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Nonlinear maps preserving sums of triple products on $*$-algebras

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be two unital complex $*$-algebras such that $\mathcal{A}$ has a nontrivial projection. In this paper, we study the structure of bijective nonlinear maps $\Phi : \mathcal{A} \to \mathcal{B}$ preserving sum of triple products $\alpha_1 abc + \alpha_2 a^*cb^* + \alpha_3 ba^*c + \alpha_4 cab^* + \alpha_5 bca + \alpha_6 cb^*a^*$, where the scalars $\{\alpha_k\}_{k=1}^6$ are complex numbers satisfying some conditions.

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be two complex $*$-algebras and $\{\alpha_k\}_{k=1}^6$ arbitrary complex numbers. We say that a nonlinear map $\Phi : \mathcal{A} \to \mathcal{B}$ *preserves sum of triple products* $\alpha_1 abc + \alpha_2 a^*cb^* + \alpha_3 ba^*c + \alpha_4 cab^* + \alpha_5 bca + \alpha_6 cb^*a^*$ if

$$\begin{aligned}&\Phi(\alpha_1 abc + \alpha_2 a^*cb^* + \alpha_3 ba^*c + \alpha_4 cab^* + \alpha_5 bca + \alpha_6 cb^*a^*)\\ &= \alpha_1\Phi(a)\Phi(b)\Phi(c) + \alpha_2\Phi(a)^*\Phi(c)\Phi(b)^* + \alpha_3\Phi(b)\Phi(a)^*\Phi(c)\\ &+ \alpha_4\Phi(c)\Phi(a)\Phi(b)^* + \alpha_5\Phi(b)\Phi(c)\Phi(a) + \alpha_6\Phi(c)\Phi(b)^*\Phi(a)^*, \qquad (1)\end{aligned}$$

for all elements $a, b, c \in \mathcal{A}$.

These kinds of maps are related to nonlinear maps preserving Lie (resp. mixed, Jordan) triple $*$-product which have been studied by many authors (for example, see the works [3], [4], [5], [6], [7], [8] and the references therein). In particular, Li et al. [3], Zhang [7] and Zhao and Li [8] studied the structure of the bijective nonlinear maps preserving Lie (mixed, Jordan) triple $*$-products on factor von Neumann algebras, respectively. These maps satisfy (1), for convenient scalars α_k ($k = 1, 2, \cdots, 6$). Motivated by these results and inspired by the works of Ferreira and Marietto [1] and [2], in this paper we will study the structure of bijective nonlinear maps Φ, from a unital prime $*$-algebra $\mathcal{A}$ having a nontrivial projection to a unital $*$-algebra $\mathcal{B}$, preserving sum of triple products $\alpha_1 abc + \alpha_2 a^*cb^* + \alpha_3 ba^*c + \alpha_4 cab^* + \alpha_5 bca + \alpha_6 cb^*a^*$, where $\{\alpha_k\}_{k=1}^6$ are complex numbers satisfying certain conditions. At the end of this paper, we make a contribution to the problem of structure characterization of the nonlinear maps preserving triple $*$-products, on unital $*$-algebras, as originated from the works [3], [7] and [8].

Our main result reads as follows.

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Theorem 1.1. *Let* $\{\alpha_k\}_{k=1}^{6}$ *be complex numbers satisfying the conditions* $\alpha_1 + \alpha_3 + \alpha_5 \neq 0$, $\alpha_2 + \alpha_4 + \alpha_6 \neq 0$ *and* $|\alpha_1 + \alpha_3 + \alpha_5| - |\alpha_2 + \alpha_4 + \alpha_6| \neq 0$, $\mathcal{A}$ *and* $\mathcal{B}$ *two unital complex* $*$*-algebras with* $1_{\mathcal{A}}$ *and* $1_{\mathcal{B}}$ *their multiplicative identities, respectively, and such that* $\mathcal{A}$ *is prime and has a nontrivial projection. Then every bijective nonlinear map* $\Phi : \mathcal{A} \to \mathcal{B}$ *preserving sum of triple products* $\alpha_1 abc + \alpha_2 a^*cb^* + \alpha_3 ba^*c + \alpha_4 cab^* + \alpha_5 bca + \alpha_6 cb^*a^*$ *is additive. In addition, if (i)* $\Phi(1_{\mathcal{A}})$ *is a projection of* $\mathcal{B}$ *and (ii)* $\Phi((\alpha_2 + \alpha_4 + \alpha_6)a) = (\alpha_2 + \alpha_4 + \alpha_6)\Phi(a)$, *for all element* $a \in \mathcal{A}$, *then* Φ *is a* $*$*-ring isomorphism.*

2. The proof of main result

In order to prove the Theorem 1.1 we need to prove several Claims. We begin with a Claim, whose proof is easy and is omitted.

Claim 2.1. $\Phi(0) = 0$.

The following well known result will be used throughout this paper: Let p_1 be any nontrivial projection of $\mathcal{A}$ and write $p_2 = 1_{\mathcal{A}} - p_1$. Then $\mathcal{A}$ has a Peirce decomposition $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$, where $\mathcal{A}_{ij} = p_i\mathcal{A}p_j$ $(i, j = 1, 2)$, satisfying the following multiplicative relations: $\mathcal{A}_{ij}\mathcal{A}_{kl} \subseteq \delta_{jk}\mathcal{A}_{il}$, where δ_{jk} is the *Kronecker delta function*.

Claim 2.2. *For every* $a_{ii} \in \mathcal{A}_{ii}$, $b_{ij} \in \mathcal{A}_{ij}$ *and* $c_{ji} \in \mathcal{A}_{ji}$ $(i \neq j; i, j = 1, 2)$ *we have: (i)* $\Phi(a_{ii} + b_{ij}) = \Phi(a_{ii}) + \Phi(b_{ij})$ *and (ii)* $\Phi(a_{ii} + c_{ji}) = \Phi(a_{ii}) + \Phi(c_{ji})$.

Proof. Let $u = u_{ii} + u_{ij} + u_{ji} + u_{jj} = \Phi^{-1}(\Phi(a_{ii} + b_{ij}) - \Phi(a_{ii}) - \Phi(b_{ij})) \in \mathcal{A}$ $(i \neq j; i, j = 1, 2)$. According to the definition of Φ, we have

$$\begin{aligned}
&\Phi(\alpha_1 1_{\mathcal{A}} u p_j + \alpha_2 1_{\mathcal{A}}^* p_j u^* + \alpha_3 u 1_{\mathcal{A}}^* p_j + \alpha_4 p_j 1_{\mathcal{A}} u^* + \alpha_5 u p_j 1_{\mathcal{A}} + \alpha_6 p_j u^* 1_{\mathcal{A}}^*)\\
&= \alpha_1\Phi(1_{\mathcal{A}})\Phi(u)\Phi(p_j) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(p_j)\Phi(u)^* + \alpha_3\Phi(u)\Phi(1_{\mathcal{A}})^*\Phi(p_j)\\
&+ \alpha_4\Phi(p_j)\Phi(1_{\mathcal{A}})\Phi(u)^* + \alpha_5\Phi(u)\Phi(p_j)\Phi(1_{\mathcal{A}}) + \alpha_6\Phi(p_j)\Phi(u)^*\Phi(1_{\mathcal{A}})^*\\
&= \alpha_1\Phi(1_{\mathcal{A}})\Phi(a_{ii} + b_{ij})\Phi(p_j) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(p_j)\Phi(a_{ii} + b_{ij})^*\\
&+ \alpha_3\Phi(a_{ii} + b_{ij})\Phi(1_{\mathcal{A}})^*\Phi(p_j) + \alpha_4\Phi(p_j)\Phi(1_{\mathcal{A}})\Phi(a_{ii} + b_{ij})^*\\
&+ \alpha_5\Phi(a_{ii} + b_{ij})\Phi(p_j)\Phi(1_{\mathcal{A}}) + \alpha_6\Phi(p_j)\Phi(a_{ii} + b_{ij})^*\Phi(1_{\mathcal{A}})^*\\
&- \alpha_1\Phi(1_{\mathcal{A}})\Phi(a_{ii})\Phi(p_j) - \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(p_j)\Phi(a_{ii})^* - \alpha_3\Phi(a_{ii})\Phi(1_{\mathcal{A}})^*\Phi(p_j)\\
&- \alpha_4\Phi(p_j)\Phi(1_{\mathcal{A}})\Phi(a_{ii})^* - \alpha_5\Phi(a_{ii})\Phi(p_j)\Phi(1_{\mathcal{A}}) - \alpha_6\Phi(p_j)\Phi(a_{ii})^*\Phi(1_{\mathcal{A}})^*\\
&- \alpha_1\Phi(1_{\mathcal{A}})\Phi(b_{ij})\Phi(p_j) - \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(p_j)\Phi(b_{ij})^* - \alpha_3\Phi(b_{ij})\Phi(1_{\mathcal{A}})^*\Phi(p_j)\\
&- \alpha_4\Phi(p_j)\Phi(1_{\mathcal{A}})\Phi(b_{ij})^* - \alpha_5\Phi(b_{ij})\Phi(p_j)\Phi(1_{\mathcal{A}}) - \alpha_6\Phi(p_j)\Phi(b_{ij})^*\Phi(1_{\mathcal{A}})^*\\
&= \Phi(\alpha_1 1_{\mathcal{A}}(a_{ii} + b_{ij})p_j + \alpha_2 1_{\mathcal{A}}^* p_j(a_{ii} + b_{ij})^* + \alpha_3(a_{ii} + b_{ij})1_{\mathcal{A}}^* p_j\\
&+ \alpha_4 p_j 1_{\mathcal{A}}(a_{ii} + b_{ij})^* + \alpha_5(a_{ii} + b_{ij})p_j 1_{\mathcal{A}} + \alpha_6 p_j(a_{ii} + b_{ij})^* 1_{\mathcal{A}}^*)\\
&- \Phi(\alpha_1 1_{\mathcal{A}} a_{ii} p_j + \alpha_2 1_{\mathcal{A}}^* p_j a_{ii}^* + \alpha_3 a_{ii} 1_{\mathcal{A}}^* p_j + \alpha_4 p_j 1_{\mathcal{A}} a_{ii}^* + \alpha_5 a_{ii} p_j 1_{\mathcal{A}}\\
&+ \alpha_6 p_j a_{ii}^* 1_{\mathcal{A}}^*) - \Phi(\alpha_1 1_{\mathcal{A}} b_{ij} p_j + \alpha_2 1_{\mathcal{A}}^* p_j b_{ij}^* + \alpha_3 b_{ij} 1_{\mathcal{A}}^* p_j + \alpha_4 p_j 1_{\mathcal{A}} b_{ij}^*\\
&+ \alpha_5 b_{ij} p_j 1_{\mathcal{A}} + \alpha_6 p_j b_{ij}^* 1_{\mathcal{A}}^*) = 0.
\end{aligned}$$

Since Φ is injective we deduce that $\alpha_1 1_{\mathcal{A}} u p_j + \alpha_2 1_{\mathcal{A}}^* p_j u^* + \alpha_3 u 1_{\mathcal{A}}^* p_j + \alpha_4 p_j 1_{\mathcal{A}} u^* + \alpha_5 u p_j 1_{\mathcal{A}} + \alpha_6 p_j u^* 1_{\mathcal{A}}^* = 0$. It follows from this that $(\alpha_1 + \alpha_3 + \alpha_5)u_{ij} + (\alpha_2 + \alpha_4 + \alpha_6)u_{ij}^* + (\alpha_1 + \alpha_3 + \alpha_5)u_{jj} + (\alpha_2 + \alpha_4 + \alpha_6)u_{jj}^* = 0$ (2). Next, applying the involution $*$ to the identity (2) we get $(\overline{\alpha_2 + \alpha_4 + \alpha_6})u_{ij} + (\overline{\alpha_1 + \alpha_3 + \alpha_5})u_{ij}^* + (\overline{\alpha_2 + \alpha_4 + \alpha_6})u_{jj} + (\overline{\alpha_1 + \alpha_3 + \alpha_5})u_{jj}^* = 0$ (3). Also, multiplying (2) by the scalar $(\overline{\alpha_1 + \alpha_3 + \alpha_5})$, (3) by the scalar $(\alpha_2 + \alpha_4 + \alpha_6)$ and subtracting the resulting identities, we arrive at $(|\alpha_1 + \alpha_3 + \alpha_5|^2 - |\alpha_2 + \alpha_4 + \alpha_6|^2)(u_{ij} + u_{jj}) = 0$. This shows that $u_{ij} + u_{jj} = 0$ which results that $u_{ij} = 0$ and $u_{jj} = 0$, by directness of the Peirce decomposition. Now, we have

$$\Phi(\alpha_1 1_{\mathcal{A}} u p_i + \alpha_2 1_{\mathcal{A}}^* p_i u^* + \alpha_3 u 1_{\mathcal{A}}^* p_i + \alpha_4 p_i 1_{\mathcal{A}} u^* + \alpha_5 u p_i 1_{\mathcal{A}} + \alpha_6 p_i u^* 1_{\mathcal{A}}^*)$$

$$= \alpha_1\Phi(1_{\mathcal{A}})\Phi(u)\Phi(p_i) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(p_i)\Phi(u)^* + \alpha_3\Phi(u)\Phi(1_{\mathcal{A}})^*\Phi(p_i)$$
$$+ \alpha_4\Phi(p_i)\Phi(1_{\mathcal{A}})\Phi(u)^* + \alpha_5\Phi(u)\Phi(p_i)\Phi(1_{\mathcal{A}}) + \alpha_6\Phi(p_i)\Phi(u)^*\Phi(1_{\mathcal{A}})^*$$
$$= \alpha_1\Phi(1_{\mathcal{A}})\Phi(a_{ii} + b_{ij})\Phi(p_i) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(p_i)\Phi(a_{ii} + b_{ij})^*$$
$$+ \alpha_3\Phi(a_{ii} + b_{ij})\Phi(1_{\mathcal{A}})^*\Phi(p_i) + \alpha_4\Phi(p_i)\Phi(1_{\mathcal{A}})\Phi(a_{ii} + b_{ij})^*$$
$$+ \alpha_5\Phi(a_{ii} + b_{ij})\Phi(p_i)\Phi(1_{\mathcal{A}}) + \alpha_6\Phi(p_i)\Phi(a_{ii} + b_{ij})^*\Phi(1_{\mathcal{A}})^*$$
$$- \alpha_1\Phi(1_{\mathcal{A}})\Phi(a_{ii})\Phi(p_i) - \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(p_i)\Phi(a_{ii})^* - \alpha_3\Phi(a_{ii})\Phi(1_{\mathcal{A}})^*\Phi(p_i)$$
$$- \alpha_4\Phi(p_i)\Phi(1_{\mathcal{A}})\Phi(a_{ii})^* - \alpha_5\Phi(a_{ii})\Phi(p_i)\Phi(1_{\mathcal{A}}) - \alpha_6\Phi(p_i)\Phi(a_{ii})^*\Phi(1_{\mathcal{A}})^*$$
$$- \alpha_1\Phi(1_{\mathcal{A}})\Phi(b_{ij})\Phi(p_i) - \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(p_i)\Phi(b_{ij})^* - \alpha_3\Phi(b_{ij})\Phi(1_{\mathcal{A}})^*\Phi(p_i)$$
$$- \alpha_4\Phi(p_i)\Phi(1_{\mathcal{A}})\Phi(b_{ij})^* - \alpha_5\Phi(b_{ij})\Phi(p_i)\Phi(1_{\mathcal{A}}) - \alpha_6\Phi(p_i)\Phi(b_{ij})^*\Phi(1_{\mathcal{A}})^*$$
$$= \Phi(\alpha_1 1_{\mathcal{A}}(a_{ii} + b_{ij})p_i + \alpha_2 1^*_{\mathcal{A}}p_i(a_{ii} + b_{ij})^* + \alpha_3(a_{ii} + b_{ij})1^*_{\mathcal{A}}p_i$$
$$+ \alpha_4 p_i 1_{\mathcal{A}}(a_{ii} + b_{ij})^* + \alpha_5(a_{ii} + b_{ij})p_i 1_{\mathcal{A}} + \alpha_6 p_i(a_{ii} + b_{ij})^* 1^*_{\mathcal{A}})$$
$$- \Phi(\alpha_1 1_{\mathcal{A}}a_{ii}p_i + \alpha_2 1^*_{\mathcal{A}}p_i a^*_{ii} + \alpha_3 a_{ii}1^*_{\mathcal{A}}p_i + \alpha_4 p_i 1_{\mathcal{A}}a^*_{ii} + \alpha_5 a_{ii}p_i 1_{\mathcal{A}}$$
$$+ \alpha_6 p_i a^*_{ii}1^*_{\mathcal{A}}) - \Phi(\alpha_1 1_{\mathcal{A}}b_{ij}p_i + \alpha_2 1^*_{\mathcal{A}}p_i b^*_{ij} + \alpha_3 b_{ij}1^*_{\mathcal{A}}p_i + \alpha_4 p_i 1_{\mathcal{A}}b^*_{ij}$$
$$+ \alpha_5 b_{ij}p_i 1_{\mathcal{A}} + \alpha_6 p_i b^*_{ij}1^*_{\mathcal{A}}) = 0$$

which leads directly to identity $\alpha_1 1_{\mathcal{A}}up_i + \alpha_2 1^*_{\mathcal{A}}p_i u^* + \alpha_3 u1^*_{\mathcal{A}}p_i + \alpha_4 p_i 1_{\mathcal{A}}u^* + \alpha_5 up_i 1_{\mathcal{A}} + \alpha_6 p_i u^* 1^*_{\mathcal{A}} = 0$. This imples that $(\alpha_1+\alpha_3+\alpha_5)u_{ii}+(\alpha_2+\alpha_4+\alpha_6)u^*_{ii}+(\alpha_1+\alpha_3+\alpha_5)u_{ji}+(\alpha_2+\alpha_4+\alpha_6)u^*_{ji} = 0$ (4). Next, applying the involution $*$ to the identity (4) we get $(\overline{\alpha_2 + \alpha_4 + \alpha_6})u_{ii} + (\overline{\alpha_1 + \alpha_3 + \alpha_5})u^*_{ii} + (\overline{\alpha_2 + \alpha_4 + \alpha_6})u_{ji} + (\overline{\alpha_1 + \alpha_3 + \alpha_5})u^*_{ji} = 0$ (5). Also, multiplying (4) by the scalar $(\overline{\alpha_1 + \alpha_3 + \alpha_5})$, (5) by the scalar $(\alpha_2 + \alpha_4 + \alpha_6)$ and subtracting the resulting identities, we obtain $(|\alpha_1 + \alpha_3 + \alpha_5|^2 - |\alpha_2 + \alpha_4 + \alpha_6|^2)(u_{ii} + u_{ji}) = 0$. This results that $u_{ii} + u_{ji} = 0$ which shows that $u_{ii} = 0$ and $u_{ji} = 0$. As a consequence, we conclude that $u = 0$. This proves the case (i). In order to prove the case (ii), let $u = u_{ii} + u_{ij} + u_{ji} + u_{jj} = \Phi^{-1}(\Phi(a_{ii} + c_{ji}) - \Phi(a_{ii}) - \Phi(c_{ji})) \in \mathcal{A}$ $(i \neq j; i, j = 1, 2)$. Then, we have

$$\Phi(\alpha_1 1_{\mathcal{A}}up_j + \alpha_2 1^*_{\mathcal{A}}p_j u^* + \alpha_3 u1^*_{\mathcal{A}}p_j + \alpha_4 p_j 1_{\mathcal{A}}u^* + \alpha_5 up_j 1_{\mathcal{A}} + \alpha_6 p_j u^* 1^*_{\mathcal{A}})$$
$$= \alpha_1\Phi(1_{\mathcal{A}})\Phi(u)\Phi(p_j) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(p_j)\Phi(u)^* + \alpha_3\Phi(u)\Phi(1_{\mathcal{A}})^*\Phi(p_j)$$
$$+ \alpha_4\Phi(p_j)\Phi(1_{\mathcal{A}})\Phi(u)^* + \alpha_5\Phi(u)\Phi(p_j)\Phi(1_{\mathcal{A}}) + \alpha_6\Phi(p_j)\Phi(u)^*\Phi(1_{\mathcal{A}})^*$$
$$= \alpha_1\Phi(1_{\mathcal{A}})\Phi(a_{ii} + c_{ji})\Phi(p_j) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(p_j)\Phi(a_{ii} + c_{ji})^*$$
$$+ \alpha_3\Phi(a_{ii} + c_{ji})\Phi(1_{\mathcal{A}})^*\Phi(p_j) + \alpha_4\Phi(p_j)\Phi(1_{\mathcal{A}})\Phi(a_{ii} + c_{ji})^*$$
$$+ \alpha_5\Phi(a_{ii} + c_{ji})\Phi(p_j)\Phi(1_{\mathcal{A}}) + \alpha_6\Phi(p_j)\Phi(a_{ii} + c_{ji})^*\Phi(1_{\mathcal{A}})^*$$
$$- \alpha_1\Phi(1_{\mathcal{A}})\Phi(a_{ii})\Phi(p_j) - \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(p_j)\Phi(a_{ii})^* - \alpha_3\Phi(a_{ii})\Phi(1_{\mathcal{A}})^*\Phi(p_j)$$
$$- \alpha_4\Phi(p_j)\Phi(1_{\mathcal{A}})\Phi(a_{ii})^* - \alpha_5\Phi(a_{ii})\Phi(p_j)\Phi(1_{\mathcal{A}}) - \alpha_6\Phi(p_j)\Phi(a_{ii})^*\Phi(1_{\mathcal{A}})^*$$
$$- \alpha_1\Phi(1_{\mathcal{A}})\Phi(c_{ji})\Phi(p_j) - \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(p_j)\Phi(c_{ji})^* - \alpha_3\Phi(c_{ji})\Phi(1_{\mathcal{A}})^*\Phi(p_j)$$
$$- \alpha_4\Phi(p_j)\Phi(1_{\mathcal{A}})\Phi(c_{ji})^* - \alpha_5\Phi(c_{ji})\Phi(p_j)\Phi(1_{\mathcal{A}}) - \alpha_6\Phi(p_j)\Phi(c_{ji})^*\Phi(1_{\mathcal{A}})^*$$
$$= \Phi(\alpha_1 1_{\mathcal{A}}(a_{ii} + c_{ji})p_j + \alpha_2 1^*_{\mathcal{A}}p_j(a_{ii} + c_{ji})^* + \alpha_3(a_{ii} + c_{ji})1^*_{\mathcal{A}}p_j$$
$$+ \alpha_4 p_j 1_{\mathcal{A}}(a_{ii} + c_{ji})^* + \alpha_5(a_{ii} + c_{ji})p_j 1_{\mathcal{A}} + \alpha_6 p_j(a_{ii} + c_{ji})^* 1^*_{\mathcal{A}})$$
$$- \Phi(\alpha_1 1_{\mathcal{A}}a_{ii}p_j + \alpha_2 1^*_{\mathcal{A}}p_j a^*_{ii} + \alpha_3 a_{ii}1^*_{\mathcal{A}}p_j + \alpha_4 p_j 1_{\mathcal{A}}a^*_{ii} + \alpha_5 a_{ii}p_j 1_{\mathcal{A}}$$
$$+ \alpha_6 p_j a^*_{ii}1^*_{\mathcal{A}}) - \Phi(\alpha_1 1_{\mathcal{A}}c_{ji}p_j + \alpha_2 1^*_{\mathcal{A}}p_j c^*_{ji} + \alpha_3 c_{ji}1^*_{\mathcal{A}}p_j + \alpha_4 p_j 1_{\mathcal{A}}c^*_{ji}$$
$$+ \alpha_5 c_{ji}p_j 1_{\mathcal{A}} + \alpha_6 p_j c^*_{ji}1^*_{\mathcal{A}}) = 0$$

which implies that $\alpha_1 1_{\mathcal{A}}up_j + \alpha_2 1^*_{\mathcal{A}}p_j u^* + \alpha_3 u1^*_{\mathcal{A}}p_j + \alpha_4 p_j 1_{\mathcal{A}}u^* + \alpha_5 up_j 1_{\mathcal{A}} + \alpha_6 p_j u^* 1^*_{\mathcal{A}} = 0$. From this last identity we get $(\alpha_1+\alpha_3+\alpha_5)u_{ij}+(\alpha_2+\alpha_4+\alpha_6)u^*_{ij}+(\alpha_1+\alpha_3+\alpha_5)u_{jj}+(\alpha_2+\alpha_4+\alpha_6)u^*_{jj} = 0$ (6) and by applying the involution on the identity (6) we get $(\overline{\alpha_2 + \alpha_4 + \alpha_6})u_{ij} + (\overline{\alpha_1 + \alpha_3 + \alpha_5})u^*_{ij} + (\overline{\alpha_2 + \alpha_4 + \alpha_6})u_{jj} + (\overline{\alpha_1 + \alpha_3 + \alpha_5})u^*_{jj} = 0$ (7). Multiplying (6) by the scalar $(\overline{\alpha_1 + \alpha_3 + \alpha_5})$, (7) by the scalar $(\alpha_2 + \alpha_4 + \alpha_6)$ and subtracting the resulting identities we arrive at identity $(|\alpha_1 + \alpha_3 + \alpha_5|^2 - |\alpha_2 + \alpha_4 + \alpha_6|^2)(u_{ij} + u_{jj}) = 0$ which shows that $u_{ij} + u_{jj} = 0$.

Next, note that

$$
\begin{aligned}
&\Phi(\alpha_1 1_{\mathcal{A}} p_j u + \alpha_2 1_{\mathcal{A}}^* u p_j^* + \alpha_3 p_j 1_{\mathcal{A}}^* u + \alpha_4 u 1_{\mathcal{A}} p_j^* + \alpha_5 p_j u 1_{\mathcal{A}} + \alpha_6 u p_j^* 1_{\mathcal{A}}^*)\\
&= \alpha_1\Phi(1_{\mathcal{A}})\Phi(p_j)\Phi(u) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(u)\Phi(p_j)^* + \alpha_3\Phi(p_j)\Phi(1_{\mathcal{A}})^*\Phi(u)\\
&+ \alpha_4\Phi(u)\Phi(1_{\mathcal{A}})\Phi(p_j)^* + \alpha_5\Phi(p_j)\Phi(u)\Phi(1_{\mathcal{A}}) + \alpha_6\Phi(u)\Phi(p_j)^*\Phi(1_{\mathcal{A}})^*\\
&= \alpha_1\Phi(1_{\mathcal{A}})\Phi(p_j)\Phi(a_{ii} + c_{ji}) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(a_{ii} + c_{ji})\Phi(p_j)^*\\
&+ \alpha_3\Phi(p_j)\Phi(1_{\mathcal{A}})^*\Phi(a_{ii} + c_{ji}) + \alpha_4\Phi(a_{ii} + c_{ji})\Phi(1_{\mathcal{A}})\Phi(p_j)^*\\
&+ \alpha_5\Phi(p_j)\Phi(a_{ii} + c_{ji})\Phi(1_{\mathcal{A}}) + \alpha_6\Phi(a_{ii} + c_{ji})\Phi(p_j)^*\Phi(1_{\mathcal{A}})^*\\
&- \alpha_1\Phi(1_{\mathcal{A}})\Phi(p_j)\Phi(a_{ii}) - \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(a_{ii})\Phi(p_j)^* - \alpha_3\Phi(p_j)\Phi(1_{\mathcal{A}})^*\Phi(a_{ii})\\
&- \alpha_4\Phi(a_{ii})\Phi(1_{\mathcal{A}})\Phi(p_j)^* - \alpha_5\Phi(p_j)\Phi(a_{ii})\Phi(1_{\mathcal{A}}) - \alpha_6\Phi(a_{ii})\Phi(p_j)^*\Phi(1_{\mathcal{A}})^*\\
&- \alpha_1\Phi(1_{\mathcal{A}})\Phi(p_j)\Phi(c_{ji}) - \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(c_{ji})\Phi(p_j)^* - \alpha_3\Phi(p_j)\Phi(1_{\mathcal{A}})^*\Phi(c_{ji})\\
&- \alpha_4\Phi(c_{ji})\Phi(1_{\mathcal{A}})\Phi(p_j)^* - \alpha_5\Phi(p_j)\Phi(c_{ji})\Phi(1_{\mathcal{A}}) - \alpha_6\Phi(c_{ji})\Phi(p_j)^*\Phi(1_{\mathcal{A}})^*\\
&= \Phi(\alpha_1 1_{\mathcal{A}} p_j(a_{ii} + c_{ji}) + \alpha_2 1_{\mathcal{A}}^*(a_{ii} + c_{ji})p_j^* + \alpha_3 p_j 1_{\mathcal{A}}^*(a_{ii} + c_{ji})\\
&+ \alpha_4(a_{ii} + c_{ji})1_{\mathcal{A}} p_j^* + \alpha_5 p_j(a_{ii} + c_{ji})1_{\mathcal{A}} + \alpha_6(a_{ii} + c_{ji})p_j^* 1_{\mathcal{A}}^*)\\
&- \Phi(\alpha_1 1_{\mathcal{A}} p_j a_{ii} + \alpha_2 1_{\mathcal{A}}^* a_{ii} p_j^* + \alpha_3 p_j 1_{\mathcal{A}}^* a_{ii} + \alpha_4 a_{ii} 1_{\mathcal{A}} p_j^* + \alpha_5 p_j a_{ii} 1_{\mathcal{A}}\\
&+ \alpha_6 a_{ii} p_j^* 1_{\mathcal{A}}^*) - \Phi(\alpha_1 1_{\mathcal{A}} p_j c_{ji} + \alpha_2 1_{\mathcal{A}}^* c_{ji} p_j^* + \alpha_3 p_j 1_{\mathcal{A}}^* c_{ji} + \alpha_4 c_{ji} 1_{\mathcal{A}} p_j^*\\
&+ \alpha_5 p_j c_{ji} 1_{\mathcal{A}} + \alpha_6 c_{ji} p_j^* 1_{\mathcal{A}}^*) = 0
\end{aligned}
$$

which implies the identity $\alpha_1 1_{\mathcal{A}} p_j u + \alpha_2 1_{\mathcal{A}}^* u p_j^* + \alpha_3 p_j 1_{\mathcal{A}}^* u + \alpha_4 u 1_{\mathcal{A}} p_j^* + \alpha_5 p_j u 1_{\mathcal{A}} + \alpha_6 u p_j^* 1_{\mathcal{A}}^* = 0$. This result that $(\alpha_1 + \alpha_3 + \alpha_5)u_{ji} + (\alpha_2 + \alpha_4 + \alpha_6)u_{ij} + (\sum_{k=1}^{6} \alpha_k)u_{jj} = 0$ which yields $(\alpha_1 + \alpha_3 + \alpha_5)u_{ji} = 0$. By the hypothesis that $\alpha_1 + \alpha_3 + \alpha_5 \neq 0$, we deduce that $u_{ji} = 0$. Finally, from case (i) we have

$$
\begin{aligned}
&\Phi(\alpha_1 1_{\mathcal{A}} u r_{ji} + \alpha_2 1_{\mathcal{A}}^* r_{ji} u^* + \alpha_3 u 1_{\mathcal{A}}^* r_{ji} + \alpha_4 r_{ji} 1_{\mathcal{A}} u^* + \alpha_5 u r_{ji} 1_{\mathcal{A}} + \alpha_6 r_{ji} u^* 1_{\mathcal{A}}^*)\\
&= \alpha_1\Phi(1_{\mathcal{A}})\Phi(u)\Phi(r_{ji}) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(r_{ji})\Phi(u)^* + \alpha_3\Phi(u)\Phi(1_{\mathcal{A}})^*\Phi(r_{ji})\\
&+ \alpha_4\Phi(r_{ji})\Phi(1_{\mathcal{A}})\Phi(u)^* + \alpha_5\Phi(u)\Phi(r_{ji})\Phi(1_{\mathcal{A}}) + \alpha_6\Phi(r_{ji})\Phi(u)^*\Phi(1_{\mathcal{A}})^*\\
&= \alpha_1\Phi(1_{\mathcal{A}})\Phi(a_{ii} + c_{ji})\Phi(r_{ji}) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(r_{ji})\Phi(a_{ii} + c_{ji})^*\\
&+ \alpha_3\Phi(a_{ii} + c_{ji})\Phi(1_{\mathcal{A}})^*\Phi(r_{ji}) + \alpha_4\Phi(r_{ji})\Phi(1_{\mathcal{A}})\Phi(a_{ii} + c_{ji})^*\\
&+ \alpha_5\Phi(a_{ii} + c_{ji})\Phi(r_{ji})\Phi(1_{\mathcal{A}}) + \alpha_6\Phi(r_{ji})\Phi(a_{ii} + c_{ji})^*\Phi(1_{\mathcal{A}})^*\\
&- \alpha_1\Phi(1_{\mathcal{A}})\Phi(a_{ii})\Phi(r_{ji}) - \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(r_{ji})\Phi(a_{ii})^* - \alpha_3\Phi(a_{ii})\Phi(1_{\mathcal{A}})^*\Phi(r_{ji})\\
&- \alpha_4\Phi(r_{ji})\Phi(1_{\mathcal{A}})\Phi(a_{ii})^* - \alpha_5\Phi(a_{ii})\Phi(r_{ji})\Phi(1_{\mathcal{A}}) - \alpha_6\Phi(r_{ji})\Phi(a_{ii})^*\Phi(1_{\mathcal{A}})^*\\
&- \alpha_1\Phi(1_{\mathcal{A}})\Phi(c_{ji})\Phi(r_{ji}) - \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(r_{ji})\Phi(c_{ji})^* - \alpha_3\Phi(c_{ji})\Phi(1_{\mathcal{A}})^*\Phi(r_{ji})\\
&- \alpha_4\Phi(r_{ji})\Phi(1_{\mathcal{A}})\Phi(c_{ji})^* - \alpha_5\Phi(c_{ji})\Phi(r_{ji})\Phi(1_{\mathcal{A}}) - \alpha_6\Phi(r_{ji})\Phi(c_{ji})^*\Phi(1_{\mathcal{A}})^*\\
&= \Phi(\alpha_1 1_{\mathcal{A}}(a_{ii} + c_{ji})r_{ji} + \alpha_2 1_{\mathcal{A}}^* r_{ji}(a_{ii} + c_{ji})^* + \alpha_3(a_{ii} + c_{ji})1_{\mathcal{A}}^* r_{ji}\\
&+ \alpha_4 r_{ji} 1_{\mathcal{A}}(a_{ii} + c_{ji})^* + \alpha_5(a_{ii} + c_{ji})r_{ji} 1_{\mathcal{A}} + \alpha_6 r_{ji}(a_{ii} + c_{ji})^* 1_{\mathcal{A}}^*)\\
&- \Phi(\alpha_1 1_{\mathcal{A}} a_{ii} r_{ji} + \alpha_2 1_{\mathcal{A}}^* r_{ji} a_{ii}^* + \alpha_3 a_{ii} 1_{\mathcal{A}}^* r_{ji} + \alpha_4 r_{ji} 1_{\mathcal{A}} a_{ii}^* + \alpha_5 a_{ii} r_{ji} 1_{\mathcal{A}}\\
&+ \alpha_6 r_{ji} a_{ii}^* 1_{\mathcal{A}}^*) - \Phi(\alpha_1 1_{\mathcal{A}} c_{ji} r_{ji} + \alpha_2 1_{\mathcal{A}}^* r_{ji} c_{ji}^* + \alpha_3 c_{ji} 1_{\mathcal{A}}^* r_{ji} + \alpha_4 r_{ji} 1_{\mathcal{A}} c_{ji}^*\\
&+ \alpha_5 c_{ji} r_{ji} 1_{\mathcal{A}} + \alpha_6 r_{ji} c_{ji}^* 1_{\mathcal{A}}^*) = 0.
\end{aligned}
$$

As a consequence we get $\alpha_1 1_{\mathcal{A}} u r_{ji} + \alpha_2 1_{\mathcal{A}}^* r_{ji} u^* + \alpha_3 u 1_{\mathcal{A}}^* r_{ji} + \alpha_4 r_{ji} 1_{\mathcal{A}} u^* + \alpha_5 u r_{ji} 1_{\mathcal{A}} + \alpha_6 r_{ji} u^* 1_{\mathcal{A}}^* = 0$ that yields the identity $(\alpha_2 + \alpha_4 + \alpha_6)r_{ji}u_{ii}^* = 0$. By the hypothesis that $\alpha_2 + \alpha_4 + \alpha_6 \neq 0$, we deduce that $r_{ji}u_{ii}^* = 0$ which shows that $u_{ii} = 0$. Therefore we have $u = 0$. □

Claim 2.3. *For every $a_{ii} \in \mathcal{A}_{ii}$, $b_{ij} \in \mathcal{A}_{ij}$, $c_{ji} \in \mathcal{A}_{ji}$ and $d_{jj} \in \mathcal{A}_{jj}$ $(i \neq j; i, j = 1, 2)$ we have:* $\Phi(a_{ii} + b_{ij} + c_{ji} + d_{jj}) = \Phi(a_{ii}) + \Phi(b_{ij}) + \Phi(c_{ji}) + \Phi(d_{jj})$.

Proof. Let $u = u_{ii} + u_{ij} + u_{ji} + u_{jj} = \Phi^{-1}(\Phi(a_{ii} + b_{ij} + c_{ji} + d_{jj}) - \Phi(a_{ii}) - \Phi(b_{ij}) - \Phi(c_{ji}) - \Phi(d_{jj})) = \Phi^{-1}(\Phi(a_{ii} + b_{ij} + c_{ji} + d_{jj}) - \Phi(a_{ii} + c_{ji}) - \Phi(b_{ij} + d_{jj})) \in \mathcal{A}$, by Claim 2.2. Then

$$\begin{aligned}
&\Phi(\alpha_1 1_{\mathcal{A}} u p_j + \alpha_2 1_{\mathcal{A}}^* p_j u^* + \alpha_3 u 1_{\mathcal{A}}^* p_j + \alpha_4 p_j 1_{\mathcal{A}} u^* + \alpha_5 u p_j 1_{\mathcal{A}} + \alpha_6 p_j u^* 1_{\mathcal{A}}^*)\\
&= \alpha_1 \Phi(1_{\mathcal{A}})\Phi(u)\Phi(p_j) + \alpha_2 \Phi(1_{\mathcal{A}})^*\Phi(p_j)\Phi(u)^* + \alpha_3 \Phi(u)\Phi(1_{\mathcal{A}})^*\Phi(p_j)\\
&+ \alpha_4 \Phi(p_j)\Phi(1_{\mathcal{A}})\Phi(u)^* + \alpha_5 \Phi(u)\Phi(p_j)\Phi(1_{\mathcal{A}}) + \alpha_6 \Phi(p_j)\Phi(u)^*\Phi(1_{\mathcal{A}})^*\\
&= \alpha_1 \Phi(1_{\mathcal{A}})\Phi(a_{ii} + b_{ij} + c_{ji} + d_{jj})\Phi(p_j) + \alpha_2 \Phi(1_{\mathcal{A}})^*\Phi(p_j)\Phi(a_{ii} + b_{ij} + c_{ji} + d_{jj})^*\\
&+ \alpha_3 \Phi(a_{ii} + b_{ij} + c_{ji} + d_{jj})\Phi(1_{\mathcal{A}})^*\Phi(p_j) + \alpha_4 \Phi(p_j)\Phi(1_{\mathcal{A}})\Phi(a_{ii} + b_{ij} + c_{ji} + d_{jj})^*\\
&+ \alpha_5 \Phi(a_{ii} + b_{ij} + c_{ji} + d_{jj})\Phi(p_j)\Phi(1_{\mathcal{A}}) + \alpha_6 \Phi(p_j)\Phi(a_{ii} + b_{ij} + c_{ji} + d_{jj})^*\Phi(1_{\mathcal{A}})^*\\
&- \alpha_1 \Phi(1_{\mathcal{A}})\Phi(a_{ii} + c_{ji})\Phi(p_j) - \alpha_2 \Phi(1_{\mathcal{A}})^*\Phi(p_j)\Phi(a_{ii} + c_{ji})^*\\
&- \alpha_3 \Phi(a_{ii} + c_{ji})\Phi(1_{\mathcal{A}})^*\Phi(p_j) - \alpha_4 \Phi(p_j)\Phi(1_{\mathcal{A}})\Phi(a_{ii} + c_{ji})^*\\
&- \alpha_5 \Phi(a_{ii} + c_{ji})\Phi(p_j)\Phi(1_{\mathcal{A}}) - \alpha_6 \Phi(p_j)\Phi(a_{ii} + c_{ji})^*\Phi(1_{\mathcal{A}})^*\\
&- \alpha_1 \Phi(1_{\mathcal{A}})\Phi(b_{ij} + d_{jj})\Phi(p_j) - \alpha_2 \Phi(1_{\mathcal{A}})^*\Phi(p_j)\Phi(b_{ij} + d_{jj})^*\\
&- \alpha_3 \Phi(b_{ij} + d_{jj})\Phi(1_{\mathcal{A}})^*\Phi(p_j) - \alpha_4 \Phi(p_j)\Phi(1_{\mathcal{A}})\Phi(b_{ij} + d_{jj})^*\\
&- \alpha_5 \Phi(b_{ij} + d_{jj})\Phi(p_j)\Phi(1_{\mathcal{A}}) - \alpha_6 \Phi(p_j)\Phi(b_{ij} + d_{jj})^*\Phi(1_{\mathcal{A}})^*\\
&= \Phi(\alpha_1 1_{\mathcal{A}}(a_{ii} + b_{ij} + c_{ji} + d_{jj})p_j + \alpha_2 1_{\mathcal{A}}^* p_j(a_{ii} + b_{ij} + c_{ji} + d_{jj})^*\\
&+ \alpha_3(a_{ii} + b_{ij} + c_{ji} + d_{jj})1_{\mathcal{A}}^* p_j + \alpha_4 p_j 1_{\mathcal{A}}(a_{ii} + b_{ij} + c_{ji} + d_{jj})^*\\
&+ \alpha_5(a_{ii} + b_{ij} + c_{ji} + d_{jj})p_j 1_{\mathcal{A}} + \alpha_6 p_j(a_{ii} + b_{ij} + c_{ji} + d_{jj})^* 1_{\mathcal{A}}^*)\\
&- \Phi(\alpha_1 1_{\mathcal{A}}(a_{ii} + c_{ji})p_j + \alpha_2 1_{\mathcal{A}}^* p_j(a_{ii} + c_{ji})^* + \alpha_3(a_{ii} + c_{ji})1_{\mathcal{A}}^* p_j\\
&+ \alpha_4 p_j 1_{\mathcal{A}}(a_{ii} + c_{ji})^* + \alpha_5(a_{ii} + c_{ji})p_j 1_{\mathcal{A}} + \alpha_6 p_j(a_{ii} + c_{ji})^* 1_{\mathcal{A}}^*)\\
&- \Phi(\alpha_1 1_{\mathcal{A}}(b_{ij} + d_{jj})p_j + \alpha_2 1_{\mathcal{A}}^* p_j(b_{ij} + d_{jj})^* + \alpha_3(b_{ij} + d_{jj})1_{\mathcal{A}}^* p_j\\
&+ \alpha_4 p_j 1_{\mathcal{A}}(b_{ij} + d_{jj})^* + \alpha_5(b_{ij} + d_{jj})p_j 1_{\mathcal{A}} + \alpha_6 p_j(b_{ij} + d_{jj})^* 1_{\mathcal{A}}^*) = 0.
\end{aligned}$$

This implies that $\alpha_1 1_{\mathcal{A}} u p_j + \alpha_2 1_{\mathcal{A}}^* p_j u^* + \alpha_3 u 1_{\mathcal{A}}^* p_j + \alpha_4 p_j 1_{\mathcal{A}} u^* + \alpha_5 u p_j 1_{\mathcal{A}} + \alpha_6 p_j u^* 1_{\mathcal{A}}^* = 0$. It follows that $(\alpha_1 + \alpha_3 + \alpha_5)(u_{ij} + u_{jj}) + (\alpha_2 + \alpha_4 + \alpha_6)(u_{ij} + u_{jj})^* = 0$ (8). Also, by the application of the involution $*$ on (8) we obtain the identity $\overline{(\alpha_2 + \alpha_4 + \alpha_6)}(u_{ij} + u_{jj}) + \overline{(\alpha_1 + \alpha_3 + \alpha_5)}(u_{ij} + u_{jj})^* = 0$ (9). Thus, multiplying (8) by the scalar $\overline{(\alpha_1 + \alpha_3 + \alpha_5)}$, (9) by the scalar $(\alpha_2 + \alpha_4 + \alpha_6)$ and subtracting the resulting identities, we arrive at $(|\alpha_1 + \alpha_3 + \alpha_5|^2 - |\alpha_2 + \alpha_4 + \alpha_6|^2)(u_{ij} + u_{jj}) = 0$ which leads to $u_{ij} = 0$ and $u_{jj} = 0$. Next, we have

$$\begin{aligned}
&\Phi(\alpha_1 1_{\mathcal{A}} u p_i + \alpha_2 1_{\mathcal{A}}^* p_i u^* + \alpha_3 u 1_{\mathcal{A}}^* p_i + \alpha_4 p_i 1_{\mathcal{A}} u^* + \alpha_5 u p_i 1_{\mathcal{A}} + \alpha_6 p_i u^* 1_{\mathcal{A}}^*)\\
&= \alpha_1 \Phi(1_{\mathcal{A}})\Phi(u)\Phi(p_i) + \alpha_2 \Phi(1_{\mathcal{A}})^*\Phi(p_i)\Phi(u)^* + \alpha_3 \Phi(u)\Phi(1_{\mathcal{A}})^*\Phi(p_i)\\
&+ \alpha_4 \Phi(p_i)\Phi(1_{\mathcal{A}})\Phi(u)^* + \alpha_5 \Phi(u)\Phi(p_i)\Phi(1_{\mathcal{A}}) + \alpha_6 \Phi(p_i)\Phi(u)^*\Phi(1_{\mathcal{A}})^*\\
&= \alpha_1 \Phi(1_{\mathcal{A}})\Phi(a_{ii} + b_{ij} + c_{ji} + d_{jj})\Phi(p_i) + \alpha_2 \Phi(1_{\mathcal{A}})^*\Phi(p_i)\Phi(a_{ii} + b_{ij} + c_{ji} + d_{jj})^*\\
&+ \alpha_3 \Phi(a_{ii} + b_{ij} + c_{ji} + d_{jj})\Phi(1_{\mathcal{A}})^*\Phi(p_i) + \alpha_4 \Phi(p_i)\Phi(1_{\mathcal{A}})\Phi(a_{ii} + b_{ij} + c_{ji} + d_{jj})^*\\
&+ \alpha_5 \Phi(a_{ii} + b_{ij} + c_{ji} + d_{jj})\Phi(p_i)\Phi(1_{\mathcal{A}}) + \alpha_6 \Phi(p_i)\Phi(a_{ii} + b_{ij} + c_{ji} + d_{jj})^*\Phi(1_{\mathcal{A}})^*\\
&- \alpha_1 \Phi(1_{\mathcal{A}})\Phi(a_{ii} + c_{ji})\Phi(p_i) - \alpha_2 \Phi(1_{\mathcal{A}})^*\Phi(p_i)\Phi(a_{ii} + c_{ji})^*\\
&- \alpha_3 \Phi(a_{ii} + c_{ji})\Phi(1_{\mathcal{A}})^*\Phi(p_i) - \alpha_4 \Phi(p_i)\Phi(1_{\mathcal{A}})\Phi(a_{ii} + c_{ji})^*\\
&- \alpha_5 \Phi(a_{ii} + c_{ji})\Phi(p_i)\Phi(1_{\mathcal{A}}) - \alpha_6 \Phi(p_i)\Phi(a_{ii} + c_{ji})^*\Phi(1_{\mathcal{A}})^*\\
&- \alpha_1 \Phi(1_{\mathcal{A}})\Phi(b_{ij} + d_{jj})\Phi(p_i) - \alpha_2 \Phi(1_{\mathcal{A}})^*\Phi(p_i)\Phi(b_{ij} + d_{jj})^*\\
&- \alpha_3 \Phi(b_{ij} + d_{jj})\Phi(1_{\mathcal{A}})^*\Phi(p_i) - \alpha_4 \Phi(p_i)\Phi(1_{\mathcal{A}})\Phi(b_{ij} + d_{jj})^*\\
&- \alpha_5 \Phi(b_{ij} + d_{jj})\Phi(p_i)\Phi(1_{\mathcal{A}}) - \alpha_6 \Phi(p_i)\Phi(b_{ij} + d_{jj})^*\Phi(1_{\mathcal{A}})^*\\
&= \Phi(\alpha_1 1_{\mathcal{A}}(a_{ii} + b_{ij} + c_{ji} + d_{jj})p_i + \alpha_2 1_{\mathcal{A}}^* p_i(a_{ii} + b_{ij} + c_{ji} + d_{jj})^*\\
&+ \alpha_3(a_{ii} + b_{ij} + c_{ji} + d_{jj})1_{\mathcal{A}}^* p_i + \alpha_4 p_i 1_{\mathcal{A}}(a_{ii} + b_{ij} + c_{ji} + d_{jj})^*\\
&+ \alpha_5(a_{ii} + b_{ij} + c_{ji} + d_{jj})p_i 1_{\mathcal{A}} + \alpha_6 p_i(a_{ii} + b_{ij} + c_{ji} + d_{jj})^* 1_{\mathcal{A}}^*)\\
&- \Phi(\alpha_1 1_{\mathcal{A}}(a_{ii} + c_{ji})p_i + \alpha_2 1_{\mathcal{A}}^* p_i(a_{ii} + c_{ji})^* + \alpha_3(a_{ii} + c_{ji})1_{\mathcal{A}}^* p_i
\end{aligned}$$

$$
\begin{aligned}
&+\alpha_4 p_i 1_{\mathcal{A}}(a_{ii}+c_{ji})^* + \alpha_5(a_{ii}+c_{ji})p_i 1_{\mathcal{A}} + \alpha_6 p_i(a_{ii}+c_{ji})^* 1^*_{\mathcal{A}})\\
&-\Phi(\alpha_1 1_{\mathcal{A}}(b_{ij}+d_{jj})p_i + \alpha_2 1^*_{\mathcal{A}} p_i(b_{ij}+d_{jj})^* + \alpha_3(b_{ij}+d_{jj})1^*_{\mathcal{A}} p_i\\
&+\alpha_4 p_i 1_{\mathcal{A}}(b_{ij}+d_{jj})^* + \alpha_5(b_{ij}+d_{jj})p_i 1_{\mathcal{A}} + \alpha_6 p_i(b_{ij}+d_{jj})^* 1^*_{\mathcal{A}})\\
&=0
\end{aligned}
$$

from which we immediately deduce the identity $\alpha_1 1_{\mathcal{A}} u p_i + \alpha_2 1^*_{\mathcal{A}} p_i u^* + \alpha_3 u 1^*_{\mathcal{A}} p_i + \alpha_4 p_i 1_{\mathcal{A}} u^* + \alpha_5 u p_i 1_{\mathcal{A}} + \alpha_6 p_i u^* 1^*_{\mathcal{A}} = 0$. This results in the identity $(\alpha_1+\alpha_3+\alpha_5)(u_{ii}+u_{ji}) + (\alpha_2+\alpha_4+\alpha_6)(u_{ii}+u_{ji})^* = 0$ (10). Also, we get $\overline{(\alpha_2+\alpha_4+\alpha_6)}(u_{ii}+u_{ji}) + \overline{(\alpha_1+\alpha_3+\alpha_5)}(u_{ii}+u_{ji})^* = 0$ (11), by the application of the involution $*$ on (10). As a consequence, multiplying (10) by the scalar $\overline{(\alpha_1+\alpha_3+\alpha_5)}$, (11) by the scalar $(\alpha_2+\alpha_4+\alpha_6)$ and subtracting the resulting identities, we arrive at $(|\alpha_1+\alpha_3+\alpha_5|^2 - |\alpha_2+\alpha_4+\alpha_6|^2)(u_{ii}+u_{ji}) = 0$ which shows that $u_{ii}+u_{ji}=0$. Consequently, we obtain $u_{ii}=0$ and $u_{ji}=0$. Therefore $u=0$. □

Claim 2.4. *For every $a_{ij}, b_{ij} \in \mathcal{A}_{ij}$ $(i \neq j; i,j = 1,2)$ we have: $\Phi(a_{ij}+b_{ij}) = \Phi(a_{ij}) + \Phi(b_{ij})$.*

Proof. First, note that the following identity holds:

$$
\begin{aligned}
&(\alpha_1+\alpha_3+\alpha_5)(a_{ij}+b_{ij}) + (\alpha_2+\alpha_4+\alpha_6)(a^*_{ij}+b_{ij}a^*_{ij})\\
&=\alpha_1 1_{\mathcal{A}}(p_i+a_{ij})(p_j+b_{ij}) + \alpha_2 1^*_{\mathcal{A}}(p_j+b_{ij})(p_i+a_{ij})^*\\
&+\alpha_3(p_i+a_{ij})1^*_{\mathcal{A}}(p_j+b_{ij}) + \alpha_4(p_j+b_{ij})1_{\mathcal{A}}(p_i+a_{ij})^*\\
&+\alpha_5(p_j+a_{ij})(p_i+b_{ij})1_{\mathcal{A}} + \alpha_6(p_j+b_{ij})(p_i+a_{ij})^* 1^*_{\mathcal{A}},
\end{aligned}
$$

for all elements $a_{ij}, b_{ij} \in \mathcal{A}_{ij}$. Hence, by Claim 2.3 we have

$$
\begin{aligned}
&\Phi((\alpha_1+\alpha_3+\alpha_5)(a_{ij}+b_{ij})) + \Phi((\alpha_2+\alpha_4+\alpha_6)(a^*_{ij}+b_{ij}a^*_{ij}))\\
&=\Phi((\alpha_1+\alpha_3+\alpha_5)(a_{ij}+b_{ij}) + (\alpha_2+\alpha_4+\alpha_6)(a^*_{ij}+b_{ij}a^*_{ij}))\\
&=\Phi(\alpha_1 1_{\mathcal{A}}(p_i+a_{ij})(p_j+b_{ij}) + \alpha_2 1^*_{\mathcal{A}}(p_j+b_{ij})(p_i+a_{ij})^*\\
&+\alpha_3(p_i+a_{ij})1^*_{\mathcal{A}}(p_j+b_{ij}) + \alpha_4(p_j+b_{ij})1_{\mathcal{A}}(p_i+a_{ij})^*\\
&+\alpha_5(p_j+a_{ij})(p_i+b_{ij})1_{\mathcal{A}} + \alpha_6(p_j+b_{ij})(p_i+a_{ij})^* 1^*_{\mathcal{A}})\\
&=\alpha_1\Phi(1_{\mathcal{A}})\Phi(p_i+a_{ij})\Phi(p_j+b_{ij}) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(p_j+b_{ij})\Phi(p_i+a_{ij})^*\\
&+\alpha_3\Phi(p_i+a_{ij})\Phi(1_{\mathcal{A}})^*\Phi(p_j+b_{ij}) + \alpha_4\Phi(p_j+b_{ij})\Phi(1_{\mathcal{A}})\Phi(p_i+a_{ij})^*\\
&+\alpha_5\Phi(p_i+a_{ij})\Phi(p_j+b_{ij})\Phi(1_{\mathcal{A}}) + \alpha_6\Phi(p_j+b_{ij})\Phi(p_i+a_{ij})^*\Phi(1_{\mathcal{A}})^*\\
&=\alpha_1\Phi(1_{\mathcal{A}})(\Phi(p_i)+\Phi(a_{ij}))(\Phi(p_j)+\Phi(b_{ij}))\\
&+\alpha_2\Phi(1_{\mathcal{A}})^*(\Phi(p_j)+\Phi(b_{ij}))(\Phi(p_i)^*+\Phi(a_{ij})^*)\\
&+\alpha_3(\Phi(p_i)+\Phi(a_{ij}))\Phi(1_{\mathcal{A}})^*(\Phi(p_j)+\Phi(b_{ij}))\\
&+\alpha_4(\Phi(p_j)+\Phi(b_{ij}))\Phi(1_{\mathcal{A}})(\Phi(p_i)^*+\Phi(a_{ij})^*)\\
&+\alpha_5(\Phi(p_i)+\Phi(a_{ij}))(\Phi(p_j)+\Phi(b_{ij}))\Phi(1_{\mathcal{A}})\\
&+\alpha_6(\Phi(p_j)+\Phi(b_{ij}))(\Phi(p_i)^*+\Phi(a_{ij})^*)\Phi(1_{\mathcal{A}})^*\\
&=\alpha_1\Phi(1_{\mathcal{A}})\Phi(p_i)\Phi(p_j) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(p_j)\Phi(p_i)^* + \alpha_3\Phi(p_i)\Phi(1_{\mathcal{A}})^*\Phi(p_j)\\
&+\alpha_4\Phi(p_j)\Phi(1_{\mathcal{A}})\Phi(p_i)^* + \alpha_5\Phi(p_i)\Phi(p_j)\Phi(1_{\mathcal{A}}) + \alpha_6\Phi(p_j)\Phi(p_i)^*\Phi(1_{\mathcal{A}})^*\\
&+\alpha_1\Phi(1_{\mathcal{A}})\Phi(a_{ij})\Phi(p_j) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(p_j)\Phi(a_{ij})^* + \alpha_3\Phi(a_{ij})\Phi(1_{\mathcal{A}})^*\Phi(p_j)\\
&+\alpha_4\Phi(p_j)\Phi(1_{\mathcal{A}})\Phi(a_{ij})^* + \alpha_5\Phi(a_{ij})\Phi(p_j)\Phi(1_{\mathcal{A}}) + \alpha_6\Phi(p_j)\Phi(a_{ij})^*\Phi(1_{\mathcal{A}})^*\\
&+\alpha_1\Phi(1_{\mathcal{A}})\Phi(p_i)\Phi(b_{ij}) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(b_{ij})\Phi(p_i)^* + \alpha_3\Phi(p_i)\Phi(1_{\mathcal{A}})^*\Phi(b_{ij})\\
&+\alpha_4\Phi(b_{ij})\Phi(1_{\mathcal{A}})\Phi(p_i)^* + \alpha_5\Phi(p_i)\Phi(b_{ij})\Phi(1_{\mathcal{A}}) + \alpha_6\Phi(b_{ij})\Phi(p_i)^*\Phi(1_{\mathcal{A}})^*\\
&+\alpha_1\Phi(1_{\mathcal{A}})\Phi(a_{ij})\Phi(b_{ij}) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(b_{ij})\Phi(a_{ij})^* + \alpha_3\Phi(a_{ij})\Phi(1_{\mathcal{A}})^*\Phi(b_{ij})\\
&+\alpha_4\Phi(b_{ij})\Phi(1_{\mathcal{A}})\Phi(a_{ij})^* + \alpha_5\Phi(a_{ij})\Phi(b_{ij})\Phi(1_{\mathcal{A}}) + \alpha_6\Phi(b_{ij})\Phi(a_{ij})^*\Phi(1_{\mathcal{A}})^*\\
&=\Phi(\alpha_1 1_{\mathcal{A}} p_i p_j + \alpha_2 1^*_{\mathcal{A}} p_j p_i^* + \alpha_3 p_i 1^*_{\mathcal{A}} p_j + \alpha_4 p_j 1_{\mathcal{A}} p_i^* + \alpha_5 p_i p_j 1_{\mathcal{A}}
\end{aligned}
$$

$$+ \alpha_6 p_j p_i^* 1_{\mathcal{A}}^*) + \Phi(\alpha_1 1_{\mathcal{A}} a_{ij} p_j + \alpha_2 1_{\mathcal{A}}^* p_j a_{ij}^* + \alpha_3 a_{ij} 1_{\mathcal{A}}^* p_j + \alpha_4 p_j 1_{\mathcal{A}} a_{ij}^*$$
$$+ \alpha_5 a_{ij} p_j 1_{\mathcal{A}} + \alpha_6 p_j a_{ij}^* 1_{\mathcal{A}}^*) + \Phi(\alpha_1 1_{\mathcal{A}} p_i b_{ij} + \alpha_2 1_{\mathcal{A}}^* b_{ij} p_i^* + \alpha_3 p_i 1_{\mathcal{A}}^* b_{ij}$$
$$+ \alpha_4 b_{ij} 1_{\mathcal{A}} p_i^* + \alpha_5 p_i b_{ij} 1_{\mathcal{A}} + \alpha_6 b_{ij} p_i^* 1_{\mathcal{A}}^*) + \Phi(\alpha_1 1_{\mathcal{A}} a_{ij} b_{ij} + \alpha_2 1_{\mathcal{A}}^* b_{ij} a_{ij}^*$$
$$+ \alpha_3 a_{ij} 1_{\mathcal{A}}^* b_{ij} + \alpha_4 b_{ij} 1_{\mathcal{A}} a_{ij}^* + \alpha_5 a_{ij} b_{ij} 1_{\mathcal{A}} + \alpha_6 b_{ij} a_{ij}^* 1_{\mathcal{A}}^*)$$
$$= \Phi((\alpha_1 + \alpha_3 + \alpha_5) a_{ij}) + \Phi((\alpha_2 + \alpha_4 + \alpha_6) a_{ij}^*) + \Phi((\alpha_1 + \alpha_3 + \alpha_5) b_{ij})$$
$$+ \Phi((\alpha_2 + \alpha_4 + \alpha_6) b_{ij} a_{ij}^*).$$

It therefore follows that $\Phi((\alpha_1 + \alpha_3 + \alpha_5)(a_{ij} + b_{ij})) = \Phi((\alpha_1 + \alpha_3 + \alpha_5)a_{ij}) + \Phi((\alpha_1 + \alpha_3 + \alpha_5)b_{ij})$ This leads to the conclusion that $\Phi(a_{ij} + b_{ij}) = \Phi(a_{ij}) + \Phi(b_{ij})$, for all elements $a_{ij}, b_{ij} \in \mathcal{A}_{ij}$. □

Claim 2.5. *For every $a_{ii}, b_{ii} \in \mathcal{A}_{ii}$ $(i = 1, 2)$, we have: $\Phi(a_{ii} + b_{ii}) = \Phi(a_{ii}) + \Phi(b_{ii})$.*

Proof. Let $u = u_{ii} + u_{ij} + u_{ji} + u_{jj} = \Phi^{-1}(\Phi(a_{ii} + b_{ii}) - \Phi(a_{ii}) - \Phi(b_{ii})) \in \mathcal{A}$ $(i \neq j; i, j = 1, 2)$. Then

$$\Phi(\alpha_1 1_{\mathcal{A}} u p_j + \alpha_2 1_{\mathcal{A}}^* p_j u^* + \alpha_3 u 1_{\mathcal{A}}^* p_j + \alpha_4 p_j 1_{\mathcal{A}} u^* + \alpha_5 u p_j 1_{\mathcal{A}} + \alpha_6 p_j u^* 1_{\mathcal{A}}^*)$$
$$= \alpha_1 \Phi(1_{\mathcal{A}})\Phi(u)\Phi(p_j) + \alpha_2 \Phi(1_{\mathcal{A}})^* \Phi(p_j)\Phi(u)^* + \alpha_3 \Phi(u)\Phi(1_{\mathcal{A}})^* \Phi(p_j)$$
$$+ \alpha_4 \Phi(p_j)\Phi(1_{\mathcal{A}})\Phi(u)^* + \alpha_5 \Phi(u)\Phi(p_j)\Phi(1_{\mathcal{A}}) + \alpha_6 \Phi(p_j)\Phi(u)^* \Phi(1_{\mathcal{A}})^*$$
$$= \alpha_1 \Phi(1_{\mathcal{A}})\Phi(a_{ii} + b_{ii})\Phi(p_j) + \alpha_2 \Phi(1_{\mathcal{A}})^* \Phi(p_j)\Phi(a_{ii} + b_{ii})^*$$
$$+ \alpha_3 \Phi(a_{ii} + b_{ii})\Phi(1_{\mathcal{A}})^* \Phi(p_j) + \alpha_4 \Phi(p_j)\Phi(1_{\mathcal{A}})\Phi(a_{ii} + b_{ii})^*$$
$$+ \alpha_5 \Phi(a_{ii} + b_{ii})\Phi(p_j)\Phi(1_{\mathcal{A}}) + \alpha_6 \Phi(p_j)\Phi(a_{ii} + b_{ii})^* \Phi(1_{\mathcal{A}})^*$$
$$- \alpha_1 \Phi(1_{\mathcal{A}})\Phi(a_{ii})\Phi(p_j) - \alpha_2 \Phi(1_{\mathcal{A}})^* \Phi(p_j)\Phi(a_{ii})^*$$
$$- \alpha_3 \Phi(a_{ii})\Phi(1_{\mathcal{A}})^* \Phi(p_j) - \alpha_4 \Phi(p_j)\Phi(1_{\mathcal{A}})\Phi(a_{ii})^*$$
$$- \alpha_5 \Phi(a_{ii})\Phi(p_j)\Phi(1_{\mathcal{A}}) - \alpha_6 \Phi(p_j)\Phi(a_{ii})^* \Phi(1_{\mathcal{A}})^*$$
$$- \alpha_1 \Phi(1_{\mathcal{A}})\Phi(b_{ii})\Phi(p_j) - \alpha_2 \Phi(1_{\mathcal{A}})^* \Phi(p_j)\Phi(b_{ii})^*$$
$$- \alpha_3 \Phi(b_{ii})\Phi(1_{\mathcal{A}})^* \Phi(p_j) - \alpha_4 \Phi(p_j)\Phi(1_{\mathcal{A}})\Phi(b_{ii})^*$$
$$- \alpha_5 \Phi(b_{ii})\Phi(p_j)\Phi(1_{\mathcal{A}}) - \alpha_6 \Phi(p_j)\Phi(b_{ii})^* \Phi(1_{\mathcal{A}})^*$$
$$= \Phi(\alpha_1 1_{\mathcal{A}}(a_{ii} + b_{ii})p_j + \alpha_2 1_{\mathcal{A}}^* p_j (a_{ii} + b_{ii})^* + \alpha_3 (a_{ii} + b_{ii}) 1_{\mathcal{A}}^* p_j$$
$$+ \alpha_4 p_j 1_{\mathcal{A}} (a_{ii} + b_{ii})^* + \alpha_5 (a_{ii} + b_{ii}) p_j 1_{\mathcal{A}} + \alpha_6 p_j (a_{ii} + b_{ii})^* 1_{\mathcal{A}}^*)$$
$$- \Phi(\alpha_1 1_{\mathcal{A}} a_{ii} p_j + \alpha_2 1_{\mathcal{A}}^* p_j a_{ii}^* + \alpha_3 a_{ii} 1_{\mathcal{A}}^* p_j + \alpha_4 p_j 1_{\mathcal{A}} a_{ii}^* + \alpha_5 a_{ii} p_j 1_{\mathcal{A}}$$
$$+ \alpha_6 p_j a_{ii}^* 1_{\mathcal{A}}^*) - \Phi(\alpha_1 1_{\mathcal{A}} b_{ii} p_j + \alpha_2 1_{\mathcal{A}}^* p_j b_{ii}^* + \alpha_3 b_{ii} 1_{\mathcal{A}}^* p_j + \alpha_4 p_j 1_{\mathcal{A}} b_{ii}^*$$
$$+ \alpha_5 b_{ii} p_j 1_{\mathcal{A}} + \alpha_6 p_j b_{ii}^* 1_{\mathcal{A}}^*) = 0$$

which leads directly to the identity $\alpha_1 1_{\mathcal{A}} u p_j + \alpha_2 1_{\mathcal{A}}^* p_j u^* + \alpha_3 u 1_{\mathcal{A}}^* p_j + \alpha_4 p_j 1_{\mathcal{A}} u^* + \alpha_5 u p_j 1_{\mathcal{A}} + \alpha_6 p_j u^* 1_{\mathcal{A}}^* = 0$. It therefore follows that $(\alpha_1 + \alpha_3 + \alpha_5)u_{ij} + (\alpha_2 + \alpha_4 + \alpha_6)u_{ij}^* + (\alpha_1 + \alpha_3 + \alpha_5)u_{jj} + (\alpha_2 + \alpha_4 + \alpha_6)u_{jj}^* = 0$ (12) and hence the identity $(\overline{\alpha_2 + \alpha_4 + \alpha_6})u_{ij} + (\overline{\alpha_1 + \alpha_3 + \alpha_5})u_{ij}^* + (\overline{\alpha_2 + \alpha_4 + \alpha_6})u_{jj} + (\overline{\alpha_1 + \alpha_3 + \alpha_5})u_{jj}^* = 0$ (13). From (12) and (13), we get $(|\alpha_1 + \alpha_3 + \alpha_5|^2 - |\alpha_2 + \alpha_4 + \alpha_6|^2)(u_{ij} + u_{jj}) = 0$ which implies that $u_{ij} + u_{jj} = 0$. This results that $u_{ij} = 0$ and $u_{jj} = 0$. Next, for all element $t_{ij} \in \mathcal{A}_{ij}$ we have

$$\Phi(\alpha_1 1_{\mathcal{A}} t_{ij} u + \alpha_2 1_{\mathcal{A}}^* u t_{ij}^* + \alpha_3 t_{ij} 1_{\mathcal{A}}^* u + \alpha_4 u 1_{\mathcal{A}} t_{ij}^* + \alpha_5 t_{ij} u 1_{\mathcal{A}} + \alpha_6 u t_{ij}^* 1_{\mathcal{A}}^*)$$
$$= \alpha_1 \Phi(1_{\mathcal{A}})\Phi(t_{ij})\Phi(u) + \alpha_2 \Phi(1_{\mathcal{A}})^* \Phi(u)\Phi(t_{ij})^* + \alpha_3 \Phi(t_{ij})\Phi(1_{\mathcal{A}})^* \Phi(u)$$
$$+ \alpha_4 \Phi(u)\Phi(1_{\mathcal{A}})\Phi(t_{ij})^* + \alpha_5 \Phi(t_{ij})\Phi(u)\Phi(1_{\mathcal{A}}) + \alpha_6 \Phi(u)\Phi(t_{ij})^* \Phi(1_{\mathcal{A}})^*$$
$$= \alpha_1 \Phi(1_{\mathcal{A}})\Phi(t_{ij})\Phi(a_{ii} + b_{ii}) + \alpha_2 \Phi(1_{\mathcal{A}})^* \Phi(a_{ii} + b_{ii})\Phi(t_{ij})^*$$
$$+ \alpha_3 \Phi(t_{ij})\Phi(1_{\mathcal{A}})^* \Phi(a_{ii} + b_{ii}) + \alpha_4 \Phi(a_{ii} + b_{ii})\Phi(1_{\mathcal{A}})\Phi(t_{ij})^*$$
$$+ \alpha_5 \Phi(t_{ij})\Phi(a_{ii} + b_{ii})\Phi(1_{\mathcal{A}}) + \alpha_6 \Phi(a_{ii} + b_{ii})\Phi(t_{ij})^* \Phi(1_{\mathcal{A}})^*$$
$$- \alpha_1 \Phi(1_{\mathcal{A}})\Phi(t_{ij})\Phi(a_{ii}) - \alpha_2 \Phi(1_{\mathcal{A}})^* \Phi(a_{ii})\Phi(t_{ij})^* - \alpha_3 \Phi(t_{ij})\Phi(1_{\mathcal{A}})^* \Phi(a_{ii})$$

$$- \alpha_4\Phi(a_{ii})\Phi(1_{\mathcal{A}})\Phi(t_{ij})^* - \alpha_5\Phi(t_{ij})\Phi(a_{ii})\Phi(1_{\mathcal{A}}) - \alpha_6\Phi(a_{ii})\Phi(t_{ij})^*\Phi(1_{\mathcal{A}})^*$$
$$- \alpha_1\Phi(1_{\mathcal{A}})\Phi(t_{ij})\Phi(b_{ii}) - \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(b_{ii})\Phi(t_{ij})^* - \alpha_3\Phi(t_{ij})\Phi(1_{\mathcal{A}})^*\Phi(b_{ii})$$
$$- \alpha_4\Phi(b_{ii})\Phi(1_{\mathcal{A}})\Phi(t_{ij})^* - \alpha_5\Phi(t_{ij})\Phi(b_{ii})\Phi(1_{\mathcal{A}}) - \alpha_6\Phi(b_{ii})\Phi(t_{ij})^*\Phi(1_{\mathcal{A}})^*$$
$$= \Phi(\alpha_1 1_{\mathcal{A}}t_{ij}(a_{ii}+b_{ii}) + \alpha_2 1^*_{\mathcal{A}}(a_{ii}+b_{ii})t^*_{ij} + \alpha_3 t_{ij}1^*_{\mathcal{A}}(a_{ii}+b_{ii})$$
$$+ \alpha_4(a_{ii}+b_{ii})1_{\mathcal{A}}t^*_{ij} + \alpha_5 t_{ij}(a_{ii}+b_{ii})1_{\mathcal{A}} + \alpha_6(a_{ii}+b_{ii})t^*_{ij}1^*_{\mathcal{A}})$$
$$- \Phi(\alpha_1 1_{\mathcal{A}}t_{ij}a_{ii} + \alpha_2 1^*_{\mathcal{A}}a_{ii}t^*_{ij} + \alpha_3 t_{ij}1^*_{\mathcal{A}}a_{ii} + \alpha_4 a_{ii}1_{\mathcal{A}}t^*_{ij} + \alpha_5 t_{ij}a_{ii}1_{\mathcal{A}}$$
$$+ \alpha_6 a_{ii}t^*_{ij}1^*_{\mathcal{A}}) - \Phi(\alpha_1 1_{\mathcal{A}}t_{ij}b_{ii} + \alpha_2 1^*_{\mathcal{A}}b_{ii}t^*_{ij} + \alpha_3 t_{ij}1^*_{\mathcal{A}}b_{ii} + \alpha_4 b_{ii}1_{\mathcal{A}}t^*_{ij}$$
$$+ \alpha_5 t_{ij}b_{ii}1_{\mathcal{A}} + \alpha_6 b_{ii}t^*_{ij}1^*_{\mathcal{A}}) = 0.$$

It follows immediately from this that $\alpha_1 1_{\mathcal{A}}t_{ij}u + \alpha_2 1^*_{\mathcal{A}}ut^*_{ij} + \alpha_3 t_{ij}1^*_{\mathcal{A}}u + \alpha_4 u1_{\mathcal{A}}t^*_{ij} + \alpha_5 t_{ij}u1_{\mathcal{A}} + \alpha_6 ut^*_{ij}1^*_{\mathcal{A}} = 0$ which yields $(\alpha_1+\alpha_3+\alpha_5)t_{ij}u_{ji} = 0$. As a consequence, we have $(\alpha_1+\alpha_3+\alpha_5)u_{ji} = 0$, because of the primeness of $\mathcal{A}$. Therefore $u_{ji} = 0$. Also, by Claims 2.3 and 2.4, for all element $t_{ji} \in \mathcal{A}_{ji}$ we have

$$\Phi(\alpha_1 1_{\mathcal{A}}t_{ji}u + \alpha_2 1^*_{\mathcal{A}}ut^*_{ji} + \alpha_3 t_{ji}1^*_{\mathcal{A}}u + \alpha_4 u1_{\mathcal{A}}t^*_{ji} + \alpha_5 t_{ji}u1_{\mathcal{A}} + \alpha_6 ut^*_{ji}1^*_{\mathcal{A}})$$
$$= \alpha_1\Phi(1_{\mathcal{A}})\Phi(t_{ji})\Phi(u) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(u)\Phi(t_{ji})^* + \alpha_3\Phi(t_{ji})\Phi(1_{\mathcal{A}})^*\Phi(u)$$
$$+ \alpha_4\Phi(u)\Phi(1_{\mathcal{A}})\Phi(t_{ji})^* + \alpha_5\Phi(t_{ji})\Phi(u)\Phi(1_{\mathcal{A}}) + \alpha_6\Phi(u)\Phi(t_{ji})^*\Phi(1_{\mathcal{A}})^*$$
$$= \alpha_1\Phi(1_{\mathcal{A}})\Phi(t_{ji})\Phi(a_{ii}+b_{ii}) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(a_{ii}+b_{ii})\Phi(t_{ji})^*$$
$$+ \alpha_3\Phi(t_{ji})\Phi(1_{\mathcal{A}})^*\Phi(a_{ii}+b_{ii}) + \alpha_4\Phi(a_{ii}+b_{ii})\Phi(1_{\mathcal{A}})\Phi(t_{ji})^*$$
$$+ \alpha_5\Phi(t_{ji})\Phi(a_{ii}+b_{ii})\Phi(1_{\mathcal{A}}) + \alpha_6\Phi(a_{ii}+b_{ii})\Phi(t_{ji})^*\Phi(1_{\mathcal{A}})^*$$
$$- \alpha_1\Phi(1_{\mathcal{A}})\Phi(t_{ji})\Phi(a_{ii}) - \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(a_{ii})\Phi(t_{ji})^* - \alpha_3\Phi(t_{ji})\Phi(1_{\mathcal{A}})^*\Phi(a_{ii})$$
$$- \alpha_4\Phi(a_{ii})\Phi(1_{\mathcal{A}})\Phi(t_{ji})^* - \alpha_5\Phi(t_{ji})\Phi(a_{ii})\Phi(1_{\mathcal{A}}) - \alpha_6\Phi(a_{ii})\Phi(t_{ji})^*\Phi(1_{\mathcal{A}})^*$$
$$- \alpha_1\Phi(1_{\mathcal{A}})\Phi(t_{ji})\Phi(b_{ii}) - \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(b_{ii})\Phi(t_{ji})^* - \alpha_3\Phi(t_{ji})\Phi(1_{\mathcal{A}})^*\Phi(b_{ii})$$
$$- \alpha_4\Phi(b_{ii})\Phi(1_{\mathcal{A}})\Phi(t_{ji})^* - \alpha_5\Phi(t_{ji})\Phi(b_{ii})\Phi(1_{\mathcal{A}}) - \alpha_6\Phi(b_{ii})\Phi(t_{ji})^*\Phi(1_{\mathcal{A}})^*$$
$$= \Phi(\alpha_1 1_{\mathcal{A}}t_{ji}(a_{ii}+b_{ii}) + \alpha_2 1^*_{\mathcal{A}}(a_{ii}+b_{ii})t^*_{ji} + \alpha_3 t_{ji}1^*_{\mathcal{A}}(a_{ii}+b_{ii})$$
$$+ \alpha_4(a_{ii}+b_{ii})1_{\mathcal{A}}t^*_{ji} + \alpha_5 t_{ji}(a_{ii}+b_{ii})1_{\mathcal{A}} + \alpha_6(a_{ii}+b_{ii})t^*_{ji}1^*_{\mathcal{A}})$$
$$- \Phi(\alpha_1 1_{\mathcal{A}}t_{ji}a_{ii} + \alpha_2 1^*_{\mathcal{A}}a_{ii}t^*_{ji} + \alpha_3 t_{ji}1^*_{\mathcal{A}}a_{ii} + \alpha_4 a_{ii}1_{\mathcal{A}}t^*_{ji} + \alpha_5 t_{ji}a_{ii}1_{\mathcal{A}}$$
$$+ \alpha_6 a_{ii}t^*_{ji}1^*_{\mathcal{A}}) - \Phi(\alpha_1 1_{\mathcal{A}}t_{ji}b_{ii} + \alpha_2 1^*_{\mathcal{A}}b_{ii}t^*_{ji} + \alpha_3 t_{ji}1^*_{\mathcal{A}}b_{ii} + \alpha_4 b_{ii}1_{\mathcal{A}}t^*_{ji}$$
$$+ \alpha_5 t_{ji}b_{ii}1_{\mathcal{A}} + \alpha_6 b_{ii}t^*_{ji}1^*_{\mathcal{A}}) = 0$$

which results in the identity $\alpha_1 1_{\mathcal{A}}t_{ji}u + \alpha_2 1^*_{\mathcal{A}}ut^*_{ji} + \alpha_3 t_{ji}1^*_{\mathcal{A}}u + \alpha_4 u1_{\mathcal{A}}t^*_{ji} + \alpha_5 t_{ji}u1_{\mathcal{A}} + \alpha_6 ut^*_{ji}1^*_{\mathcal{A}} = 0$. This shows that $(\alpha_1+\alpha_3+\alpha_5)t_{ji}u_{ii} + (\alpha_2+\alpha_4+\alpha_6)u_{ii}t^*_{ji} = 0$ that implies $(\alpha_1+\alpha_3+\alpha_5)t_{ji}u_{ii} = 0$. As a consequence we get $(\alpha_1+\alpha_3+\alpha_5)u_{ii} = 0$ which yields $u_{ii} = 0$. It follows from all that $u = 0$. □

Claim 2.6. Φ *is an additive map.*

Proof. The result is a direct consequence of Claims 2.3, 2.4 and 2.5. □

In what follows, we prove the second part of the Theorem 1.1. In the remainder of this paper, all Claims satisfy the conditions (i)-(ii).

Claim 2.7. *(i)* $\Phi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$ *and (ii)* $\Phi((\sum_{k=1}^{6}\alpha_k)c) = (\sum_{k=1}^{6}\alpha_k)\Phi(c)$, *for all element* $c \in \mathcal{A}$.

Proof. First, note that

$$\Phi((\sum_{k=1}^{6}\alpha_k)1_{\mathcal{A}}) = \Phi(\alpha_1 1_{\mathcal{A}}1_{\mathcal{A}}1_{\mathcal{A}} + \alpha_2 1^*_{\mathcal{A}}1_{\mathcal{A}}1^*_{\mathcal{A}} + \alpha_3 1_{\mathcal{A}}1^*_{\mathcal{A}}1_{\mathcal{A}} + \alpha_4 1_{\mathcal{A}}1_{\mathcal{A}}1^*_{\mathcal{A}}$$
$$+ \alpha_5 1_{\mathcal{A}}1_{\mathcal{A}}1_{\mathcal{A}} + \alpha_6 1_{\mathcal{A}}1^*_{\mathcal{A}}1^*_{\mathcal{A}}) = \alpha_1\Phi(1_{\mathcal{A}})\Phi(1_{\mathcal{A}})\Phi(1_{\mathcal{A}}) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(1_{\mathcal{A}})\Phi(1_{\mathcal{A}})^*$$

$$+ \alpha_3\Phi(1_{\mathcal{A}})\Phi(1_{\mathcal{A}})^*\Phi(1_{\mathcal{A}}) + \alpha_4\Phi(1_{\mathcal{A}})\Phi(1_{\mathcal{A}})\Phi(1_{\mathcal{A}})^* + \alpha_5\Phi(1_{\mathcal{A}})\Phi(1_{\mathcal{A}})\Phi(1_{\mathcal{A}})$$
$$+ \alpha_6\Phi(1_{\mathcal{A}})\Phi(1_{\mathcal{A}})^*\Phi(1_{\mathcal{A}})^* = (\textstyle\sum_{k=1}^{6}\alpha_k)\Phi(1_{\mathcal{A}}).$$

Hence, choose an element $c \in \mathcal{A}$, such that $\phi(c) = 1_{\mathcal{B}}$. Then

$$\Phi((\textstyle\sum_{k=1}^{6}\alpha_k)c) = \Phi(\alpha_1 1_{\mathcal{A}}1_{\mathcal{A}}c + \alpha_2 1_{\mathcal{A}}^* c 1_{\mathcal{A}}^* + \alpha_3 1_{\mathcal{A}}1_{\mathcal{A}}^* c + \alpha_4 c 1_{\mathcal{A}}1_{\mathcal{A}}^*$$
$$+ \alpha_5 1_{\mathcal{A}} c 1_{\mathcal{A}} + \alpha_6 c 1_{\mathcal{A}}^* 1_{\mathcal{A}}^*) = \alpha_1\Phi(1_{\mathcal{A}})\Phi(1_{\mathcal{A}})\Phi(c) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(c)\Phi(1_{\mathcal{A}})^*$$
$$+ \alpha_3\Phi(1_{\mathcal{A}})\Phi(1_{\mathcal{A}})^*\Phi(c) + \alpha_4\Phi(c)\Phi(1_{\mathcal{A}})\Phi(1_{\mathcal{A}})^* + \alpha_5\Phi(1_{\mathcal{A}})\Phi(c)\Phi(1_{\mathcal{A}})$$
$$+ \alpha_6\Phi(c)\Phi(1_{\mathcal{A}})^*\Phi(1_{\mathcal{A}})^* = (\textstyle\sum_{k=1}^{6}\alpha_k)\Phi(1_{\mathcal{A}})^2 = \Phi((\textstyle\sum_{k=1}^{6}\alpha_k)1_{\mathcal{A}}).$$

This shows that $c = 1_{\mathcal{A}}$. As a consequence of this last result, for an arbitrary element $c \in \mathcal{A}$, we have

$$\Phi((\textstyle\sum_{k=1}^{6}\alpha_k)c) = \Phi(\alpha_1 1_{\mathcal{A}}1_{\mathcal{A}}c + \alpha_2 1_{\mathcal{A}}^* c 1_{\mathcal{A}}^* + \alpha_3 1_{\mathcal{A}}1_{\mathcal{A}}^* c + \alpha_4 c 1_{\mathcal{A}}1_{\mathcal{A}}^*$$
$$+ \alpha_5 1_{\mathcal{A}} c 1_{\mathcal{A}} + \alpha_6 c 1_{\mathcal{A}}^* 1_{\mathcal{A}}^*) = \alpha_1\Phi(1_{\mathcal{A}})\Phi(1_{\mathcal{A}})\Phi(c) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(c)\Phi(1_{\mathcal{A}})^*$$
$$+ \alpha_3\Phi(1_{\mathcal{A}})\Phi(1_{\mathcal{A}})^*\Phi(c) + \alpha_4\Phi(c)\Phi(1_{\mathcal{A}})\Phi(1_{\mathcal{A}})^* + \alpha_5\Phi(1_{\mathcal{A}})\Phi(c)\Phi(1_{\mathcal{A}})$$
$$+ \alpha_6\Phi(c)\Phi(1_{\mathcal{A}})^*\Phi(1_{\mathcal{A}})^* = (\textstyle\sum_{k=1}^{6}\alpha_k)\Phi(c).$$

□

Claim 2.8. *(i)* $\Phi((\alpha_1 + \alpha_3 + \alpha_5)a) = (\alpha_1 + \alpha_3 + \alpha_5)\Phi(a)$, *for all element* $a \in \mathcal{A}$, *and (ii)* $\Phi(b)^* = \Phi(b)^*$, *for all element* $b \in \mathcal{A}$.

Proof. It is clear that $\Phi((\alpha_1 + \alpha_3 + \alpha_5)a) = (\alpha_1 + \alpha_3 + \alpha_5)\Phi(a)$, for all element $a \in \mathcal{A}$, because of hypothesis (ii), of the Theorem 1.1, and Claims 2.6 and 2.7(ii). Thus, for an arbitrary element $b \in \mathcal{A}$ we have

$$\Phi(\alpha_1 1_{\mathcal{A}} b 1_{\mathcal{A}} + \alpha_2 1_{\mathcal{A}}^* 1_{\mathcal{A}} b^* + \alpha_3 b 1_{\mathcal{A}}^* 1_{\mathcal{A}} + \alpha_4 1_{\mathcal{A}}1_{\mathcal{A}} b^* + \alpha_5 b 1_{\mathcal{A}}1_{\mathcal{A}} + \alpha_6 1_{\mathcal{A}} b^* 1_{\mathcal{A}}^*)$$
$$= \alpha_1\Phi(1_{\mathcal{A}})\Phi(b)\Phi(1_{\mathcal{A}}) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(1_{\mathcal{A}})\Phi(b)^* + \alpha_3\Phi(b)\Phi(1_{\mathcal{A}})^*\Phi(1_{\mathcal{A}})$$
$$+ \alpha_4\Phi(1_{\mathcal{A}})\Phi(1_{\mathcal{A}})\Phi(b)^* + \alpha_5\Phi(b)\Phi(1_{\mathcal{A}})\Phi(1_{\mathcal{A}}) + \alpha_6\Phi(1_{\mathcal{A}})\Phi(b)^*\Phi(1_{\mathcal{A}})^*$$

that leads to the identity $(\alpha_1 + \alpha_3 + \alpha_5)\Phi(b) + (\alpha_2 + \alpha_4 + \alpha_6)\Phi(b^*) = (\alpha_1 + \alpha_3 + \alpha_5)\Phi(b) + (\alpha_2 + \alpha_4 + \alpha_6)\Phi(b)^*$. Consequently, we get $\Phi(b^*) = \Phi(b)^*$. □

Claim 2.9. Φ *is a multiplicative map.*

Proof. For arbitrary elements $b, c \in \mathcal{A}$, replace a by $1_{\mathcal{A}}$ in the identity (1). Then

$$\Phi(\alpha_1 1_{\mathcal{A}} bc + \alpha_2 1_{\mathcal{A}}^* c b^* + \alpha_3 b 1_{\mathcal{A}}^* c + \alpha_4 c 1_{\mathcal{A}} b^* + \alpha_5 bc 1_{\mathcal{A}} + \alpha_6 c b^* 1_{\mathcal{A}}^*)$$
$$= \alpha_1\Phi(1_{\mathcal{A}})\Phi(b)\Phi(c) + \alpha_2\Phi(1_{\mathcal{A}})^*\Phi(c)\Phi(b)^* + \alpha_3\Phi(b)\Phi(1_{\mathcal{A}})^*\Phi(c)$$
$$+ \alpha_4\Phi(c)\Phi(1_{\mathcal{A}})\Phi(b)^* + \alpha_5\Phi(b)\Phi(c)\Phi(1_{\mathcal{A}}) + \alpha_6\Phi(c)\Phi(b)^*\Phi(1_{\mathcal{A}})^*$$

This results in the identity

$$(\alpha_1 + \alpha_3 + \alpha_5)\Phi(bc) + (\alpha_2 + \alpha_4 + \alpha_6)\Phi(cb^*) = (\alpha_1 + \alpha_3 + \alpha_5)\Phi(b)\Phi(c)$$
$$+ (\alpha_2 + \alpha_4 + \alpha_6)\Phi(c)\Phi(b)^*. \tag{14}$$

By applying involution to the identity (14), we get

$$(\overline{\alpha_1 + \alpha_3 + \alpha_5})\Phi(c^*b^*) + (\overline{\alpha_2 + \alpha_4 + \alpha_6})\Phi(bc^*) = (\overline{\alpha_1 + \alpha_3 + \alpha_5})\Phi(c)^*\Phi(b)^*$$
$$+ (\overline{\alpha_2 + \alpha_4 + \alpha_6})\Phi(b)\Phi(c)^* \tag{15}$$

and, replacing in (15) c^* by c, we obtain

$$(\overline{\alpha_2 + \alpha_4 + \alpha_6})\Phi(bc) + (\overline{\alpha_1 + \alpha_3 + \alpha_5})\Phi(cb^*) = (\overline{\alpha_2 + \alpha_4 + \alpha_6})\Phi(b)\Phi(c)$$

$$+ (\overline{\alpha_1 + \alpha_3 + \alpha_5})\Phi(c)\Phi(b)^*. \tag{16}$$

Multiplying (14) by the scalar $\overline{\alpha_1 + \alpha_3 + \alpha_5}$, (16) by the scalar $\alpha_2 + \alpha_4 + \alpha_6$ and subtracting the resulting identities, we arrive at $(|\alpha_1 + \alpha_3 + \alpha_5|^2 - |\alpha_2 + \alpha_4 + \alpha_6|^2)\Phi(bc) = (|\alpha_1 + \alpha_3 + \alpha_5|^2 - |\alpha_2 + \alpha_4 + \alpha_6|^2)\Phi(b)\Phi(c)$ which results in $\Phi(bc) = \Phi(b)\Phi(c)$. This shows that Φ is multiplicative. □

Therefore, by Claims 2.6, 2.8(ii) and 2.9 we conclude that Φ is a $*$-ring isomorphism.

The proof of the Theorem 1.1 is complete.

From Theorem 1.1 we can deduce the following result. However, we first present the necessary definitions and notations.

Let $\mathcal{A}$ and $\mathcal{B}$ be two complex $*$-algebras and η a non-zero complex number. For $a, b \in \mathcal{A}$ (resp., $a, b \in \mathcal{B}$) denote by $a\blacklozenge_\eta b = ab + \eta ba^*$, the *Jordan η-$*$-product*. We say that a nonlinear map $\Phi : \mathcal{A} \to \mathcal{B}$ *preserves Jordan triple $*$-product $a\blacklozenge_\eta b\blacklozenge_\nu c$*, where $a\blacklozenge_\eta b\blacklozenge_\nu c = (a\blacklozenge_\eta b)\blacklozenge_\nu c$ and η, ν are non-zero complex numbers, if $\Phi(a\blacklozenge_\eta b\blacklozenge_\nu c) = \Phi(a)\blacklozenge_\eta\Phi(b)\blacklozenge_\nu\Phi(c)$, for all elements $a, b, c \in \mathcal{A}$.

From the above definition, we can easily verify that nonlinear maps preserving Lie (mixed, Jordan) triple $*$-products, as defined in [3], [7] and [8], are nonlinear maps preserving Jordan triple $*$-products $a\blacklozenge_{-1}b\blacklozenge_{-1}c$, $a\blacklozenge_{-1}b\blacklozenge_1 c$ and $a\blacklozenge_1 b\blacklozenge_1 c$, respectively, and nonlinear maps preserving Jordan triple $*$-product $a\blacklozenge_\eta b\blacklozenge_\nu c$ are nonlinear maps that preserve sum of triple products $1abc + 0a^*cb^* + \eta ba^*c + \nu\overline{\eta}cab^* + 0bca + \nu cb^*a^*$.

In view of this, we have the following corollary.

Corollary 2.10. *Let $\mathcal{A}$ and $\mathcal{B}$ be two unital complex $*$-algebras with $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ their multiplicative identities, respectively, and such that $\mathcal{A}$ is prime and has a nontrivial projection. Then every bijective nonlinear map $\Phi : \mathcal{A} \to \mathcal{B}$ preserving triple $*$-product $a\blacklozenge_\eta b\blacklozenge_\nu c$, where η, ν are non-zero complex numbers satisfying the conditions $\eta \neq -1$ and $|\nu| \neq 1$, is additive. In addition, if (i) $\Phi(1_{\mathcal{A}})$ is a projection of $\mathcal{B}$ and (ii) $\Phi(\nu(\overline{\eta} + 1)a) = \nu(\overline{\eta} + 1)\Phi(a)$, for all element $a \in \mathcal{A}$, then Φ is a $*$-ring isomorphism. In particular, if $\Phi(1_{\mathcal{A}})$ is a projection of $\mathcal{B}$ and η and ν are non-zero complex numbers such that $\nu(\overline{\eta} + 1)$ is a rational number, then Φ is a $*$-ring isomorphism.*

References

[1] J.C.M. Ferreira, M.G.B. Marietto, *Nonlinear maps preserving a family of sum of triple products on $*$-algebras*, Bull. Iran. Math. Soc. **48** (2022), 3687–3699.

[2] J.C.M. Ferreira, M.G.B. Marietto, *Nonlinear maps of $*$-derivation-type on a family of sum of triple products on $*$-algebras*, Comm. Algebra **51** (2023), 3394–3404.

[3] C. Li, Q. Chen, T. Wang, *Nonlinear Maps Preserving the Jordan Triple $*$-Product on Factor von Neumann Algebras*, Chin. Ann. Math. Ser. B **39** (2018), 633–642.

[4] C. Li, F. Lu, *Nonlinear Maps Preserving the Jordan Triple* 1-$*$-*Product on Von Neumann Algebras*, Complex Anal. Oper. Theory **11** (2017), 109–117.

[5] C. Li, F. Lu, T. Wang, *Nonlinear maps preserving the Jordan triple $*$-product on von Neumann algebras*, Ann. Funct. Anal. **7** (2016), 496–507.

[6] C. Li, Y. Zhao, F. Zhao, *Nonlinear maps preserving the mixed product $[A \bullet B, C]_*$ on von Neumann algebras*, Filomat **35** (2021), 2775–2781.

[7] F. Zhang, *Nonlinear maps preserving the mixed triple $*$-product between factors*, Filomat **37** (2023), 2397–2403.

[8] F. Zhao, C. Li, *Nonlinear maps preserving the Jordan triple $*$-product between factors*, Indag. Math. **29** (2018), 619–627.