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On the curvatures of timelike circular surfaces in Lorentz-Minkowski space

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Abstract. In this paper, using the classical methods of differential geometry, we define invariants of timelike circular surfaces in Lorentz-Minkowski space $\mathbb{R}^3_1$, called curvature functions, and show kinematic meaning of these invariants. Then we discuss the properties of these invariants and give a kind of classification of the surfaces with the theories of these invariants. Besides, to demonstrate our theoretical results some computational examples are given and plotted.

1. Introduction

A circular surface is a special surface generated by a continuously moving of a circle with its center following a curve, which acts as the spine curve. Circular surfaces have the most important positions and applications in the study of design problems in spatial mechanisms and physics, kinematics and computer-aided design. Also, circular surfaces are one of the important topics of differential geometry. Because of this position, geometers have been studied on these surfaces in Euclidean and Minkowski 3-spaces and they have investigated many properties of these surfaces. R. Lopez[27] presented the Weingarten surface includes circular surface when its Gaussian curvature and mean curvature satisfy the formula: $aK + bH = c$ (a, b, c are all constants). M. P. Do. Carmo [4] discussed several geometric features of circular canal surfaces and proved two important theorems in the differential geometry concerning the total curvature of space curves, namely, Fenchel's theorem and the Fary–Milnor theorem. Lu and Pottmann [29] verified that a circular surface with a rational spine curve always admits a rational parametrization and proposed an algorithm for its computation. Patrikalakis and Maekawa [30] provided a thorough representation of local and global singularities of a circular surface. Luo [13, 14] delineated the characteristics of the double envelope of circular surface of constant diameter by means of concepts of parallel surfaces and parallel conjugate tooth surfaces and investigated systematically two examples which are the Riemann

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worm drive and arc surface worm drive with the double envelope of cylindrical surface. Izumiya et al. [10, 11] applied the method of moving frames to investigate the circular surfaces with various radii, their work concentrated on some corresponding properties of circular surfaces with classical ruled surfaces, and explored the singularities of circular surfaces. In addition, a team of researchers referred to as Li et al. [15–26] carried out theoretical research and advancement on submanifold theory, soliton theory, and other related areas. Additional motivations can be found in the papers referenced [1–14, 27–35]. Their work has contributed to the advancement of related research topics. Inspired by the above studies, the geometric properties of circular surfaces are helpful to the trajectory planning of the cutting tool and the determination of the radius of the cutter head in the processing of circular surfaces or the envelop surfaces of circular surface. By paying attention to this fact, Lei et al. [3] introduced differential geometry to analyze and synthesis of spatial mechanisms and studied extensively the invariants of some typical constraint circular surfaces. W. Wang, and D. Wang [32] extend the Serret–Frenet frame of ruled surface to that of circular surface, and the kinematic invariants introduced to show the differential structure of circular surfaces. It is well known that, Minkowski geometry provides the theoretical model in mathematics for Einstein's relativity theory. Especially, $\mathbb{R}^4_1$ (Minkowski space-time) has a solid physical background. The Lorentz-Minkowski space $\mathbb{R}^3_1 \subseteq \mathbb{R}^4_1$ has many properties which are different from Euclidean 3-space. Some basic concepts, such as vector, frame, and the motion of point, have qualitative changes. Since the metric in $\mathbb{R}^3_1$ is not positive definite metric, the distance function $\langle , \rangle$ can be positive, negative or zero, whereas the distance function in the Euclidean space can only be positive. Therefore, the vectors in $\mathbb{R}^3_1$ can be classified into three types: spacelike, timelike and lightlike vectors according to the sign of the distance function. Especially, emergence of lightlike vectors makes the results of some problems are surprising. As the spatiotemporal model of relativity theory, Minkowski space gains much attention in the field of mathematics and physics. However, the differential geometry of the ruled and circular surfaces in the Minkowski space has been studied thoroughly [1, 2, 5, 6, 12, 28, 31, 33].

This paper aims to study geometric properties and singularities of timelike circular surfaces with constant radius in Minkowski 3-space $\mathbb{R}^3_1$. We classify such timelike circular surfaces into a timelike canal surface, a Lorentzian sphere, a special kind of timelike surfaces or a timelike surface that is smoothly connected to three surfaces. The third surface in the classification is called a timelike tangent circular surface which is analogue to the tangent developable of a space curve.

2. Basic concepts

In this section, we list some notions, formulas and conclusions for curves and surfaces in $\mathbb{R}^3_1$ which can be found in the textbooks on differential geometry (See for instance [4, 28, 33–35]). Let $\mathbb{R}^3$ denote the real vector space with its usual vector structure. We denote (x_1, x_2, x_3) the coordinates of a vector with respect to the canonical basis of $\mathbb{R}^3$. The three-dimensional Minkowski 3-space is the metric space $\mathbb{R}^3_1 = (\mathbb{R}^3, \langle , \rangle)$, where the metric $\langle , \rangle$ is

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2 - x_3 y_3, \ \mathbf{x} = (x_1, x_2, x_3), \ \mathbf{y} = (y_1, y_2, y_3),$$

which is called the Lorentzian metric. A vector $\mathbf{x} \in \mathbb{R}^3_1$ is said to be spacelike if $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ or $\mathbf{x} = \mathbf{0}$, timelike if $\langle \mathbf{x}, \mathbf{x} \rangle < 0$ and lightlike or null if $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ and $\mathbf{x} \neq \mathbf{0}$. A timelike or light-like vector in $\mathbb{R}^3_1$ is said to be causal. We point out that the null vector $\mathbf{x} = \mathbf{0}$ is considered as a spacelike type although it satisfies $\langle \mathbf{x}, \mathbf{x} \rangle = 0$.

For $\mathbf{x} \in \mathbb{R}^3_1$ the norm is defined by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x} \rangle|}$, the vector $\mathbf{x}$ is called a spacelike unit vector if $\langle \mathbf{x}, \mathbf{x} \rangle = 1$ and a timelike unit vector if $\langle \mathbf{x}, \mathbf{x} \rangle = -1$. Similarly, a regular curve in $\mathbb{R}^3_1$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors are spacelike, timelike or null (lightlike), respectively. For any two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ of $\mathbb{R}^3_1$, the vector product is defined by

$$\mathbf{x} \times \mathbf{y} = ((x_2 y_3 - x_3 y_2), (x_3 y_1 - x_1 y_3), -(x_1 y_2 - x_2 y_1)).$$

The angle between two vectors in $\mathbb{R}^3_1$ is defined by [28, 33]:

Definition 2.1. *i) Spacelike angle: Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be spacelike vectors in* $\mathbb{R}^3_1$ *that span a spacelike vector subspace; we have* $\left|\langle \mathbf{x}, \mathbf{y}\rangle\right| \leq \|\mathbf{x}\| \left\|\mathbf{y}\right\|$*, and hence, there is a unique real number* $\theta \geq 0$*, such that* $\langle \mathbf{x}, \mathbf{y}\rangle = \|\mathbf{x}\| \left\|\mathbf{y}\right\| \cos\theta$*. This number is called the spacelike angle between the vectors* $\mathbf{x}$ *and* $\mathbf{y}$*.*
ii) Central angle: Let $\mathbf{x}$ *and* $\mathbf{y}$ *be spacelike vectors in* $\mathbb{R}^3_1$ *that span a timelike vector subspace; we have* $\left|\langle \mathbf{x}, \mathbf{y}\rangle\right| > \|\mathbf{x}\| \left\|\mathbf{y}\right\|$*, and hence, there is a unique real number* $\theta \geq 0$*, such that* $\langle \mathbf{x}, \mathbf{y}\rangle = \|\mathbf{x}\| \left\|\mathbf{y}\right\| \cosh\theta$*. This number is called the central angle between the vectors* $\mathbf{x}$ *and* $\mathbf{y}$*.*
iii) Lorentzian timelike angle: Let $\mathbf{x}$ *be spacelike vector and* $\mathbf{y}$ *be timelike vector in* $\mathbb{R}^3_1$*. Then, there is a unique real number* $\theta \geq 0$*, such that* $\langle \mathbf{x}, \mathbf{y}\rangle = \|\mathbf{x}\| \left\|\mathbf{y}\right\| \sinh\theta$*. This number is called the Lorentzian timelike angle between the vectors* $\mathbf{x}$ *and* $\mathbf{y}$*.*

The hyperbolic and Lorentzian unit spheres, respectively, are:

$$H^2_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3_1 \mid x_1^2 + x_2^2 - x_3^2 = -1,\ x_3 \geq 1\},$$

and

$$S^2_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3_1 \mid x_1^2 + x_2^2 - x_3^2 = 1\}.$$

We denote a surface M in $\mathbb{R}^3_1$ by

$$\boldsymbol{r}(u,\theta) = (x_1(u,\theta), x_2(u,\theta), x_3(u,\theta)),\ \ (u,\theta) \in D \subseteq \mathbb{R}^2.$$

Let $\mathbf{N}$ be the standard unit normal vector field on a surface M defined by $\boldsymbol{N} = \frac{\boldsymbol{r}_u \times \boldsymbol{r}_\theta}{\|\boldsymbol{r}_u \times \boldsymbol{r}_\theta\|}$, where, $\boldsymbol{r}_i = \frac{\partial \boldsymbol{r}}{\partial i}(i = u, \theta)$. Then the metric (first fundamental form) I of a surface M is defined by

$$I = g_{11}du^2 + 2g_{12}dud\theta + g_{22}d\theta^2,$$

where $g_{11} = \langle \boldsymbol{r}_u, \boldsymbol{r}_u\rangle$, $g_{12} = \langle \boldsymbol{r}_u, \boldsymbol{r}_\theta\rangle$, $g_{22} = \langle \boldsymbol{r}_\theta, \boldsymbol{r}_\theta\rangle$. We define the second fundamental form II of M by

$$II = h_{11}du^2 + 2h_{12}dud\theta + h_{22}d\theta^2,$$

where $h_{11} = \langle \boldsymbol{r}_{uu}, \boldsymbol{N}\rangle$, $h_{12} = \langle \boldsymbol{r}_{u\theta}, \boldsymbol{N}\rangle$, $h_{22} = \langle \boldsymbol{r}_{\theta\theta}, \boldsymbol{N}\rangle$.
M is called a timelike (spacelike) surface if the induced metric on M is a Lorentzian (Riemannian) metric on each tangent plane. This is equivalent to saying that the unit normal vector $\boldsymbol{N}$ is spacelike (timelike) at each point of M. Moreover, the Gaussian curvature K and the mean curvature H are given by

$$K = \langle \boldsymbol{N}, \boldsymbol{N}\rangle \frac{h_{11}h_{22} - h_{12}^2}{g_{11}g_{22} - g_{12}^2},\ \ H = \langle \boldsymbol{N}, \boldsymbol{N}\rangle \frac{h_{11}g_{11} - 2h_{12}g_{12} + h_{22}g_{22}}{2(g_{11}g_{22} - g_{12}^2)}.$$

Since Brioschi's formulas in Euclidean and Minkowski 3-spaces are the same, we are able to define the second Gaussian curvature K_{II} by [1, 2, 12]:

$$K_{II} = \frac{1}{h^2}\left(\begin{vmatrix} -\frac{h_{11,22}}{2} + h_{12,12} - \frac{h_{22,11}}{2} & \frac{h_{11,1}}{2} & h_{12,1} - \frac{h_{11,2}}{2} \\ h_{12,2} - \frac{h_{22,1}}{2} & h_{11} & h_{12} \\ \frac{h_{22,2}}{2} & h_{12} & h_{22} \end{vmatrix} - \begin{vmatrix} 0 & \frac{h_{11,2}}{2} & \frac{h_{22,1}}{2} \\ \frac{h_{11,2}}{2} & h_{11} & h_{12} \\ \frac{h_{22,1}}{2} & h_{12} & h_{22} \end{vmatrix} \right), \tag{1}$$

where $h = \det(h_{ij})$, $h_{ij,\alpha} = \frac{\partial h_{ij}}{\partial u^\alpha}$, and $h_{ij,\alpha\beta} = \frac{\partial^2 h_{ij}}{\partial u^\alpha \partial u^\beta}$. Furthermore, the second mean curvature H_{II} is given by:

$$H_{II} = H - \frac{1}{2}\Delta\left(\ln \sqrt{|K|}\right), \tag{2}$$

where Δ is the Laplacian with respect to the second fundamental form of M, expressed as:

$$\Delta = -\frac{1}{\sqrt{|h|}}\frac{\partial}{\partial u^i}\left[\sqrt{|h|}h^{ij}\frac{\partial}{\partial u^j}\right],\ \left(h^{ij}\right) = \left(h_{ij}\right)^{-1}.$$

As it is known, a timelike canal surface in $\mathbb{R}^3_1$ satisfying the Jacobian equation $f(K, H) = 0$ is called a Weingarten surface or a W-surface. Also, if a surface satisfies a linear equation with respect to K and H, that is, $aK + bH = c$ $(a, b, c \in \mathbb{R},\ (a, b, c) \neq (0, 0, 0))$, then, it is said to be a linear Weingarten surface or a LW-surface.

3. Circular surface

In this section, we define the notion of circular surfaces in Minkowski 3-space $\mathbb{R}^3_1$: Given a non null curve $\boldsymbol{\gamma} = \boldsymbol{\gamma}(u)$; that is, a smooth regular curve whose tangent vectors $\boldsymbol{t} = \boldsymbol{\gamma}'$ such that $\left\|\boldsymbol{\gamma}'\right\| \neq 0$ for every $u \in$ I, and a positive number $r > 0$, a circular surface is defined to be the surface that is swept out by a set of circles with their center points following the curve $\boldsymbol{\gamma}$. Each circle is called a generating circle, which lies on a plane named circle plane.

Let $\boldsymbol{e}_1$ be the timelike unit normal vector of the circle plane and u be the arc length of the spherical image curve $\boldsymbol{e}_1(u) \in H^2_+$. We can get: $\boldsymbol{e}_2 = \boldsymbol{e}_1'(u)$, and $\boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2$. So $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ constitute orthogonal vectors lies on the circular plane. The Blaschke moving frame $\{\boldsymbol{e}_1(u),\ \boldsymbol{e}_2(u),\ \boldsymbol{e}_3(u)\}$ whose origin point is on the spine curve, is set up, the differential operation equation is:

$$\begin{pmatrix} \boldsymbol{e}_1' \\ \boldsymbol{e}_2' \\ \boldsymbol{e}_3' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \gamma \\ 0 & -\gamma & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{e}_1 \\ \boldsymbol{e}_2 \\ \boldsymbol{e}_3 \end{pmatrix}, \tag{3}$$

where $\gamma(u)$ is called the geodesic (spherical) curvature function of the spherical curve $\boldsymbol{e}_1(u) \in H^2_+$. From now on, we assume such a parametrization and indicate its differentiation with respect to u with primes. One can easily have:

$$\begin{aligned} &\langle \boldsymbol{e}_1, \boldsymbol{e}_1\rangle = -1,\ \langle \boldsymbol{e}_2, \boldsymbol{e}_2\rangle = \langle \boldsymbol{e}_3, \boldsymbol{e}_3\rangle = 1, \\ &\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3, \boldsymbol{e}_3 \times \boldsymbol{e}_1 = \boldsymbol{e}_2, \boldsymbol{e}_2 \times \boldsymbol{e}_3 = -\boldsymbol{e}_1. \end{aligned} \tag{4}$$

Let the vector equation: $\boldsymbol{\gamma} = \boldsymbol{\gamma}(u)$, $u_1 \leq u \leq u_2$ be the general equation of the spine non-null curve. In this work we will assume that M is a timelike surface, we can get the vector equation of the circular surface:

$$\left.\begin{aligned} &M:\ \boldsymbol{r}(u, \theta) = \boldsymbol{\gamma}(u) + r\left(\cos\theta \boldsymbol{e}_2(u) + \sin\theta \boldsymbol{e}_3(u)\right), \\ &u_1 \leq u \leq u_2,\ 0 \leq \theta \leq 2\pi, \end{aligned}\right\} \tag{I}$$

where the positive number $r > 0$ is the radius of the generating circle. The tangent vector $\boldsymbol{\gamma}'$ can be expressed by the moving frame attached to each of its points as

$$\boldsymbol{\gamma}' = \alpha\boldsymbol{e}_1 + \sigma\boldsymbol{e}_2 + \eta\boldsymbol{e}_3. \tag{5}$$

The four functions $\gamma(u)$, $\alpha(u)$, $\sigma(u)$ and $\eta(u)$ constitute a complete system of curvature functions (invariants) of the surface M. When $\alpha \neq 0$, and $\sigma = \eta = 0$, the spine curve is perpendicular to the circular plane and when $\alpha = 0$, and σ, η don't equal to zero simultaneously, the spine curve is tangent to the circular plane. It is easily checked that the two tangent vectors of M are given by:

$$\left.\begin{aligned} &\boldsymbol{r}_u = (r\cos\theta + \alpha)\boldsymbol{e}_1 + (\sigma - r\gamma\sin\theta)\boldsymbol{e}_2 + (\eta + r\gamma\cos\theta)\boldsymbol{e}_3, \\ &\boldsymbol{r}_\theta = r(-\sin\theta\boldsymbol{e}_2 + \cos\theta\boldsymbol{e}_3). \end{aligned}\right\} \tag{6}$$

Thus, we have

$$\left.\begin{aligned} &g_{11} = -(r\cos\theta + \alpha)^2 + (\sigma - r\gamma\sin\theta)^2 + (\eta + r\gamma\cos\theta)^2, \\ &g_{12} = r(r\gamma - \sigma\sin\theta + \eta\cos\theta),\ g_{22} = r^2, \\ &g_{11}g_{22} - g_{12}^2 = r^2\left\{-(\alpha + r\cos\theta)^2 + (\sigma\cos\theta + \eta\sin\theta)^2\right\}. \end{aligned}\right\} \tag{7}$$

So, the unit normal vector is:

$$N(u,\theta) = \frac{-(\sigma\cos\theta + \eta\sin\theta)\boldsymbol{e}_1 - (r\cos\theta + \alpha)(\cos\theta\boldsymbol{e}_2 + \sin\theta\boldsymbol{e}_3)}{\sqrt{\left|-(\sigma\cos\theta + \eta\sin\theta)^2 + (\alpha + r\cos\theta)^2\right|}}. \tag{8}$$

By a straightforward calculation, we get:

$$\left.\begin{aligned} \boldsymbol{r}_{uu} &= (\alpha' - \sigma)\boldsymbol{e}_1 + (\sigma' + r\cos\theta + \alpha)\boldsymbol{e}_2 + (\eta' + \sigma\gamma)\boldsymbol{e}_3 + \gamma\boldsymbol{r}_{\theta u}, \\ \boldsymbol{r}_{u\theta} &= -r(\sin\theta\boldsymbol{e}_1 + \gamma\cos\theta\boldsymbol{e}_2 + \gamma\sin\theta\boldsymbol{e}_3), \\ \boldsymbol{r}_{\theta\theta} &= -r(\cos\theta\boldsymbol{e}_2 + \sin\theta\boldsymbol{e}_3). \end{aligned}\right. \tag{9}$$

Then, the coefficients of the second fundamental form are:

$$\left.\begin{aligned} h_{11} &= \frac{\begin{array}{c}(r\cos\theta + \alpha)[r\gamma - (\eta' + \sigma\gamma)\sin\theta - (\alpha + r\cos\theta + \sigma')\cos\theta] \\ +(\alpha' - \sigma - r\gamma\sin\theta)(\eta\sin\theta + \sigma\cos\theta)\end{array}}{\sqrt{\left|-(\sigma\cos\theta + \eta\sin\theta)^2 + (\alpha + r\cos\theta)^2\right|}}, \\ h_{12} &= \frac{r[-(\sigma\cos\theta + \eta\sin\theta)\sin\theta + \gamma(r\cos\theta + \alpha)]}{\sqrt{\left|-(\sigma\cos\theta + \eta\sin\theta)^2 + (\alpha + r\cos\theta)^2\right|}}, \\ h_{22} &= \frac{r(r\cos\theta + \alpha)}{\sqrt{\left|-(\sigma\cos\theta + \eta\sin\theta)^2 + (\alpha + r\cos\theta)^2\right|}}. \end{aligned}\right\} \tag{10}$$

It is well known that, one can give a classification to the circular surfaces M into timelike or spacelike surfaces if $g_{11}g_{22} - g_{12}^2 < 0$ or $g_{11}g_{22} - g_{12}^2 > 0$ is satisfied.

3.1. Local singularities and striction curves

Singularities and striction curves are essential for understanding the properties of circular surfaces and are investigated in the following.

It can be seen that the circular surface M has singularities if and only if

$$\|\boldsymbol{r}_u \times \boldsymbol{r}_\theta\| = \sqrt{\left|-(\sigma\cos\theta + \eta\sin\theta)^2 + (\alpha + r\cos\theta)^2\right|} = 0.$$

This is equivalent to

$$\left.\begin{aligned} &\alpha + r\cos\theta = 0, \\ &\sigma\cos\theta + \eta\sin\theta = 0. \end{aligned}\right\} \tag{11}$$

There are three cases for Eq.(11) to be discussed for all values of θ as follows:

Case 3.1. *: $\alpha \neq 0$, and $\sigma = \eta = 0$, the tangent vectors of the spine curve $\boldsymbol{\gamma}$ are always perpendicular to the circle planes. Hence, the circular surfaces become timelike canal surfaces, which have been extensively studied in [2]. In this case, if $|\alpha| > r$, there are no singular points, and If $|\alpha| < r$, the singular points occurring at $\theta = \pm\cos^{-1}(\alpha/r)$.*

Case 3.2. *: $\alpha = 0$, for a circular surface to have singular points, it is necessary that $\cos\theta = 0$, and $\eta = 0$. Hence, there are two singular points on the generating circle, occurring at $\theta = \pm\pi/2$.*

Case 3.3. *: $\sigma \neq 0$, and $\eta \neq 0$, the singular points occur at $\theta = \sin^{-1}(\alpha\sigma/\eta r)$. If $|\alpha| > r$, there are no singular points. If $\alpha = \pm r$, the singular points occur at $\theta = \pm\sin^{-1}(\sigma/\eta)$.*

We now define striction curves of circular surfaces compared with those of ruled surfaces [4, 11]:

Definition 3.4. *A curve is given by*

$$\boldsymbol{\xi}(u) = \boldsymbol{\gamma}(u) + r(\cos\theta \boldsymbol{e}_2(u) + \sin\theta \boldsymbol{e}_3(u)), \tag{12}$$

is a striction curve of the circular surface M, if $\boldsymbol{\gamma}'$, $\boldsymbol{e}_2$, *and* $\boldsymbol{e}_3$ *satisfy:*

$$\langle \boldsymbol{e}_2, \boldsymbol{\gamma}' \rangle = \langle \boldsymbol{e}_3, \boldsymbol{\gamma}' \rangle = 0 \Leftrightarrow \sigma = \eta = 0, \tag{13}$$

for all $u \in I$.

This definition means that any curves on the circular surface M transverse to generating circles satisfy the condition of striction curves. So, the class of the circular surfaces M is an analogous class to the class of cylindrical surfaces. A thorough treatment on timelike canal surfaces will be given in Section 4.
We can also define the notion of non-canal circular surfaces analogous to that of non-cylindrical ruled surfaces.

Definition 3.5. *A circular surface M is a non-canal surface of radius r, if* $\boldsymbol{\gamma}'$, $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, *and* $\boldsymbol{e}_3$ *satisfy:*

$$\langle \boldsymbol{e}_1, \boldsymbol{\gamma}' \rangle = \alpha \neq 0, \text{ and } \langle \boldsymbol{e}_2, \boldsymbol{\gamma}' \rangle = \sigma \neq 0 \text{ or } \langle \boldsymbol{e}_3, \boldsymbol{\gamma}' \rangle = \eta \neq 0, \tag{14}$$

for all $u \in I$.

4. Timelike canal surface

Timelike canal surfaces are defined to be those that the tangent vectors of spine curve are always perpendicular to the circle planes. This section examines in details the properties of timelike canal surfaces.
Let s be the arc length parameter of the timelike spine curve $\gamma(u)$, then

$$s = \int_0^u \left\| \gamma' \right\| du,$$

and $\{\boldsymbol{t}(s), \boldsymbol{n}(s), \boldsymbol{b}(s)\}$ is the moving Frenet frame along $\boldsymbol{\gamma}(u)$. Then, we have:

$$\boldsymbol{t}(s) = \frac{\gamma'}{\left\| \gamma' \right\|} = \boldsymbol{e}_1, \ \boldsymbol{n}(s) = \boldsymbol{e}_2, \ \boldsymbol{b}(s) = \boldsymbol{t}(s) \times \boldsymbol{n}(s) = \boldsymbol{e}_3. \tag{15}$$

It is interesting to note that as long as $\boldsymbol{e}_1$ is perpendicular to the circle planes at each point of the spine curve $\boldsymbol{\gamma}(u)$. Depending on the causal character of the curve $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$, we have the following Frenet formulae:

$$\frac{d}{ds}\begin{pmatrix} \boldsymbol{t}(s) \\ \boldsymbol{n}(s) \\ \boldsymbol{b}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ \kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{t}(s) \\ \boldsymbol{n}(s) \\ \boldsymbol{b}(s) \end{pmatrix}, \tag{16}$$

where the curvature $\kappa(s)$, and torsion $\tau(s)$ of $\boldsymbol{\gamma}(u)$ are given by:

$$\kappa(s) = \frac{1}{\alpha}, \ \tau(s) = \frac{\gamma}{\alpha}, \ \alpha > 0. \tag{17}$$

Then, a timelike canal surface of radius r about $\boldsymbol{\gamma}(u)$ is the surface with parametrization

$$M : \boldsymbol{r}(s, \theta) = \boldsymbol{\gamma}(s) + r(\cos\theta \boldsymbol{n}(s) + \sin\theta \boldsymbol{b}(s)). \tag{18}$$

4.1. The relation among the curvature functions

We now study timelike canal surfaces satisfying some equations in terms of the Gaussian curvature, the mean curvature, the second Gaussian curvature, and the second mean curvature as follows:
Firstly, the Gaussian curvature K and the mean curvature H are, respectively

$$K(s,\theta) = -\frac{\kappa\cos\theta}{r\zeta}, \text{ and } H(s,\theta) = \frac{2r\kappa\cos\theta + 1}{2r\zeta}, \tag{19}$$

where $\zeta = 1 + r\kappa\cos\theta$. From Eq. (19) it is interesting to note that K, and H are independent of τ. In other words, if a family of timelike canal surfaces has the same value of κ, then the values of their Gaussian, and mean curvatures are the same at the corresponding point which is a fact is geometrically nontrivial. Moreover, we also have

$$r^2K + 2rH - 1 = 0, \tag{20}$$

and this means that M is a W-surface. On the other hand, from Eqs. (1), (2), (18), it follows that

$$\begin{aligned} K_{II} &= \frac{\kappa}{4r\zeta^2}\left(6r\kappa\cos\theta - 4r^2\kappa^2\cos^2\theta - \sec^2\theta - 1\right), \\ H_{II} &= \frac{1}{4\zeta^4r^2\kappa^2}[a_0 + a_1r^2\kappa\cos\theta - 2a_2r^3\kappa^4\sec^2\theta + a_3r\kappa^3\sec\theta \\ &\quad + a_4r^3\kappa^2\sin\theta + a_5\kappa^2\tan\theta], \end{aligned} \tag{21}$$

where a_0, a_1,..., a_5 are

$$\begin{aligned} a_0 &= r^2\kappa\kappa_{ss} - 3r^2\kappa^4 - 2r\kappa^2 + r^2\kappa_s^2 - \kappa^2, \\ a_1 &= 10\kappa^2 + r\kappa^4 + 3r\kappa_s^2 - r\kappa\kappa_{ss}, \\ a_2 &= 9 + 7r\kappa\cos\theta - 2r^2\kappa^2\cos^2\theta, \\ a_3 &= 3\kappa - 2r^2\kappa\tau^2 + r^2\kappa\tau^2\cos 2\theta + r\tau^2\sec\theta, \\ a_4 &= 4\kappa_s\tau - \kappa\tau_s, \\ a_5 &= r^2\tau_s + \tan\theta. \end{aligned}$$

Differentiating K, K_{II}, H, and H_{II} with respect to s and θ respectively, then after straightforward calculations, we get

$$\begin{gathered} (K)_s = -\frac{\kappa_s\cos\theta}{r\zeta^2}, \quad (K)_\theta = -\frac{\kappa\sin\theta}{r\zeta^2}, \\ (H)_s = \frac{r\kappa_s\cos\theta}{2r\zeta^2}, \quad (H)_\theta = \frac{\kappa\sin\theta}{2r\zeta^2}, \\ (K_{II})_s = \frac{\kappa_s(8r^3\kappa^3\cos^3\theta - 18r^2\kappa^2\cos^2\theta + 12r\kappa\cos^3\theta + \cos^2\theta + 1)}{4\xi^2\cos^2\theta}, \\ (K_{II})_\theta = \frac{\kappa(8r^3\kappa^3\cos^3\theta - 18r^2\kappa^2\cos^2\theta + 12r\kappa\cos^3\theta + \sin^2\theta + 2r\kappa\cos\theta)\sin\theta}{4\xi^2\cos^2\theta}, \\ (H_{II})_s = \frac{1}{8\kappa^4\xi^4\cos^3\theta}\sum_{i=0}^{6} p_i\cos^i\theta, \end{gathered} \tag{22}$$

and where the coefficients p_i are

$$\begin{aligned} p_0 &= 3\kappa^2\tau(\kappa_s\tau - 2\kappa\tau_s), \\ p_1 &= 2\kappa\left(2\kappa_s(\kappa_s\tau - \kappa\tau_s) - \kappa\tau\kappa_{ss}\right)\sin\theta + 6r\kappa^3\tau(3\kappa\tau_s - 2k_s\tau), \\ p_2 &= 2r\kappa^2\left(9\kappa_s(\kappa_s\tau - \kappa\tau_s) + 2\kappa\tau\kappa_{ss}\right)\sin\theta + 6r^2\kappa^4\tau(3\kappa\tau_s - 2k_s\tau) \\ &\quad + \kappa_s(9\kappa_s^2 - 10\kappa\kappa_{ss}) + \kappa^2\tau(2\kappa\tau_s - k_s\tau), \\ p_3 &= 2r\kappa^3\left(\kappa_s(16\kappa_s\tau - 7\kappa\tau_s) - 4\kappa\tau\kappa_{ss}\right)\sin\theta + \\ &\quad 2r\kappa\left[15\kappa\kappa_s\kappa_{ss} - (15\kappa_s^2 + \kappa^4)\kappa_s + k^2\tau(2\tau\kappa_s - 5\kappa\tau_s)\right], \\ p_4 &= 2r^2\kappa^2\left[5\kappa_s(3\kappa_s^2 - 2\kappa\kappa_s) + 2k^2\tau(2\tau\kappa_s - 5\kappa\tau_s)\right] - 2\kappa^2\kappa_s, \\ p_5 &= 6r\kappa^5\kappa_s, \ p_6 = -4r^2\kappa^6\kappa_s. \end{aligned}$$

And, we have

$$(H_{II})_\theta = \frac{1}{8r\kappa^3\xi^4\cos^4\theta}\sum_{i=0}^{6}\mu_i\cos^i\theta, \tag{23}$$

where the coefficients μ_i are

$$\begin{aligned}
\mu_0 &= -9\kappa^2\tau\sin\theta,\\
\mu_1 &= 2\kappa^3\left(1+15r^2\tau^2\right)\sin\theta + 4r\kappa(\kappa\tau_s - k_s\tau),\\
\mu_2 &= r\left[2\kappa\kappa_{ss} - 8\kappa^4 + \kappa^2\tau^2(1-30r^2\kappa^2) - \kappa_s^2\right]\sin\theta + 6r^2\kappa^2(3\kappa\tau_s - \kappa_s\tau),\\
\mu_3 &= 4r^2\kappa\left[2\kappa^4 - \tau^2\kappa^2 - 2\kappa\kappa_{ss} + 3\kappa_s^2\right]\sin\theta +\\
&\quad 2r\kappa\left[\kappa\tau_s - \kappa_s\tau + 4r^2\kappa^2(\kappa\tau_s - 4\kappa_s\tau)\right],\\
\mu_4 &= 2r^2\kappa^3\left[3r^2(2\kappa^2\tau - \kappa^3 + \kappa_{ss}) + \kappa\right]\sin\theta + 2r^2\kappa^2\left[4(\kappa\tau_s - \kappa_s\tau) - 9r\kappa_s^3\right],\\
\mu_5 &= 6r^2\kappa^3\left[r(4\kappa\tau_s - \kappa_s\tau) - \kappa^2\sin\theta\right],\ \mu_6 = 4r^3\kappa^6\sin\theta.
\end{aligned}$$

Now, for the timelike canal surface M, one can obtain the following:
(i) $f(K, H_{II}) = (K)_s\,(H_{II})_\theta - (K)_\theta\,(H_{II})_s = 0$,
(ii) $f(H, H_{II}) = (H)_s\,(H_{II})_\theta - (H)_\theta\,(H_{II})_s = 0$,
(iii) $f(K_{II}, H_{II}) = (K_{II})_s\,(H_{II})_\theta - (K_{II})_\theta\,(H_{II})_s = 0$.
From (i) – (iii), one can get the Jacobian equations which are equivalent to $\tau_s = \kappa_s = 0$. Therefore, κ, τ are constants. Consequently, we have the following theorem:

Theorem 4.1. *Let $X, Y \in \{K,\ K_{II},\ H,\ H_{II}\}$, and let M be a timelike canal surface defined by Eq.* (18) *with non-degenerate second fundamental form. Then M is a (X, Y)-timelike canal W-surface if and only if M is a timelike canal surface around a timelike circle or a timelike helix.*

Secondly, we study (K, H_{II}), (H, H_{II}), (H_{II}, K_{II}), (K, H, H_{II}), (K, H, K_{II}), (H, K_{II}, H_{II}), (K, K_{II}, H_{II}), and (K, H, K_{II}, H_{II})-timelike canal surfaces in $\mathbb{R}^3_1$, whereas (K, H), (K, K_{II}), and (H, K_{II})-timelike canal W/LW-surfaces are studied in [32].
Let a_1, a_2, a_3, a_4, and b are constants, a linear combination of K, K_{II}, H, and H_{II} can be constructed as:

$$a_1K + a_2K_{II} + a_3H + a_4H_{II} = b. \tag{24}$$

After some algebraic manipulations, we can give the following theorems:

Theorem 4.2. *Let $X, Y \in \{(K, H_{II}),\ (H, H_{II}),\ (H_{II}, K_{II})\}$. Then, there are no (X, Y)-timelike canal LW-surfaces M defined by Eq.* (18)*with non-degenerate second fundamental form.*

Theorem 4.3. *Let $X,\ Y,\ Z \in \{(K, H, H_{II}), (K, H, K_{II}), (H, K_{II}, H_{II}),\ (K, K_{II}, H_{II})\}$. Then, there are no $(X,\ Y,\ Z)$-timelike canal LW-surfaces M defined by Eq.* (18) *with non-degenerate second fundamental form.*

Theorem 4.4. *Let M be a timelike canal surface defined by Eq.* (18) *with non-degenerate second fundamental form. Then, there are no (K, H, K_{II}, H_{II})-timelike canal LW-surface in Minkowski 3-space $\mathbb{R}^3_1$.*

Therefore, the study of timelike canal W/LW-surfaces in Minkowski 3-space $\mathbb{R}^3_1$ is completed with [2, 12]. In the following, we give simple examples.

Example 4.5. *Suppose we are given a parametric timelike circular helix*

$$\boldsymbol{\gamma}(s) = (a\cos s, a\sin s, bs),\ a > 0,\ \ b \neq 0,\ a^2 - b^2 = -1,\ 0 \leq s \leq 4\pi.$$

After simple computation, we have:

$$\boldsymbol{t}(s) = (-a\sin s, a\cos s, b),\ \boldsymbol{n}(s) = (\cos s, \sin s, 0),\ \boldsymbol{b}(s) = (-b\sin s, b\cos s, a).$$

Figure 1 shows a timelike canal W-surface with $b = \sqrt{3}$, $a = \sqrt{2}$, $r = 1$, and $0 \leq \theta \leq 2\pi$.

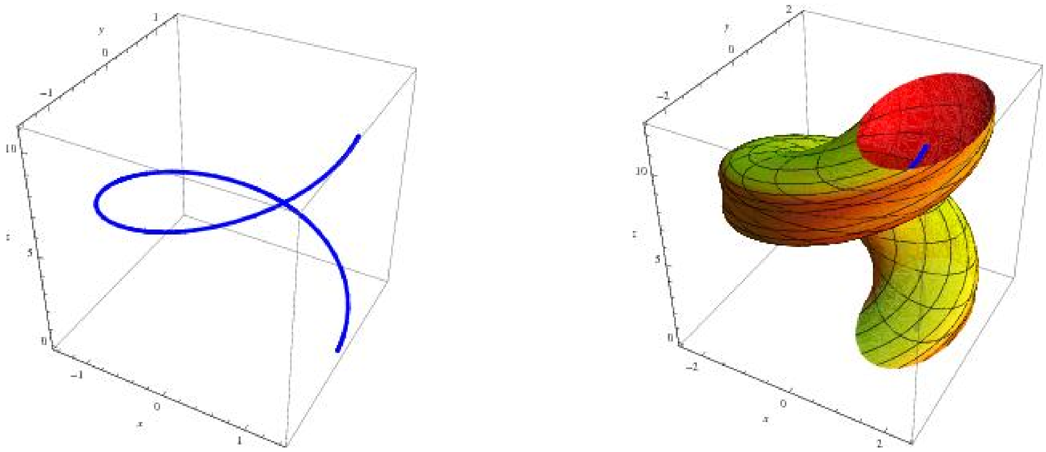

Figure 1: The timelike circular helix $\boldsymbol{\gamma}(s)$ and corresponding timelike canal W-surface $\boldsymbol{r}(s,\theta)$.

Example 4.6. *Given a parametric timelike curve,*

$$\boldsymbol{\gamma}(s) = (0, \cosh s, \sinh s),\ a > 0,\ \ -1 \leq s \leq 1.$$

By computing, we get:

$$\boldsymbol{t}(s) = (0, \sinh s, \cosh s),\ \boldsymbol{n}(s) = (0, \cosh s, \sinh s),\ \boldsymbol{b}(s) = (-1, 0, 0).$$

Figure 2 shows a timelike canal W-surface with a timelike planar curve.

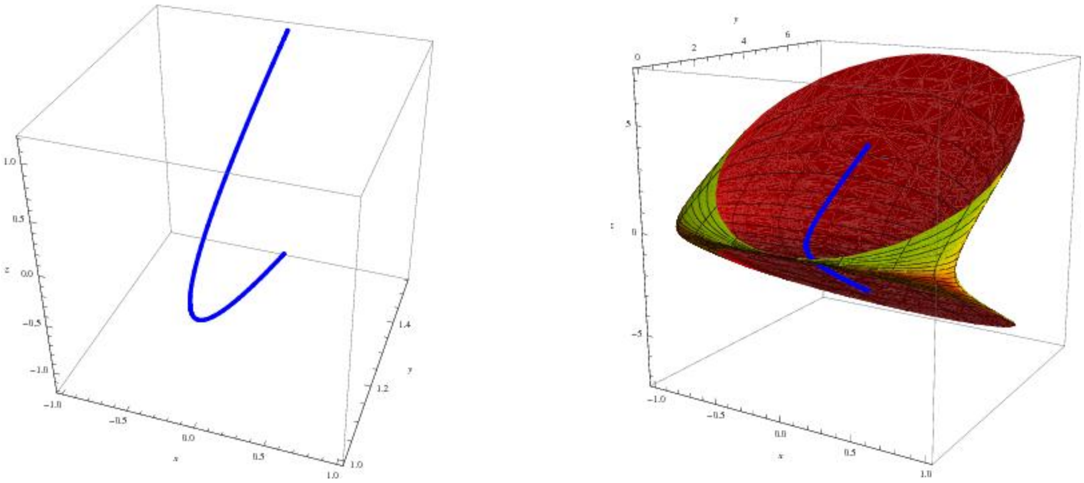

Figure 2: The timelike planar curve and corresponding timelike canal W-surface.

5. Timelike tangent circular surfaces

Since the lines of curvature are geometric features of surfaces, it is interesting to know in what conditions the generating circles are lines of curvature. This section will discuss these conditions for a timelike circular surface. For this purpose, we review a known theorem, which characterizes lines of curvature on a surface [4, 28].

Theorem 5.1. *A necessary and sufficient condition that a curve on a surface be a line of curvature is that the surface normals along the curve form a developable surface.*

According to Theorem 5, the circles are lines of curvature if and only if

$$\frac{\partial \mathbf{N}(u,\theta)}{\partial \theta} \parallel \boldsymbol{r}_\theta, \tag{25}$$

for all the values of θ. After some algebraic manipulations, the following can be obtained

$$\begin{aligned}(\alpha + r\cos\theta)\left[\alpha(\eta\cos\theta + \sigma\sin\theta) - r\eta\right] = 0,\\ (\sigma\cos\theta + \eta\sin\theta)\left[\alpha(\eta\cos\theta + \sigma\sin\theta) - r\eta\right] = 0.\end{aligned}$$

Then, for all non-singular points, we obtain

$$\alpha(\eta\cos\theta + \sigma\sin\theta) - r\eta = 0. \tag{26}$$

Since, singular points on a generating circle are at most two points, then by differentiating Eq. (26) with respect to θ, we have

$$\alpha(-\eta\sin\theta + \sigma\cos\theta) = 0. \tag{27}$$

This is equivalent to $\eta(u) = \sigma(u) = 0$ or $\alpha(u) = 0$ for all $u \in I$. From Eq. (27) we have two different cases:
(1) In the case of $\alpha = \eta = \sigma = 0$, the tangent vector of the spine curve is zero vector, that is, $\boldsymbol{\gamma}' = 0$. Thus the spine curve $\boldsymbol{\gamma}$ is a fixed point. The circular surfaces become Lorentzian spheres with radii r.
(2) In the case of $\alpha = \eta = 0$, and $\sigma \neq 0$, the tangent vectors of the spine curve $\boldsymbol{\gamma}$ lie on the circle planes at each point of $\boldsymbol{\gamma}$. From Eq. (5), it follows that

$$\boldsymbol{\gamma}' = \sigma \boldsymbol{e}_2, \tag{28}$$

when σ is a constant, integrating Eq. (28) gives

$$\boldsymbol{\gamma} = \sigma \boldsymbol{e}_1 + \boldsymbol{\gamma}_0, \tag{29}$$

where $\boldsymbol{\gamma}_0$ is a constant vector. From Eqs. (I), and (29) it can be found that:

$$\left\|\boldsymbol{r} - \boldsymbol{\gamma}_0\right\|^2 = -\sigma^2 + r^2, \tag{30}$$

which means that all the circle points lie on a Lorentzian sphere of radius $\sqrt{-\sigma^2 + r^2} < r$ with $\boldsymbol{\gamma}_0$ being its center point in $\mathbb{R}^3_1$.

A timelike circular surface with a non-constant σ, and $\alpha = \eta = 0$, is coined as the timelike tangent circular, since it is analogous to the tangent developable of a space curve [2]. Moreover, in view of Eqs. (6), the singular points can be obtained as:

$$\|\boldsymbol{r}_u \times \boldsymbol{r}_\theta\| = |\cos\theta| \sqrt{-\sigma^2 + r^2} = 0. \tag{31}$$

From Eq. (31) it follows that singularities only occur when $\theta = \pm\pi/2$ since $\sqrt{-\sigma^2 + r^2} \neq 0$. Hence, there are two singular points on every generating circle. Connecting these two sets of singular points gives two striction curves that contain all the singular points of a timelike tangent circular surface. Further, from Eq. (12) it follows that the expression of the two striction curves is

$$\boldsymbol{\zeta}_1(u) = \boldsymbol{\gamma}(u) + r\boldsymbol{e}_3(u),\ \boldsymbol{\zeta}_2(u) = \boldsymbol{\gamma}(u) - r\boldsymbol{e}_3(u). \tag{32}$$

Obviously, their curvatures κ_i and torsions τ_i $(i = 1, 2)$ can be obtained as

$$\left.\begin{aligned}\kappa_1 = \tfrac{1}{\sigma - r\gamma}\sqrt{-1+\gamma^2},\ \tau_1 = \tfrac{\gamma'}{(-1+\gamma^2)(\sigma - r\gamma)},\\ \kappa_2 = \tfrac{1}{\sigma + r\gamma}\sqrt{-1+\gamma^2},\ \tau_2 = \tfrac{\gamma'}{(-1+\gamma^2)(\sigma + r\gamma)}.\end{aligned}\right\} \tag{33}$$

From Eq. (33) it follows that if γ is a constant, then each of the torsions τ_i equals zero. Thus the striction curves are planar spacelike curves. Hence, the following conclusion can be reached.

Theorem 5.2. *In the Minkowski 3-space $\mathbb{R}^3_1$, besides the timelike circular surfaces there are two families of timelike circular surfaces whose generating circles are lines of curvature. These two families are the timelike tangent circular surfaces and the Lorentzian spheres with the radius being less than that of the generating circles.*

Since $\alpha = \eta = 0$ in the timelike tangent circular surface, the Gaussian, and mean curvatures can be obtained as:

$$\begin{aligned} K(u,\theta) &= \tfrac{1}{r^2-\sigma^2} + \tfrac{r\sigma'}{(r^2-\sigma^2)^2\cos\theta}, \\ H(u,\theta) &= \tfrac{\sigma'}{2\left(\sqrt{r^2-\sigma^2}\right)^3\cos\theta} + \tfrac{1}{\sqrt{r^2-\sigma^2}}. \end{aligned} \tag{34}$$

Because every generating circle is a line of curvature for a timelike tangent circular surface, the value of one principal curvatures is:

$$k_1 = \frac{1}{r}. \tag{35}$$

So, the other principal curvature is given by:

$$k_2 = \frac{K}{k_1} = \frac{r\left[(r^2-\sigma^2)\cos\theta + r\sigma'\right]}{(r^2-\sigma^2)^2\cos\theta}. \tag{36}$$

It is a remarkable fact that concepts such as Gaussian and mean curvatures whose definitions make essential use of the position of a surface in the space, do not depend on geodesic curvature of the spherical indicatrix of $\boldsymbol{e}_1$, but only on θ and σ. In other words, if a family of timelike tangent circular surfaces has the same value of γ, then the values of their Gaussian, and mean curvatures are the same at the corresponding point, a fact that is geometrically nontrivial. By the above calculation, we have the following Corollary.

Corollary 5.3. *If a family of timelike tangent circular surfaces have the same radius, scalar σ, and it is derivative σ', the Gaussian and the mean curvatures are the same at corresponding points. Furthermore, these values are independent of the geodesic curvature of the spherical indicatrix of the vector $\boldsymbol{e}_1$.*

However, in order to describe the kinematic geometry properties of timelike circular surface, the Serret–Frenet of the spine curve $\boldsymbol{\gamma}(u)$ is needed to be built. For this purpose, let s denote the arc length of the spine curve $\boldsymbol{\gamma}(u)$ and assume that $\sigma(u) > 0$, at any $u \in I \subseteq \mathbb{R}$, the Frenet frame of the spine curve can be obtained:

$$\boldsymbol{T}(s) = \frac{\boldsymbol{\gamma}'}{\left\|\boldsymbol{\gamma}'\right\|} = \boldsymbol{e}_2,\ \boldsymbol{N}(s) = \frac{\boldsymbol{T}'}{\|\boldsymbol{T}'\|} = \frac{\boldsymbol{e}_1 + \gamma\boldsymbol{e}_3}{\sqrt{-1+\gamma^2}},\ \boldsymbol{B}(s) = \frac{\gamma\boldsymbol{e}_1 + \boldsymbol{e}_3}{\sqrt{-1+\gamma^2}}. \tag{37}$$

Consequently, the following relations exist:

$$\begin{pmatrix} \boldsymbol{T}(s) \\ \boldsymbol{N}(s) \\ \boldsymbol{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ \sinh\varphi & 0 & \cosh\varphi \\ \cosh\varphi & 0 & \sinh\varphi \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_1(s) \\ \boldsymbol{e}_2(s) \\ \boldsymbol{e}_3(s) \end{pmatrix}, \tag{38}$$

where

$$\cosh\varphi = \frac{\gamma}{\sqrt{-1+\gamma^2}},\ \sinh\varphi = \frac{1}{\sqrt{-1+\gamma^2}},\ \ \gamma > 1.$$

By taking the derivative of Eq. (38) with respect to s, and using the inverse transformation, we obtain:

$$\frac{d}{ds}\begin{pmatrix} \boldsymbol{T}(s) \\ \boldsymbol{N}(s) \\ \boldsymbol{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{T}(s) \\ \boldsymbol{N}(s) \\ \boldsymbol{B}(s) \end{pmatrix}, \tag{39}$$

where the curvature and torsion of spine curve $\boldsymbol{\gamma}$ can be obtained in terms of σ and γ as:

$$\kappa(s) = \frac{\sqrt{-1+\gamma^2}}{\sigma}, \ \tau(s) - \frac{d\varphi}{ds} = 0, \ \frac{d\varphi}{ds} = \frac{\gamma'}{\sigma(-1+\gamma^2)}. \tag{40}$$

That is

$$\varphi(s) = -\int_{s_0}^{s} \tau ds + \varphi_0,$$

where s_0 is the starting value of the arclength. In terms of the Frenet frame $\{\boldsymbol{T}(s), \boldsymbol{N}(s), \boldsymbol{B}(s)\}$, the parametric expression of the timelike tangent circular surface can be given as follows:

$$M : \boldsymbol{r}(s,\theta) = \boldsymbol{\gamma} + r[\cos\theta \boldsymbol{T} + \sin\theta(\cosh\varphi \boldsymbol{N} - \sinh\varphi \boldsymbol{B})], \tag{41}$$

In the above equation, we do not only prove the existence of the timelike tangent circular, but also give the concrete expression of the surface. This is very meaningful in practical application.
With the aid of Eqs. (34), (35), (36) and (40) we have the following theorem for timelike tangent circular surfaces.

Theorem 5.4. *For the timelike tangent circular surface M expressed by Eq.* (41), *we have the following results:*
1- The Gaussian $K(s,\theta)$ and mean $H(s,\theta)$ curvatures of M can be obtained as:

$$\begin{aligned} K(s,\theta) &= \frac{\kappa\sinh\varphi\left[(-1+r^2\kappa^2\cosh^2\varphi)\kappa\cosh\varphi\cos\theta + r(\kappa_s\sinh\varphi + \kappa\tau\cosh\varphi)\right]}{\left(-1+r^2\kappa^2\cosh^2\varphi\right)^2\cos\theta}, \\ H(s,\theta) &= \frac{2r^2\kappa^3\sinh^3\varphi\cos\theta + \kappa(2\sinh\varphi\cos\theta + r\tau\cosh\varphi) + r\kappa_s\cosh\varphi}{2\left(\sqrt{-1+r^2\kappa^2\cosh^2\varphi}\right)^3\cosh\theta}. \end{aligned}$$

2- The principle curvatures k_1 and k_2 are given, respectively, as:

$$\begin{aligned} k_1(u,\theta) &= \frac{\kappa\sinh\varphi}{\sqrt{-1+r^2\kappa^2\cosh^2\varphi}}, \\ k_2(u,\theta) &= \frac{(-1+r^2\kappa^2\cosh^2\varphi)\kappa\cosh\varphi\cos\theta - r(\kappa_s\sinh\varphi + \kappa\tau\cosh\varphi)}{\left(\sqrt{-1+r^2\kappa^2\cosh^2\varphi}\right)^3\cos\theta}. \end{aligned}$$

In particular, the principle direction of k_1 points the direction of the generating circle and this curvature is constant along the generating circle, i.e., $k_1(u,\theta) = k_1(u)$.
3- Two striction curves coincide singular locus and their curvatures κ_i and torsions τ_i $(i = 1, 2)$ are given as follows:

$$\left.\begin{aligned} \kappa_1 = \frac{\kappa}{\sqrt{1-r\kappa\cosh\varphi}}, \ \tau_1 = \frac{\tau}{\sqrt{1-r\kappa\cosh\varphi}}, \\ \kappa_2 = \frac{\kappa}{\sqrt{1+r\kappa\cosh\varphi}}, \ \tau_2 = \frac{\tau}{\sqrt{1+r\kappa\cosh\varphi}}. \end{aligned}\right\}$$

It is known that a regular surface is flat if and only if its Gaussian curvature vanishes identically. Therefore, we immediately derive that:

$$\kappa\sinh\varphi\left[(-1+r^2\kappa^2\cosh^2\varphi)\kappa\cosh\varphi\cos\theta + r(\kappa_s\sinh\varphi + \kappa\tau\cosh\varphi)\right] = 0. \tag{42}$$

Thus in a neighborhood of any point on M with $\kappa \neq 0$, we get $\kappa_s \sinh\varphi + \kappa\tau\cosh\varphi = 0$ for any $s \in \mathbb{R}$. This is equivalent to that $\kappa_s = \tau = 0$ which means that τ is an identically zero function. By Eqs. (42), we have $\sinh\varphi = 0$. A timelike tangent circular surface satisfying this condition is a part of a Lorentzian plane. In the same manner, we get that M is a minimal flat surface.

Hence, we state that: Every flat (minimal) timelike tangent circular surfaces are subsets of Lorentzian planes.

We now give some examples of timelike tangent circular surfaces. They also serve to verify the correctness of the formulae derived above.

Example 5.5. *Given a parametric spacelike helix, whose normal vector and binormal vector are spacelike and timelike respectively,*

$$\boldsymbol{\gamma}(s) = (a\cosh\frac{s}{c}, b\frac{s}{c}, a\sinh\frac{s}{c}),\ a > 0,\ \ b \neq 0,\ b^2 - a^2 = c^2,\ -2 \leq s \leq 2.$$

After simple computation, we have

$$\left.\begin{array}{r}\boldsymbol{T}(s) = (\frac{a}{c}\sinh\frac{s}{c}, \frac{b}{c}, \frac{a}{c}\cosh\frac{s}{c}),\\ \boldsymbol{N}(s) = (\cosh\frac{s}{c}, 0, \sinh\frac{s}{c}),\\ \boldsymbol{B}(s) = (-\frac{b}{c}\sinh\frac{s}{c}, -\frac{a}{c}, -\frac{b}{c}\cosh\frac{s}{c}),\end{array}\right\}$$

and $\tau = -\frac{b}{c^2}$, *then* $\varphi(s) = -\frac{b}{c^2}s + \varphi_0$. *If we choose* $\varphi_0 = 0$, $a = 1$, $b = 2$, *and* $r = 1$, *then the timelike tangent circular surface is shown in Figure 3.*

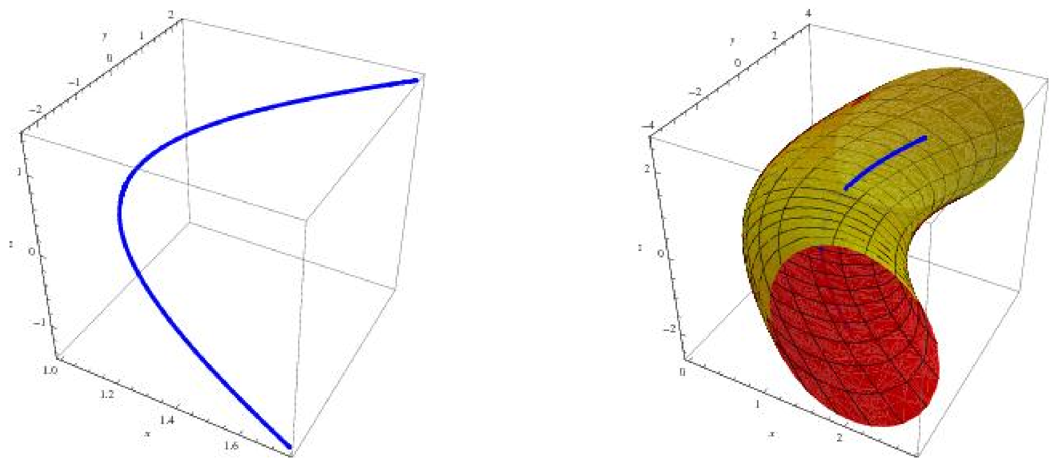

Figure 3: The spacelike helix and corresponding timelike tangent circular surface.

Example 5.6. *Suppose we are given a parametric spacelike curve*

$$\boldsymbol{\gamma}(s) = (0, \sinh s, \cosh s), \quad -2 \leq s \leq 2.$$

It is easy to show that

$$\boldsymbol{T}(s) = (0, \cosh s, \sinh s),\ \boldsymbol{N}(s) = (0, \sinh s, \cosh s),\ \boldsymbol{B}(s) = (1, 0, 0),$$

and $\tau = 0$ *which follows* $\varphi(s) = \varphi_0$ *is a constant. If* $\varphi_0 = 3/2$, *and* $r = 1$, *the timelike tangent circular surface is shown in Figure 4.*

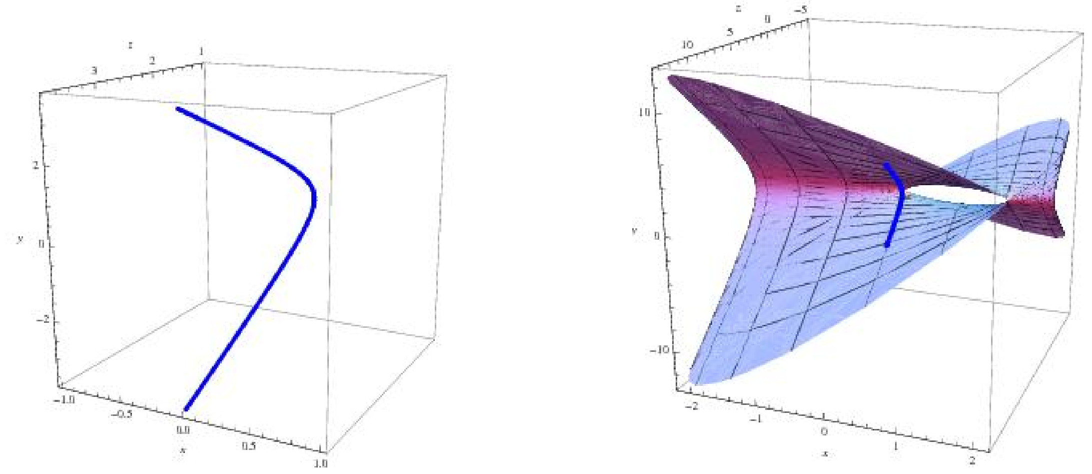

Figure 4: The spacelike curve $\gamma(s)$ and corresponding timelike tangent circular surface.

6. Conclusion

In this paper, we extend the work of Izumiya et al. [11] to Minkowski 3-space $\mathbb{R}^3_1$, and derive the invariants of circular surface, by setting up an orthonormal moving frame to each point of the spine curve and applying the moving frame method. A new type of timelike circular surfaces was identified and coined as the timelike tangent circular surface. The new timelike circular surface has the property that all generating circles being lines of curvature and its Gaussian and mean curvatures being independent of the geodesic curvature of the spherical indicatrix. This study is intended to clear away to conduct the geometric analysis of circular surfaces through the spine curve and the spherical indicatrix of the normal vector of circle planes.

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