Filomat 38:4 (2024), 1465–1477
https://doi.org/10.2298/FIL2404465L

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Some properties of the matrix related to q-coloured coordination number

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Abstract. The q-coloured coordination number $S_{n,k}(q)$ counts the number of lattice paths from (0,0) to (n,k) using steps $(0,1),(1,0)$ and $(1,1)$ without east-steps on the x-axis, among which the $(1,1)$ steps are coloured with q colours. We investigate some properties of the polynomial matrix $S(q)=[s_{n,k}(q)]_{n,k\geq 0}=[S_{n-k,k}(q)]_{n,k\geq 0}$, including the unimodality problems of sequences located over rays in $S(q)$ and the q-total positivity of $S(q)$. We show that the zeros of all row sums $R_n(q)=\sum_{k=0}^{n}s_{n,k}(q)=\sum_i r_{n,i}q^i$ are in $(-\infty,-1)$ and are dense in the corresponding semi-closed interval. We also prove that the coefficients $r_{n,i}$ are asymptotically normal (by central and local limit theorems).

1. Introduction

Following Conway and Sloane [10], the *coordination sequence* of an infinite vertex-transitive graph $\mathfrak{G}$ is the sequence $(S(0),S(1),S(2),\ldots)$, where $S(n)$ is the number of vertices at distance n from some fixed vertex of $\mathfrak{G}$. O'Keeffe [18] have shown that the coordination sequence can be used as a fingerprint to identity structures of $\mathfrak{G}$ (see [1, 10, 18] for details). For the k-dimensional integer lattice $\mathbb{Z}^k$, Conway and Sloane [10] gave the generating functions $S_k(x)=(1+x)^k/(1-x)^k$ of the coordination sequences. Denote $S_k(x)=\sum_{n\geq 0}S(n,k)x^n$, we can know that the coordination number $S(n,k)=[x^n]\,S_k(x)=[x^n]\,(1+x)^k/(1-x)^k$. The first few terms of the coordination number $S(n,k)$ are as follows:

$n\setminus k$	0	1	2	3	4
0	1	1	1	1	1
1	0	2	4	6	8
2	0	2	8	18	32
3	0	2	12	38	88
4	0	2	16	66	192

2020 *Mathematics Subject Classification*. 05A15; 05A16; 05A20.
Keywords. Asymptotic normality; q-total positivity; log-concave.
Received: 05 July 2023; Accepted: 22 August 2023
Communicated by Paola Bonacini
Research supported by the National Natural Science Foundation of China (No. 12001301), the Graduate students' research & Innovation fund of Inner Mongolia Normal University (CXJJS22097) and the Fundamental Research Funds for the Inner Mongolia Normal University (2022JBXC013) optionally.
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Liang *et al.* [14] have investigated the analytic behaviors of coordination number. On the other hand, coordination number has a nice combinatorial interpretation from the viewpoint of lattice paths. Let D be those lattice paths starting from $(0,0)$ that use the steps $(1,0),(0,1)$, and $(1,1)$ without east-steps on the x-axis (see [26]). The coordination number $S(n,k)$ corresponds to the number of lattice paths D ending at the point (n,k). Then it follows that

$$S(n,k) = S(n-1,k) + S(n,k-1) + S(n-1,k-1) \tag{1}$$

with the initial values $S(n,0) = 0$ for $n > 1$, and $S(0,k) = 1$ for $n \geq 0$, or a further expression

$$S(n,k) = \sum_i \binom{k}{i}\binom{n-1}{i-1}2^i = \sum_i \binom{k}{i}\binom{n+k-i-1}{k-1}. \tag{2}$$

Let D' be those lattice paths that all diagonal steps of paths D are coloured with q colours ($q \geq 0$). The q-coloured coordination number $S_{n,k}(q)$ denotes the number of the paths D' ending at the point (n,k) in this case. Then, analogous to (1) and (2), respectively, we have

$$S_{n,k}(q) = S_{n-1,k}(q) + S_{n,k-1}(q) + qS_{n-1,k-1}(q), \tag{3}$$

and

$$S_{n,k}(q) = \sum_i \binom{k}{i}\binom{n-1}{i-1}(q+1)^i = \sum_i \binom{k}{i}\binom{n+k-i-1}{k-1}q^i.$$

As a polynomial, $S_{n,k}(q)$ has some nice properties, which is partly due to the fact that it is a special Jacobi polynomial $P_n^{(-1,-n-k)}(-2q-1)$. Moreover, $S_{n,k}(q)$ can be proved to have only real zeros by the Maló Theorem [16], which states that if both $\sum_{i=0}^{n} a_i q^i$ and $\sum_{j=0}^{m} b_j q^j$ have only real zeros then $\sum_{k=0}^{\min\{n,m\}} a_k b_k q^k$ has only real zeros. It is also worth noting that many well-known combinatorial counting sequences are q-coloured coordination number. For example, $S_{n,k}(0)$ is related to the binomial coefficients and $S_{n,k}(1)$ is the coordination number. In a sense, that endowing the diagonal steps with being q-coloured pleasantly brings more research materials to the existing setting. Our paper is to study some properties of the matrix related to q-coloured coordination number. The q-coloured coordination number constitutes the square matrix

$$[S_{n,k}(q)]_{n,k\geq 0} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 0 & q+1 & 2q+2 & 3q+3 & 4q+4 & \\ 0 & q+1 & q^2+4q+3 & 3q^2+9q+6 & 6q^2+16q+10 & \\ 0 & q+1 & 2q^2+6q+4 & q^3+9q^2+18q+10 & 4q^3+24q^2+40q+20 & \\ 0 & q+1 & 3q^2+8q+5 & 3q^3+18q^2+30q+15 & q^4+16q^3+60q^2+80q+35 & \\ \vdots & & & & & \ddots \end{bmatrix},$$

whereas our paper focuses on the following triangular matrix

$$S(q) := [s_{n,k}(q)]_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & q+1 & 1 & & & & \\ 0 & q+1 & 2q+2 & 1 & & & \\ 0 & q+1 & q^2+4q+3 & 3q+3 & 1 & & \\ 0 & q+1 & 2q^2+6q+4 & 3q^2+9q+6 & 4q+4 & 1 & \\ \vdots & & & & & & \ddots \end{bmatrix},$$

which is derived by arranging the q-coloured coordination number in a triangle array, i.e., $s_{n,k}(q) = S_{n-k,k}(q)$. This matrix is more convenient for the following investigation than the former one (albeit more natural), and therefore is our protagonist here. It is interesting to mention in passing that $S(q)$ can unify some combinatorial triangles. For example, $S(0)$ is related to the well-known Pascal triangle, $S(1)$ is the coordination triangle, $S(2)$ is Riordan array (see [20, A122016]).

The main objective of this paper is to investigate properties of $S(q)$. The paper is organized as follows. In section 2, we show that the polynomial sequences located in a ray or a transversal line of $S(q)$ are strongly q-log-concave and the polynomial sequences located over rays in $S(1)$ are unimodal. In section 3, We also prove the q-total positivity of matrix satisfying a special recurrence in a unified approach. So we show that both $[S_{n,k}(q)]_{n,k\geq 0}$ and $S(q)$ are q-totally positive. In Section 4, we show, for the row sums $R_n(q)$, that all their zeros lie in the open interval $(-\infty,-1)$ and are dense in the semi-closed interval $(-\infty,-1]$. In the final section, we prove that the coefficients $r_{n,i}$ are asymptotically normal by central and local limit theorems.

2. Unimodality problems of sequences located over rays in S(q)

In this section, we investigate strong q-log-concavity of $S(q)$ and the unimodality of sequences located over rays in $S(1)$.

2.1. Strong q-log-concavity of S(q)

Let $f(q)$ and $g(q)$ be two real polynomials in q. We say that $f(q)$ is *q-nonnegative* if $f(q)$ has nonnegative coefficients. Denote $f(q) \geq_q g(q)$ if the difference $f(q) - g(q)$ is *q-nonnegative*. For a polynomial sequence $(f_n(q))_{n\geq 0}$, it is called *q-log-concave* (or *q-log-convex*) if

$$f_n(q)^2 \geq_q f_{n+1}(q)f_{n-1}(q) \quad \left(\text{or } f_n(q)^2 \leq_q f_{n+1}(q)f_{n-1}(q)\right)$$

for $n \geq 1$. It is called *strongly q-log-concave* (or *strongly q-log-convex*) if

$$f_n(q)f_m(q) \geq_q f_{n+1}(q)f_{m-1}(q) \quad \left(\text{or } f_n(q)f_m(q) \leq_q f_{n+1}(q)f_{m-1}(q)\right)$$

for $n \geq m \geq 1$. Clearly, the strong q-log-concavity (strong q-log-convexity) of polynomial sequences implies the q-log-concavity (q-log-convexity), which further implies the log-concavity (log-convexity) for any fixed $q \geq 0$, not vice versa. The (strong) q-log-concavity has been extensively studied (see [6, 13, 19]).

It is known that $S(0)$ is related to the Pascal triangle P. Su and Wang [23] proved the log-concavity of the sequence located in a transversal line of P or a line parallel to the boundary of P. The q-coloured Delannoy number $D_{n,k}(q)$ count the number of lattice paths from (0,0) to (n,k) using steps $(0,1), (1,0)$ and $(1,1)$, among which the $(1,1)$ steps are coloured with q colours. The polynomial matrix $[D_{n-k,k}(q)]_{n,k\geq 0}$ is denoted as $D(q)$. Yu [27] pointed out that such properties also hold in the Delannoy triangle $D(1)$. Recently, Mu and Zheng [17] studied the strong q-log-concavity of polynomial sequences located in a ray or a transversal line of $D(q)$. Next we investigate the strong q-log-concavity of the sequence $(s_{n_0+ai,k_0+bi}(q))_{i\geq 0}$ in $S(q)$ for nonnegative integers a and b in the following theorem (the sequence shown in Figure 1).

The lattice path interpretations of q-coloured Delannoy number and q-coloured coordination number differs only by the east-steps on the x-axis. Very recently, Mu and Zheng [17] showed that polynomial sequences located in a ray or a transversal line of $D(q)$ are *strongly q-log-concave*. The following result can be obtained by means of the same idea used in the proof of Theorem 2.1 in [17]. We omit the details for the sake of brevity.

Theorem 2.1. *Let n_0, k_0, a, b be four nonnegative integers and $n_0 \geq k_0, a + b \neq 0$. Define the sequence*

$$S_i(q) = s_{n_0+ai,k_0+bi}(q), \quad i = 0, 1, 2, \ldots.$$

If $a \leq b$, then the polynomial sequence $(S_i(q))_{i\geq 0}$ is strongly q-log-concave.

From Theorem 2.1, we have the following results immediately.

Corollary 2.2. *All the polynomial sequences located in a transversal of $S(q)$ or in a line parallel to the boundary of $S(q)$ are strongly q-log-concave.*

Corollary 2.3. *All the sequences located in a transversal of $S(1)$ (or $S(2)$) or in a line parallel to the boundary of $S(1)$ (or $S(2)$) are log-concave.*

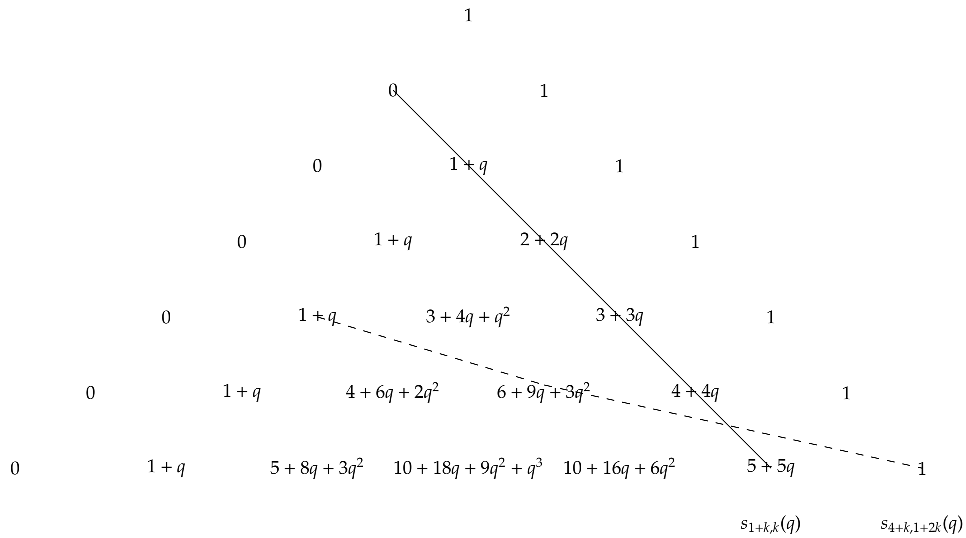

Figure 1: The triangular array of q-coloured coordination number.

Remark 2.4. *A polynomial sequence* $(a_i(q))_{i\geq 0}$ *is called a q-Pólya frequency (q-PF for short) sequence if all minors of the corresponding Toeplitz matrix* $\left[a_{i-j}(q)\right]_{i,j\geq 0}$ *are q-nonnegative. In fact, the polynomial sequence* $(S_i(q))_{i\geq 0}$ *forms a q-PF sequence, which could be proved by the same technique used in the proof of Theorem 2 in [27].*

2.2. *Unimodality of sequences located over rays in* $S(1)$

Let $(a_k)_{k\geq 0}$ be a sequence of nonnegative numbers. We say that the sequence is log-concave if $a_{k-1}a_{k+1} \leq a_k^2$ for $k \geq 1$, and unimodal if

$$a_0 \leq a_1 \leq \cdots \leq a_m \geq a_{m+1} \geq \cdots$$

for some m. It is well known [5] that a log-concave sequence without internal zeros is unimodal.

Following Karlin [12], a (finite or infinite) matrix is called *totally positive* (TP for short) if all its minors are nonnegative. Let $(a_k)_{k\geq 0}$ be an infinite sequence of nonnegative numbers (we identify a finite sequence $a_0, a_1, \ldots, a_n$ with the infinite sequence $a_0, a_1, \ldots, a_n, 0, 0, \ldots$). Define its Toeplitz matrix

$$[a_{i-j}]_{i,j\geq 0} = \begin{bmatrix} a_0 & & & & \\ a_1 & a_0 & & & \\ a_2 & a_1 & a_0 & & \\ a_3 & a_2 & a_1 & a_0 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

We say that the sequence is a *Pólya frequency* (PF for short) sequence if the corresponding Toeplitz matrix is TP. A fundamental characterization for PF sequences is due to Schoenberg and Edrei as following (see [12, p. 412] for instance).

Schoenberg-Edrei Theorem. *A sequence* $(a_k)_{k\geq 0}$ *of nonnegative numbers is PF if and only if its generating function has the form*

$$\sum_{k\geq 0} a_k x^k = a x^m e^{\gamma x} \frac{\prod_{j\geq 0}(1+\alpha_j x)}{\prod_{j\geq 0}(1-\beta_j x)},$$

where $a > 0, m \in \mathbb{N}, \alpha_j, \beta_j, \gamma \geq 0$ *and* $\sum_{j\geq 0}(\alpha_j + \beta_j) < +\infty$.

In this case, the generating function is called a *Pólya frequency* formal power series.

Let $s(n,k) = S(n-k,k)$. Then the *coordination triangle* $[s(n,k)]_{n,k\geq 0}$ is derived by arranging the coordination number in a triangular array and $s(n,k)$ can be obtained by the following recurrence relation:

$$s(n,k) = s(n-1,k) + s(n-1,k-1) + s(n-2,k-1), \quad n \geq 2, k \geq 1, \tag{4}$$

where $s(0,0) = 1, s(n,n) = 1$ and $s(n,0) = 0$ for $n \geq 1$. We use the convention $s(n,k) = 0$ for $k \notin \{0,\dots,n\}$. What's more, the following identity can be obtained from (2),

$$s(n,k) = \sum_i \binom{k}{i}\binom{n-k-1}{i-1}2^i. \tag{5}$$

Especially, the triangular array $S(1) = [s(n,k)]_{n,k\geq 0}$ is *coordination triangle*. To explore the properties of combinatorial triangles, the properties of row sequences and central sequences are generally considered [9, 25]. The log-concavity of sequences $(s(n_0 + k, k))_{k\geq 0}$ located on a line parallel to the right-hand boundary of $S(1)$ can be obtained from Corollary 2.3, so we can know that the sequences are unimodal and its algebraic proof is given in following Theorem. More generally, we aim to study unimodality of a sequence located over rays of S(1). For all fixed n_0, k_0, a, b, the sequences over rays of the coordination triangle are obtained by this expression $s_k = (s(n_0 + ka, k_0 + kb))_{k\geq 0}$ (see Figure 2). We present unimodality of the sequences $s_k = (s(n_0 + ka, k_0 + kb))_{k\geq 0}$ in following Theorem.

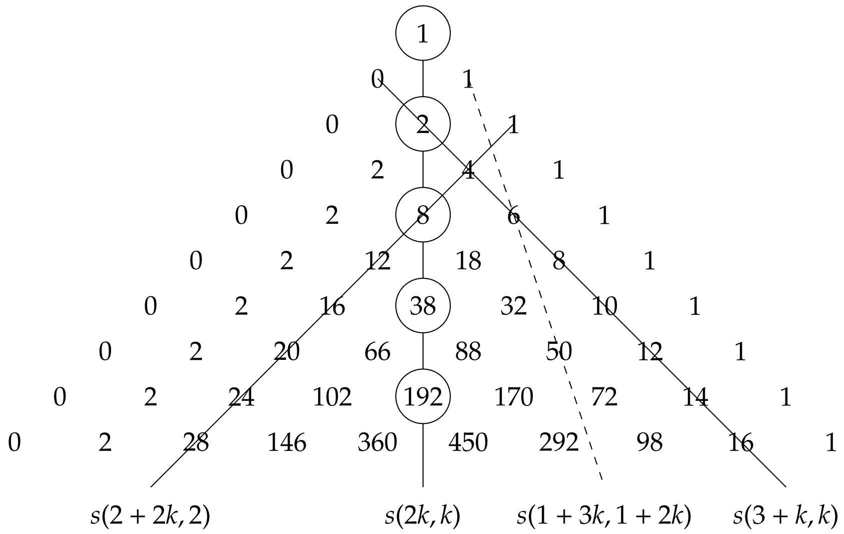

Figure 2: The triangular array of coordination number.

Lemma 2.5 ([24]). *If the sequence* $(x_n)_{n\geq 0}$ *is log-concave, then the linear transformation*

$$y_n := \sum_{k=0}^{n} \binom{n}{k} x_k, n = 0, 1, 2 \dots$$

preserves the log-concavity property.

Lemma 2.6 ([23, Lemma 1]). *If a sequence* $(a_k)_{k\geq 0}$ *of positive numbers is unimodal (resp. increasing, decreasing, concave, convex, log-concave, log-convex), then so is its subsequence* $(a_{n+kd})_{k\geq 0}$ *for arbitrary fixed nonnegative integers n and d.*

Theorem 2.7. *Let n_0, k_0, a, b be four nonnegative integers and $n_0 \geq k_0$. Define the sequence $s_k = (s(n_0 + ka, k_0 + kb))_{k\geq 0}$, therefore*

(i) *if $k_0 = 0$ and $a = b = 1$, the sequence $(s(n_0 + k, k))_{k\geq 0}$ is log-concave and thus unimodal.*

(ii) *if $b = 0$, the sequence $(s(n_0 + ka, k_0))_{k\geq 0}$ is increasing, log-concave and hence unimodal.*

(iii) *if $b > k_0, a - b \geq 0$, the sequence $(s(n_0 + ka, k_0 + kb))_{k\geq 0}$ is nondecreasing and hence unimodal.*

Proof. (i) According to (5), we can know that if $k_0 = 0$ and $a = b = 1$, then the sequence $s(n_0 + k, k) = \sum_i \binom{k}{i}\binom{n_0-1}{i-1}2^i$. Since the sequence $x_j := \binom{n_0-1}{i-1}2^i$ is trivially log-concave, then the sequence $(s(n_0 + k, k))_{k\geq 0}$ is log-concave and thus unimodal from Lemma 2.5.

(ii) The log-concavity of $(s\,(n_0 + ka, k_0))_{k\geq 0}$ are obtained from the log-concavity of $(s\,(k, k_0))_{k\geq 0}$ by Lemma 2.6.

It's easy to see that the generating function of the k_0th column of $S(1)$ is $x^{k_0}\dfrac{(1+x)^{k_0}}{(1-x)^{k_0}}$ and is PF from Schoenberg-Edrei Theorem, thus the sequence $(s\,(k, k_0))_{k\geq 0}$ is increasing and log-concave in k. Then so is sequence $(s\,(n_0 + ka, k_0))_{k\geq 0}$. It's clearly that sequence $(s\,(n_0 + ka, k_0))_{k\geq 0}$ is unimodal.

(iii) In order to prove that the sequence $(s_k)_{k\geq 0} = (s(n_0 + ka, k_0 + kb))_{k\geq 0}$ are nondecreasing, one has to prove that the following relation is satisfied

$$s\,(n_0 + (k+1)a, k_0 + (k+1)b) = s(n_0 + ka, k_0 + kb) + \sigma$$

with $\sigma \geq 0$.

Using the relation (4), we have

$$\begin{aligned}
&s(n_0 + (k+1)a, k_0 + (k+1)b) \\
&= s\,(n_0 + (k+1)a - 1, k_0 + (k+1)b) + s\,(n_0 + (k+1)a - 1, k_0 + (k+1)b - 1) \\
&+ s\,(n_0 + (k+1)a - 2, k_0 + (k+1)b - 1) \\
&= s\,(n_0 + (k+1)a - 2, k_0 + (k+1)b) + s\,(n_0 + (k+1)a - 2, k_0 + (k+1)b - 1) \\
&+ s\,(n_0 + (k+1)a - 3, k_0 + (k+1)b - 1) + s\,(n_0 + (k+1)a - 1, k_0 + (k+1)b - 1) \\
&+ (n_0 + (k+1)a - 2, k_0 + (k+1)b - 1)\,.
\end{aligned}$$

If we use the relation (4) $(a - b)$ times, we obtain a relation of the following form

$$s\,(n_0 + (k+1)a, k_0 + (k+1)b) = s\,(n_0 + ak + b, k_0 + (k+1)b) + \sigma' \tag{6}$$

with $\sigma' \geq 0$. One can show that the relation (6) can be rewritten as follows

$$\begin{aligned}
&s\,(n_0 + (k+1)a, k_0 + (k+1)b) \\
&= s\,(n_0 + ak + b - 1, k_0 + (k+1)b) + (n_0 + ak + b - 1, k_0 + (k+1)b - 1) \\
&+ s\,(n_0 + ak + b - 2, k_0 + (k+1)b - 1) + \sigma' \\
&= s\,(n_0 + ak + b - 1, k_0 + (k+1)b) + s\,(n_0 + ak + b - 2, k_0 + (k+1)b - 1) \\
&+ s\,(n_0 + ak + b - 2, k_0 + (k+1)b - 2) + s\,(n_0 + ak + b - 3, k_0 + (k+1)b - 2) \\
&+ (n_0 + ak + b - 2, k_0 + (k+1)b - 1) + \sigma'.
\end{aligned}$$

Repeating this process b times we get the desired result

$$s\,(n_0 + (k+1)a, k_0 + (k+1)b) = s\,(n_0 + ak, k_0 + kb) + \sigma$$

with $\sigma \geq 0$. Consequently the sequences $(s_k)_{k\geq 0}$ are nondecreasing and hence unimodal. □

3. q-total positivity

We use

$$A\left[\begin{array}{c} i_0,\ldots,i_k \\ j_0,\ldots,j_k \end{array}\right] \quad \text{and} \quad A\left(\begin{array}{c} i_0,\ldots,i_k \\ j_0,\ldots,j_k \end{array}\right)$$

to denote the submatrix and minor of the matrix A determined by the rows indexed $i_0 < i_1 < \cdots < i_k$ and columns indexed $j_0 < j_1 < \cdots < j_k$ respectively.

Let $f(q)$ and $g(q)$ be two real polynomials in q. Let $M(q) = [m_{n,k}(q)]_{n,k\geq 0}$ be the matrix whose entries are all real polynomials in q. We say that $M(q)$ is *q-totally positive* (q-TP for short) if all minors are q-nonnegative. We also need the following classical results.

Cauchy-Binet Formula. *Let A, B, C be three matrices and $C = AB$. Then*

$$C\binom{i_1,\ldots,i_k}{j_1,\ldots,j_k} = \sum_{\ell_1<\cdots<\ell_k} A\binom{i_1,\ldots,i_k}{\ell_1,\ldots,\ell_k}\cdot B\binom{\ell_1,\ldots,\ell_k}{j_1,\ldots,j_k}.$$

Remark 3.1. *It immediately follows that the product of TP matrices is still TP.*

We may investigate the q-coloured coordination number and the q-coloured Delannoy number [17] in a unified approach. Let $m \in \mathbb{N}$, define the numbers $L^{(m)}(n,k)$ by

$$L^{(m)}(n,k) = L^{(m)}(n-1,k-1) + L^{(m)}(n-1,k) + L^{(m)}(n,k-1) \tag{7}$$

with the initial values $L^{(m)}(0,k) = 1$ for $k \geq 0$ and $L^{(m)}(n,0) = m$ for $n \geq 1$. Let $D(n,k)$ denote Delannoy number. Note also that

$$L^{(m)}(n,k) = D(n,k) + (m-1)D(n-1,k) \tag{8}$$

for $n,k \geq 0$, hence we have

$$\begin{aligned} L^{(m)}(n,k) &= \sum_i \binom{k}{i}\left[\binom{n-1}{i-1} + m\binom{n-1}{i}\right]2^i \\ &= \sum_i \binom{k}{i}\left[\binom{n+k-i-1}{k-1} + m\binom{n+i-1}{k}\right]. \end{aligned} \tag{9}$$

Let $L^{(m)}_{n,k}(q)$ is the q-analogues of $L^{(m)}(n,k)$, satisfying recurrence:

$$L^{(m)}_{n,k}(q) = L^{(m)}_{n-1,k}(q) + L^{(m)}_{n,k-1}(q) + qL^{(m)}_{n-1,k-1}(q),$$

analogous to (9), we have

$$\begin{aligned} L^{(m)}_{n,k}(q) &= \sum_i \binom{k}{i}\left[\binom{n-1}{i-1} + m\binom{n-1}{i}\right](1+q)^i \\ &= \sum_i \binom{k}{i}\left[\binom{n+k-i-1}{k-1} + m\binom{n+i-1}{k}\right]q^i. \end{aligned} \tag{10}$$

Clearly, the q-coloured coordination number $S_{n,k}(q) = L^{(0)}_{n,k}(q)$, the q-coloured Delannoy number $D_{n,k}(q) = L^{(1)}_{n,k}(q)$ and the binomial coefficients $P_{n,k} = L^{(1)}_{n,k}(0)$. $L^{(m)}_{n,k}(q)$ constitute the square matrix

$$[L^{(m)}_{n,k}(q)]_{n,k\geq 0} = \left[\begin{array}{cccc} 1 & 1 & 1 & \cdots \\ m & 1+m+q & 2+m+2q & \\ m & 1+2m+qm+q & 3+3m+4q+2qm+q^2 & \\ \vdots & & & \ddots \end{array}\right], \tag{11}$$

we focus on the following triangular matrix

$$L^{(m)}(q) := [l^{(m)}_{n,k}(q)]_{n,k\geq 0} = \begin{bmatrix} 1 & & & & \\ m & 1 & & & \\ m & 1+m+q & 1 & & \\ m & 1+2m+q+qm & 2+m+2q & 1 & \\ \vdots & & & & \ddots \end{bmatrix},$$

which is derived by arranging the numbers $L^{(m)}_{n,k}(q)$ in a triangle array, where $l^{(m)}_{n,k}(q) = L^{(m)}_{n-k,k}(q)$.

A (proper) Riordan array, denoted by $R(d(x), h(x))$, is an infinite lower triangular matrix whose generating function of the kth column is $d(x)h^k(x)$ for $k = 0, 1, 2, \ldots$, where $d(0) = 1, h(0) = 0$ and $h'(0) \neq 0$.

Chen and Wang [8, Theorem 2.1] gave the following criterion for the total positivity of Riordan arrays (see also [21, Theorem 1.2]).

Lemma 3.2 ([8, Theorem 2.1]). *Let $R = R(d(x), h(x))$ be a Riordan array. If both $d(x)$ and $h(x)$ are PF formal power series, then R is TP.*

Theorem 3.3. *The square matrix $\left[L^{(m)}_{n,k}(q)\right]_{n,k\geq 0}$ is q-TP.*

Proof. Note that, since (9), the square matrix $\left[L^{(m)}_{n,k}(q)\right]_{n,k\geq 0} = MDP^T$, where P is the Pascal triangle, $D = \mathrm{diag}\left(1, 1+q, (1+q)^2, (1+q)^3, \ldots\right)$ and

$$M = \begin{bmatrix} 1 & & & & \\ m & 1 & & & \\ m & m+1 & 1 & & \\ m & 2m+1 & m+2 & 1 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

Next we want to show that the triangle M is TP. Note that $M = R\left(\frac{1+(m-1)x}{1-x}, \frac{x}{1-x}\right)$, it's clearly that Riordan array M is TP by Lemma 3.2.

Hence the q-total positivity of $\left[L^{(m)}_{n,k}(q)\right]_{n,k\geq 0}$ follows immediately from the Cauchy-Binet formula and the total positivity of the Pascal triangle. □

It follows immediately from Theorem 3.3 that the square matrix $[S_{n,k}(q)]_{n,k\geq 0}$ is q-TP. Moreover, the the q-total positivity of the square matrix $[D_{n,k}(q)]_{n,k\geq 0}$ follows immediately from Theorem 3.3, which has been shown in [17].

Lemma 3.4 ([17, Lemma 3.1]). *Let $M(q) = R(d(x), h(x))$ be a Riordan array, where $d(x) = \sum_{n\geq 0} d_n(q)x^n$ and $h(x) = \sum_{n\geq 0} h_n(q)x^n$. If the matrix*

$$\begin{bmatrix} d_0(q) & h_0(q) & & & \\ d_1(q) & h_1(q) & h_0(q) & & \\ d_2(q) & h_2(q) & h_1(q) & h_0(q) & \\ \vdots & & & & \ddots \end{bmatrix}$$

is q-TP, then so is the Riordan array $M(q)$.

Theorem 3.5. *The triangle $L^{(m)}(q)$ is q-TP.*

Proof. Note that $L^{(m)}(q) = R(d(x), h(x)) = R\left(\frac{1+(m-1)x}{1-x}, \frac{x+qx^2}{1-x}\right)$. Let $T(q) = R(h(x), x)$ and $v(q) = (d_0(q), d_1(q), d_2(q), \ldots)^T$. By Lemma 3.4, it suffices to show that $(v(q), T(q))$ is q-TP. We have

$$(v(q), T(q)) = \begin{bmatrix} 1 & & & & \\ m & 1 & & & \\ m & 1+q & 1 & & \\ m & 1+q & 1+q & 1 & \\ \vdots & & & & \ddots \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 1 & 1 & & \\ 0 & 1 & 1 & 1 & \\ \vdots & & & & \ddots \end{bmatrix} \begin{bmatrix} 1 & & & & \\ m & 1 & & & \\ & q & 1 & & \\ & & q & 1 & \\ & & & \ddots & \ddots \end{bmatrix}.$$

One can check that both matrices on the right-hand side are q-TP. Therefore, $(v(q), T(q))$ is q-TP by the classic Cauchy-Binet formula, as required. □

An immediate consequence of Theorem 3.5 is that both the q-coloured coordination triangle $S(q)$ and q-coloured Delannoy triangle $D(q)$ are q-TP. What's more, the the q-total positivity of $D(q)$ has been shown in [17].

4. Zeros of row sums

Let $R_n(q) = \sum_i r_{n,i} q^i$ be the sum of the nth row of $S(q)$, i.e.,

$$R_n(q) = \sum_{k=0}^{n} s_{n,k}(q).$$

The first few entries of $(R_n(q))_{n\geq 0}$ are $(1, 1, 2+q, 4+3q, \ldots)$. The coefficient matrix of $R_n(q)$ is defined by the matrix

$$[r_{n,i}]_{n,i\geq 0} = \begin{bmatrix} 1 & & \\ 1 & & \\ 2 & 1 & \\ 4 & 3 & \\ \vdots & & \ddots \end{bmatrix}.$$

Note that the q-coloured coordination number $S_{n,k}(q)$ satisfies the recurrence (3), hence

$$s_{n,k}(q) = s_{n-1,k-1}(q) + s_{n-1,k}(q) + q s_{n-2,k-1}(q).$$

Thus, we can get the following proposition.

Proposition 4.1. *The row sum $R_n(q)$ satisfies the simple recurrence*

$$R_n(q) = 2R_{n-1}(q) + qR_{n-2}(q) \tag{12}$$

with $R_1(q) = 1, R_2(q) = 1$ and has the Binet form

$$R_n(q) = \frac{(1-\lambda_2)\lambda_1^n - (\lambda_1 - 1)\lambda_2^n}{\lambda_1 - \lambda_2} = \frac{\lambda_1^n + \lambda_2^n}{2}, \tag{13}$$

where

$$\lambda_{1,2} = 1 \pm \sqrt{1+q}.$$

Let $(f_n(x))_{n\geq 0}$ be a sequence of complex polynomials. We say that the complex number x to be a *limit of zeros* of the sequence $(f_n(x))_{n\geq 0}$ if there is a sequence $(z_n)_{n\geq 0}$ such that $f_n(z_n) = 0$ and $z_n \to x$ as $n \to +\infty$. Suppose now that $(f_n(x))_{n\geq 0}$ is a sequence of polynomials satisfying the recursion

$$f_{n+k}(x) = -\sum_{j=1}^{k} p_j(x) f_{n+k-j}(x),$$

where $p_j(x)$ are polynomials in x. Let $\lambda_j(x)$ be all roots of the associated characteristic equation $\lambda^k + \sum_{j=1}^{k} p_j(x)\lambda^{k-j} = 0$. It is well known that if $\lambda_j(x)$ are distinct, then

$$f_n(x) = \sum_{j=1}^{k} \alpha_j(x)\lambda_j^n(x), \tag{14}$$

where $\alpha_j(x)$ are determined from the initial conditions.

Beraha-Kahane-Weiss Theorem [3, Theorem]. *Under the non-degeneracy requirements that in* (14) *no $\alpha_j(x)$ is identically zero and that for no pair $i \neq j$ is $\lambda_i(x) \equiv \omega\lambda_j(x)$ for some $\omega \in \mathbb{C}$ of unit modulus, then x is a limit of zeros of $(f_n(x))_{n\geq 0}$ if and only if either*

(i) *two or more of the $\lambda_i(x)$ are of equal modulus, and strictly greater (in modulus) than the others; or*

(ii) *for some j, $\lambda_j(x)$ has modulus strictly greater than all the other $\lambda_i(x)$ have, and $\alpha_j(x) = 0$.*

Theorem 4.2. (i) *Zeros of $R_n(q)$ are real, distinct and in the open interval* $(-\infty, -1)$.

(ii) *All the zeros of $R_n(q)$ are dense in the semi-closed interval* $(-\infty, -1]$.

Proof. (i) We next show that

$$R_n(q) = \frac{1}{2}\prod_{k=1}^{\lfloor n/2\rfloor}\left(4 + 4q\cos^2\frac{(2k-1)\pi}{2n}\right). \tag{15}$$

We do this only for n even since the case n odd is similar. Let $w_k = e^{\frac{(2k-1)\pi i}{n}}$. Then $\lambda^n + 1 = \prod_{k=1}^{n}(\lambda - w_k)$. Denote $C_k = \cos\frac{(2k-1)\pi}{2n}$. Note that

$$(\lambda - w_k)(\lambda - w_{n-k+1}) = \lambda^2 - 2\lambda\cos\frac{(2k-1)\pi}{n} + 1 = (\lambda+1)^2 - 4\lambda C_k^2,$$

since $\cos\frac{(2k-1)\pi}{n} = 2C_k^2 - 1$. Hence for n even,

$$\lambda^n + 1 = \prod_{k=1}^{n/2}\left[(\lambda+1)^2 - 4\lambda C_k^2\right],$$

and so

$$\lambda_1^n + \lambda_2^n = \prod_{k=1}^{n/2}\left[(\lambda_1+\lambda_2)^2 - 4\lambda_1\lambda_2 C_k^2\right].$$

Clearly, $\lambda_1 + \lambda_2 = 2$ and $\lambda_1\lambda_2 = -q$. We have by (13)

$$R_n(q) = \frac{\lambda_1^n + \lambda_2^n}{2} = \frac{1}{2}\prod_{k=1}^{n/2}\left[4 + 4q\cos^2\frac{(2k-1)\pi}{2n}\right].$$

This proves (15), as desired.

Denote $z_{n,k} = -1/\cos^2 \frac{(2k-1)\pi}{2n}, k = 1, 2, \cdots, n/2$. Then the polynomial $R_n(q)$ has distinct real zeros $z_{n,1} > z_{n,2} > \cdots > z_{n,n/2}$. Since

$$\lim_{n\to\infty} z_{n,1} = -\infty \text{ and } \lim_{n\to\infty} z_{n,n/2} = -1,$$

all zeros of $R_n(q)$ are in $(-\infty, -1)$.

(ii) Next we prove a stronger result: each $q \in (-\infty, -1]$ is a limit of zeros of the sequence $(R_n(q))_{n\geq 0}$. Recall that the Binet form of $R_n(q)$ is

$$R_n(q) = \frac{(\lambda_1 - 1)\lambda_1^n - (\lambda_2 - 1)\lambda_2^n}{\lambda_1 - \lambda_2}, \tag{16}$$

where

$$\lambda_{1,2} = 1 \pm \sqrt{1+q}.$$

The non-degeneracy conditions of Beraha-Kahane-Weiss Theorem are clearly satisfied from (16). So the limits of zeros of $(R_n(q))_{n\geq 0}$ are those real numbers q for which $|\lambda_1| = |\lambda_2|$, i.e.,

$$\left|1 + \sqrt{1+q}\right| = \left|1 - \sqrt{1+q}\right|.$$

Thus $1 + q \leq 0$, i.e., $q \leq -1$, which is what we wanted to show. This completes the proof. □

A classical approach for attacking the unimodality and log-concavity problem of a finite sequence is to use the famous Newton inequality.

Newton Inequality. *Suppose that the polynomial $f(x) = \sum_{k=0}^n a_k x^k$ has only real zeros. Then*

$$a_k^2 \geq a_{k-1}a_{k+1}\frac{(k+1)(n-k+1)}{k(n-k)}, \qquad k = 1, 2, \ldots, n-1.$$

In particular, if all a_k are nonnegative, then the sequence $a_0, a_1, \ldots, a_n$ is log-concave and unimodal. We refer the reader to [4, 5, 22] for details.

Corollary 4.3. *For each $n \geq 1$, the sequence $r_{n,0}, \ldots, r_{n,n}$ is log-concave and unimodal.*

5. Asymptotic normality

Let $a(n, k)$ be a double-indexed sequence of nonnegative numbers and let

$$p(n,k) = \frac{a(n,k)}{\sum_{j=0}^n a(n,j)}$$

denote the normalized probabilities. Following Bender [2], we say that the sequence $a(n, k)$ is *asymptotically normal by a central limit theorem*, if

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \sum_{k\leq \mu_n + x\sigma_n} p(n,k) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} dt \right| = 0, \tag{17}$$

where μ_n and σ_n^2 are the mean and variance of $a(n, k)$, respectively. We say that $a(n, k)$ is *asymptotically normal by a local limit theorem* on $\mathbb{R}$ if

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| \sigma_n p(n, \lfloor \mu_n + x\sigma_n \rfloor) - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \right| = 0. \tag{18}$$

In this case,

$$a(n,k) \sim \frac{e^{-x^2/2}\sum_{j=0}^{n} a(n,j)}{\sigma_n\sqrt{2\pi}} \text{ as } n \to \infty, \tag{19}$$

where $k = \mu_n + x\sigma_n$ and $x = O(1)$. Clearly, the validity of (18) implies that of (17).

Many well-known combinatorial sequences enjoy central and local limit theorems. For example, the famous de Movior-Laplace theorem states that the binomial coefficients $\binom{n}{k}$ are asymptotically normal (by central and local limit theorems). Other examples include the signless Stirling numbers $\left[\begin{array}{c} n \\ k \end{array}\right]$ of the first kind, the Stirling numbers $\left\{\begin{array}{c} n \\ k \end{array}\right\}$ of the second kind, the Eulerian numbers $A(n,k)$ [7] and the Delannoy numbers $d(n,k)$ [25]. Recently, Liu et al. proved the asymptotic normality of combinatorial numbers related to Dowling lattices [15] and the Stirling-Whitney-Riordan triangle [11]. A standard approach to demonstrating asymptotic normality is the following criterion (see [2, Theorem 2] for instance).

Lemma 5.1. *Suppose that $A_n(x) = \sum_{k=0}^{n} a(n,k)x^k$ have only real zeros and $A_n(x) = \prod_{i=1}^{n}(x + t_{n,i})$, where all $t_{n,i}$ are nonnegative. Let $\mu_n = \sum_{i=1}^{n} \frac{1}{1+t_{n,i}}$ and $\sigma_n^2 = \sum_{i=1}^{n} \frac{t_{n,i}}{(1+t_{n,i})^2}$. Then if $\sigma_n^2 \to +\infty$ as $n \to +\infty$, the numbers $a(n,k)$ are asymptotically normal (by central and local limit theorems) with the mean μ_n and variance σ_n^2.*

Combining Theorem 4.2 and Lemma 5.1, we obtain the following.

Theorem 5.2. *The coefficients $r_{n,i}$ are asymptotically normal (by central and local limit theorems) with the mean $\mu_n \sim \frac{2-\sqrt{2}}{2}n$ and variance $\sigma_n^2 \sim \frac{\sqrt{2}}{8}n$.*

Proof. Combining (15) and Lemma 5.1, we have

$$\mu_n = \sum_{k=1}^{n} \frac{1}{1 + \frac{1}{\cos^2\frac{(2k-1)\pi}{2n}}} = \sum_{k=1}^{n} \frac{\cos^2\frac{(2k-1)\pi}{2n}}{1 + \cos^2\frac{(2k-1)\pi}{2n}}.$$

Hence

$$\begin{aligned}
\mu_n &\to \frac{n}{\pi}\int_0^{\pi} \frac{\cos^2\theta}{1+\cos^2\theta}\mathrm{d}\theta \\
&= \frac{2n}{\pi}\int_0^{\pi/2} \frac{\cos^2\theta}{1+\cos^2\theta}\mathrm{d}\theta \\
&= \frac{2n}{\pi}\left[\theta - \frac{\arctan\left(\frac{\tan\theta}{\sqrt{2}}\right)}{\sqrt{2}}\right]_0^{\frac{\pi}{2}} \\
&= \frac{2-\sqrt{2}}{2}n.
\end{aligned}$$

On the other hand, we have

$$\sigma_n^2 = \sum_{k=1}^{n} \frac{\frac{1}{\cos^2\frac{(2k-1)\pi}{2n}}}{\left[1 + \frac{1}{\cos^2\frac{(2k-1)\pi}{2n}}\right]^2} = \sum_{k=1}^{n} \frac{\cos^2\frac{(2k-1)\pi}{2n}}{\left[1 + \cos^2\frac{(2k-1)\pi}{2n}\right]^2}.$$

Thus

$$\begin{aligned}\sigma_n^2 &\to \frac{n}{\pi}\int_0^{\pi}\frac{\cos^2\theta}{(1+\cos^2\theta)^2}\mathrm{d}\theta\\ &= \frac{2n}{\pi}\int_0^{\pi/2}\frac{\cos^2\theta}{(1+\cos^2\theta)^2}\mathrm{d}\theta\\ &= \frac{2n}{\pi}\left[\frac{\arctan\left(\frac{\tan\theta}{\sqrt{2}}\right)}{4\sqrt{2}}+\frac{\sin 2\theta}{4(\cos 2\theta+3)}\right]_0^{\frac{\pi}{2}}\\ &= \frac{\sqrt{2}}{8}n.\end{aligned}$$

The statement follows from Lemma 5.1. □

Acknowledgement

We would like to thank the reviewers greatly for the very careful reviewing and the valuable suggestions to this paper.

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