n} f'(x) - \frac{nx + 4\alpha x^2 + 4\alpha^2 x^4}{2n^2} f''(x)\right| \\ \leq\ & \left(G_n^\alpha |\epsilon(t,x)|(e_1 - xe_0)^2\right)(x).\end{aligned} \tag{6}$$

Following [10, pp. 101], we get

$$\left|\epsilon(t,x)|(e_1 - xe_0)^2\right| \leq \frac{1}{2}\left(e^{2Ax} + e^{At}\right)\left(1 + \frac{|t-x|}{h}\right)\omega_1(f'', h, A)\,|t-x|^2.$$

Consequently

$$\begin{aligned}\left(G_n^\alpha |\epsilon(t,x)|(e_1 - xe_0)^2\right)(x) &\leq \frac{1}{2}\left[G_n^\alpha\left(\left(e^{2Ax} + e^{At}\right).\left(|t-x|^2 + \frac{|t-x|^3}{h}\right); x\right)\right]\omega_1(f'', h, A) \\ &\leq \frac{e^{2Ax}}{2}\left[(G_n^\alpha (t-x)^2)(x) + \frac{1}{h}(G_n^\alpha |t-x|^3)(x)\right] \\ &\quad + \frac{1}{2}\left[(G_n^\alpha e^{At}(t-x)^2)(x) + \frac{1}{h}(G_n^\alpha e^{At}|t-x|^3)(x)\right]\omega_1(f'', h, A) \\ &\leq \frac{e^{2Ax}}{2}\left[\mu_{n,2}^\alpha(x) + \frac{1}{h}(\mu_{n,2}^\alpha(x).\mu_{n,4}^\alpha(x))^{1/2}\right] \\ &\quad + \frac{1}{2}\left[(G_n^\alpha e^{At}(t-x)^2)(x) + \frac{1}{h}((G_n^\alpha e^{2At}(t-x)^2)(x).\mu_{n,4}^\alpha(x))^{1/2}\right]\omega_1(f'', h, A).\end{aligned}$$

Considering $h := \sqrt{\frac{\mu_{n,4}^\alpha(x)}{\mu_{n,2}^\alpha(x)}}$, using Lemma 2.2, Lemma 2.3 in above and substituting in (6), we obtain the desired result. □

Remark 3.7. *In the recent years certain operators have been studied which preserve exponential functions, see for instance [6], [8]. We can extend the results of present paper along such lines. Also other results can be achieved similar to [7]. This work may appear in forthcoming papers.*

Acknowledgements The authors are thankful to the reviewers for valuable suggestions leading to better presentation of the paper.

Conflict of interest

The authors declare that they have no conflict of interest.

References

[1] A. M. Acu, M. Heilmann, I. Rasa and A. Seserman, Poisson approximation to the binomial distribution: Extensions to the convergence of positive operators. arXiv:2208.08326.

[2] A. M. Acu, V. Gupta, I. Rasa and F. Sofonea, Convergence of special sequences of semi-exponential operators, Mathematics 10 (16)(2022), 2978: https://doi.org/10.3390/math10162978

[3] O. Agratini, A. Aral and E. Deniz, On two classes of approximation processes of integral type, Positivity 21(2017), 1189-1199.

[4] Z. Ditzian, On global inverse theorems of Szász and Baskakov operators, Can. J. Math., 31(2)(1979), 255–263.

[5] A. D. Gadzhiev, Theorems of Korovkin type, Math. Notes 20(5)(1676), 995–998.

[6] V. Gupta and A. Aral, A note on Szász–Mirakyan–Kantorovich type operators preserving e^{-x}, Positivity 22 (2018), 415–423. https://doi.org/10.1007/s11117-017-0518-5

[7] V. Gupta, C. P. Muraru and V. A. Radu, Convergence of certain hybrid operators, Rocky Mountain J. Math. 51(4)(2021), 1249–1258.

[8] V. Gupta and A-J López-Moreno, Phillips operators preserving arbitrary exponential functions, e^{at}, e^{bt}, Filomat 32 (14) (2018), 5071–5082.

[9] V. Gupta and M. Th. Rassias, Moments of Linear Positive Operators and Approximation, Series: SpringerBriefs in Mathematics, Springer Nature Switzerland AG (2019).

[10] V. Gupta and G. Tachev, Approximation with Positive Linear Operators and Linear Combinations, Series: Developments in Mathematics, Vol. **50** Springer, 2017.

[11] A. Holhoş, The rate of approximation of functions in an infinite interval by positive linear operators, Stud. Univ. Babeş-Bolyai Math. 2 (2010), 133-142.

[12] N. Ispir, On modified Baskakov operators on weighted spaces, Turkish J. Math. 25 (2001), 355–365.

[13] G. Krech, A note on some positive linear operators associated with the Hermite polynomials, Carpathian J. Math. 32(1)(2016), 71–77.