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Orthogonal F-weak contraction mapping in orthogonal metric space, fixed points and applications

Deepak Kumar[a,*], Astha Malhotra[b]

[a]*Department of Mathematics, Lovely Professional University, Jalandhar, 144411, India*
[b]*Research Scholar, Lovely Professional University, Jalandhar, 144411, India*

Abstract. In this article, we introduce the notion of orthogonal F-weak contraction mapping in an orthogonal metric space, as well as certain fixed point results. Furthermore, the examples presented in the main result illustrates that the results proved in this article are a proper extension of some of the results presented in the literature. The results are used to show the existence and uniqueness of the solution to a first order differential equation.

1. Introduction

S. Banach [1], in 1922, came up with a classical and the most celebrated outcome called *"Banach Contraction Principle"* for the existence and uniqueness of the fixed point of a self-map on a complete metric space along with a contractive condition. Thereafter, numerous generalizations of Banach Contraction Principle have been presented by the researchers (see [2–4, 7, 13, 14] and references cited therein).

D. Wardowski [13], in 2012, introduced a novel contraction condition called as F-contraction along with the fixed point result in a complete metric space. However, in 2014, D. Wardowski and N. V. Dung [14] further generalized the F-contraction condition and initiated the idea of F-weak contraction by using the maximize condition. Over the period of time, multiple attempts have been made by authors to generalize F-contraction as well as F-weak contraction ([5, 9, 10, 12]).

M. E. Gordji *et al.*, in 2017 [7] put forward the notion of an orthogonal set and subsequently orthogonal metric space and deduced fixed point results using an analogous form of Banach Contraction Principle in this setting. Many authors have generalized and extended the results obtained in [7] (see [6, 8, 11, 12, 15]).

In this article, we aim to introduce the notion of an orthogonal F-weak contraction condition and prove certain fixed point results in an orthogonal complete metric space with an application. The examples presented in the main result asserts that the outcomes presented in this article are a proper extension of the results proven in [2, 7, 12].

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* Corresponding author: Deepak Kumar
Email addresses: deepakanand@live.in (Deepak Kumar), astha.malhotra19@gmail.com (Astha Malhotra)

2. Preliminaries

In this section, we recall some definitions used in the main results. The symbol $\mathbb{R}, \mathbb{N}, \mathbb{R}^+$ denotes the set of real numbers, natural numbers and non-negative real numbers respectively.

Definition 2.1. *[7] A non-empty set X along with a binary relation $\perp$ is called an orthogonal set (denoted by O-set) if $\exists\ \rho_0 \in X$ such that $[\rho \perp \rho_0\ \forall\ \rho \in X]$ or $[\rho_0 \perp \rho\ \forall\ \rho \in X]$. The element ρ_0 is called an orthogonal element.*

Example 2.2. *Let X be the set of all $n \times n$ real matrices with non-zero determinant. For matrices $M, N \in X$, define $M \perp N$ if and only if $MN = NM$. Then, $(X, \perp)$ is an O-set.*

Example 2.3. *Consider the interval $[0, +\infty)$ then $\forall\ \rho, \sigma \in [0, +\infty)$, define $\rho \perp \sigma$ if and only if $\rho.\sigma = 0$. Then, $([0, +\infty), \perp)$ is an O-set.*

Definition 2.4. *[7] Let $(X, \perp)$ be an O-set. A sequence $\{\rho_n\}_{n\in\mathbb{N}}$ is called an orthogonal sequence (denoted by O-sequence) if $[\rho_n \perp \rho_{n+1}\ \forall\ n \in \mathbb{N}]$ or $[\rho_{n+1} \perp \rho_n\ \forall\ n \in \mathbb{N}]$.*

Example 2.5. *Let $X = \mathbb{R}$ and define $\rho \perp \sigma$ if and only if $\rho.\sigma \leq 0$ then, $\left\{\frac{(-1)^n}{n}\right\}_{n\in\mathbb{N}}$ is an O-sequence in the O-set $(X, \perp)$.*

Definition 2.6. *[7] The non-empty set X endowed with metric d and binary relation $\perp$ is called an orthogonal metric space (denoted by $(X, \perp, d)$) if $(X, \perp)$ is an orthogonal set and (X, d) is a metric space.*

Example 2.7. *In the Example 2.5, $(X, \perp)$ together with the usual metric space is an orthogonal metric space.*

Definition 2.8. *[7] For an orthogonal metric space $(X, \perp, d)$, a function $f : X \to X$ is said to be orthogonally continuous (denoted by $\perp$-continuous) at $\rho \in X$ if for each O-sequence $\{\rho_n\}_{n\in\mathbb{N}}$ with $\rho_n \to \rho$ as $n \to +\infty$ implies $f(\rho_n) \to f(\rho)$ as $n \to +\infty$. In addition, f is said to be $\perp$-continuous on X if f is $\perp$-continuous at each point $\rho \in X$.*

Definition 2.9. *[7] An orthogonal metric space $(X, \perp, d)$ is said to be orthogonally complete (denoted by O-complete) if every Cauchy O-sequence in X is convergent in X. Also, a function $f : X \to X$ is called $\perp$-preserving if $\rho \perp \sigma$ implies $f(\rho) \perp f(\sigma)$ and f is called weakly $\perp$-preserving if $\rho \perp \sigma$ implies $f(\rho) \perp f(\sigma)$ or $f(\sigma) \perp f(\rho)$.*

Definition 2.10. *[13] Let $\mathfrak{F}$ be a family of all mappings $F : (0, +\infty) \to (-\infty, +\infty)$ such that*

(F1) for $\rho, \sigma \in (0, +\infty)$ if $\rho < \sigma$ implies $F(\rho) < F(\sigma)$;

(F2) for each sequence $\{\rho_n\}_{n\in\mathbb{N}}$ of positive number such that

$$\lim_{n\to+\infty} \rho_n = 0 \text{ if and only if } \lim_{n\to+\infty} F(\rho_n) = -\infty;$$

(F3) $\exists\ r \in (0, 1)$ such that $\lim_{\zeta\to 0^+} \zeta^r F(\zeta) = 0$.

A self-map $f : X \to X$ on a metric space (X, d) is said to be F-contraction if $\exists\ F \in \mathfrak{F}$ and $\tau > 0$ such that $\forall\ \rho, \sigma \in X$ with $d(f\rho, f\sigma) > 0$ implies

$$\tau + F\Big(d(f\rho, f\sigma)\Big) \leq F\Big(d(\rho, \sigma)\Big).$$

Definition 2.11. *[14] A map $f : X \to X$ on a metric space (X, d) is said to be F-weak contraction if $\exists\ F \in \mathfrak{F}$ and $\tau > 0$ such that $\forall\ \rho, \sigma \in X$ with $d(f\rho, f\sigma) > 0$ implies*

$$\tau + F(d(f\rho, f\sigma)) \leq F\Big(\max\Big\{d(\rho, \sigma), d(\rho, f\rho), d(\sigma, f\sigma), \frac{d(\rho, f\sigma) + d(\sigma, f\rho)}{2}\Big\}\Big).$$

Definition 2.12. *[2, 12] For an orthogonal metric space $(X, \perp, d)$ with $F \in \mathfrak{F}$, a map $f : X \to X$ is called an orthogonal F-contraction map (denoted by $\perp_F$-contraction) if $\exists\ \tau > 0$ such that $\forall\ \rho, \sigma \in X$ with $\rho \perp \sigma$ and $d(f\rho, f\sigma) > 0$ implies*

$$\tau + F\Big(d(f\rho, f\sigma)\Big) \leq F(d(\rho, \sigma)).$$

3. Main Results

Inspired by the work done in [2, 12], in this section, we put forward the notion of orthogonal F-weak contraction and prove some fixed point theorem with orthogonal F-weak contraction in an O-complete metric space.

Definition 3.1. *A map $f : X \to X$, where $(X, \perp, d)$ is an orthogonal metric space and $F \in \mathfrak{F}$, is called an orthogonal F-weak contraction (denoted by $\perp_F$-weak contraction) if $\exists\ \tau > 0$ such that $\forall\ \rho, \sigma \in X$ with $\rho \perp \sigma$ and $d(f\rho, f\sigma) > 0$ implies*

$$\tau + F(d(f\rho, f\sigma)) \quad \leq \quad F\Big(\max\Big\{d(\rho,\sigma), d(\rho, f\rho), d(\sigma, f\sigma), \frac{d(\rho, f\sigma) + d(\sigma, f\rho)}{2}\Big\}\Big). \tag{1}$$

Remark 3.2. *From (1), one can infer that every $\perp_F$-contraction is $\perp_F$-weak contraction. However, following example shows that the converse need not be true.*

Example 3.3. *Let $X = \{0, 1, 2, 3, 4\}$ endowed with usual metric. Let $R = \{(0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (4, 0), (4, 1), (4, 2), (4, 3), (4, 4)\}$. Define $\rho \perp \sigma$ if and only if $(\rho, \sigma) \in R$. Clearly, $(X, \perp)$ is an orthogonal set (with 0 and 4 as orthogonal elements). Define $f : X \to X$ as $f(0) = 0 = f(1) = f(4)$, $f(2) = 1$, $f(3) = 2$. Let $F(\beta) = \ln(\beta)$. It can be verified that f is $\perp_F$-weak contraction however, f is not $\perp_F$-contraction since for $\rho = 4$ and $\sigma = 3$, $\tau + F(d(f\rho, f\sigma)) \leq F(d(\rho, \sigma))$ does not hold for any $\tau > 0$.*

Theorem 3.4. *Let $(X, \perp, d)$ be an O-complete metric space and $F \in \mathfrak{F}$. If a map $f : X \to X$ be $\perp$-continuous, $\perp_F$-weak contraction and $\perp$-preserving. Then, f has a unique fixed point in X.*

Proof. As $(X, \perp)$ is an orthogonal set, therefore $\exists$ an orthogonal element $\rho_0 \in X$ such that

$$[\rho \perp \rho_0\ \forall\ \rho \in X] \quad \text{or} \quad [\rho_0 \perp \rho\ \forall\ \rho \in X]. \tag{2}$$

As ρ_0, $f\rho_0 \in X$ then by (2), we have

$$[f\rho_0 \perp \rho_0] \quad \text{or} \quad [\rho_0 \perp f\rho_0].$$

Define a sequence$\{\rho_n\}_{n\in\mathbb{N}}$ in X, where $\rho_{n+1} = f\rho_n\ \forall\ n \in \mathbb{N}$. Since f is $\perp$-preserving. Therefore, $\{\rho_n\}_{n\in\mathbb{N}}$ is an orthogonal sequence. Let us consider $\vartheta_n = d(\rho_n, \rho_{n+1})$ for $n = 0, 1, 2, \ldots$. If for some $n_0 \in \mathbb{N}$, $\vartheta_{n_0} = 0$

$$\begin{aligned} \text{then, } d(\rho_{n_0}, \rho_{n_{0+1}}) &= 0 \\ \text{implies } \rho_{n_0} = \rho_{n_{0+1}} &= f\rho_{n_0}, \end{aligned}$$

which gives that f possesses a fixed point. On the contrary, suppose that $\vartheta_n \neq 0\ \forall\ n \in \mathbb{N}$. Since f is $\perp_F$-weak contraction so $\forall\ n \in \mathbb{N}$, we have

$$\begin{aligned} F(\vartheta_n) = F(d(\rho_n, \rho_{n+1}) &= F(d(f\rho_{n-1}, f\rho_n)) \\ &\leq F\Big(\max\Big\{d(\rho_{n-1}, \rho_n), d(\rho_{n-1}, f\rho_{n-1}), d(\rho_n, f\rho_n), \frac{d(\rho_{n-1}, f\rho_n) + d(\rho_n, f\rho_{n-1})}{2}\Big\}\Big) - \tau \\ &= F\Big(\max\Big\{d(\rho_{n-1}, \rho_n), d(\rho_{n-1}, \rho_n), d(\rho_n, \rho_{n+1}), \frac{d(\rho_{n-1}, \rho_{n+1}) + d(\rho_n, \rho_n)}{2}\Big\}\Big) - \tau \\ &\leq F\Big(\max\Big\{d(\rho_{n-1}, \rho_n), d(\rho_n, \rho_{n+1}), \frac{d(\rho_{n-1}, \rho_n) + d(\rho_n, \rho_{n+1})}{2}\Big\}\Big) - \tau \\ &= F\Big(\max\Big\{d(\rho_{n-1}, \rho_n), d(\rho_n, \rho_{n+1})\Big\}\Big) - \tau. \end{aligned}$$

If $\max\{d(\rho_{n-1}, \rho_n), d(\rho_n, \rho_{n+1})\} = d(\rho_n, \rho_{n+1})$ then from above, we have

$$F(\vartheta_n) \leq F(\vartheta_n) - \tau,$$

which is a contradiction (for $\tau > 0$). Therefore,

$$\max\Big\{d(\rho_{n-1},\rho_n), d(\rho_n,\rho_{n+1})\Big\} = d(\rho_{n-1},\rho_n) \quad \forall\, n \in \mathbb{N},$$

$$\text{and, } F(\vartheta_n) \leq F(\vartheta_{n-1}) - \tau \leq F(\vartheta_{n-2}) - 2\tau \leq \cdots \leq F(\vartheta_0) - n\tau. \tag{3}$$

Letting $n \to +\infty$ in (3), we obtain

$$\lim_{n\to+\infty} F(\vartheta_n) = -\infty.$$

Using ($F2$), we obtain

$$\lim_{n\to+\infty} \vartheta_n = 0. \tag{4}$$

By ($F3$) property $\exists\, r \in (0,1)$ such that

$$\lim_{n\to+\infty} \vartheta_n^r F(\vartheta_n) = 0. \tag{5}$$

From (3), we have

$$\vartheta_n^r F(\vartheta_n) - \vartheta_n^r F(\vartheta_0) \leq -\vartheta_n^r n\tau.$$

Taking $n \to +\infty$ and using (4) and (5), we get

$$\lim_{n\to+\infty} n\vartheta_n^r = 0. \tag{6}$$

On observing (6), we get that $\exists\, n_1 \in \mathbb{N}$ such that $n\vartheta_n^r \leq 1\ \forall\, n > n_1$. Therefore,

$$\vartheta_n \leq \frac{1}{n^{\frac{1}{r}}} \quad \forall\, n > n_1. \tag{7}$$

Next, we claim that the sequence $\{\rho_n\}_{n\in\mathbb{N}}$ is a Cauchy O-sequence. Consider $m, n \in \mathbb{N}$ such that $m > n > n_1$, using (7) and triangle inequality of metric space d, we get

$$\begin{aligned} d(\rho_m,\rho_n) &\leq d(\rho_m,\rho_{m-1}) + \cdots + d(\rho_{n+2},\rho_{n+1}) + d(\rho_{n+1},\rho_n) \\ &= \vartheta_{m-1} + \cdots + \vartheta_{n+1} + \vartheta n \\ &< \sum_{i=1}^{+\infty} \vartheta_i \leq \sum_{i=1}^{+\infty} 1/i^{1/r}. \end{aligned}$$

Using convergence of $\sum_{n=1}^{+\infty} 1/n^{1/r}$ (for $r \in (0,1)$), we get $\{\rho_n\}_{n\in\mathbb{N}}$ is Cauchy O-sequence and by O-completeness of X, $\{\rho_n\}_{n\in\mathbb{N}}$ is convergent, that is, $\exists\, \rho \in X$ such that

$$\lim_{n\to+\infty} \rho_n = \rho.$$

Using $\perp$-continuity of f, we get

$$\lim_{n\to+\infty} \rho_{n+1} = \lim_{n\to+\infty} f\rho_n = f\rho.$$

Thus, $\rho = f\rho$. Hence, ρ is a fixed point of f. Now, we show the uniqueness of the fixed point. Let ρ^* be another fixed point of f which implies, $f^n(\rho^*) = \rho^*\ \forall\, n \in \mathbb{N}$. By (2), we have

$$[\rho_0 \perp \rho^*] \quad \text{or} \quad [\rho^* \perp \rho_0].$$

Since f is $\perp$-preserving, therefore

$$[f^n(\rho_0) \perp f(\rho^*)] \quad \text{or} \quad [f(\rho^*) \perp f^n(\rho_0)].$$

Also, f is $\perp_F$-weak contraction, thus

$$\begin{aligned}
F(d(\rho_n, \rho^*)) &= F(d(f^n\rho_0, \rho^*)) \\
&= F(d(f\rho_{n-1}, f\rho^*)), \\
&\leq F\Big(\max\Big\{d(\rho_{n-1}, \rho^*), d(\rho_{n-1}, f\rho_{n-1}), d(\rho^*, f\rho^*), \frac{d(\rho_{n-1}, f\rho^*) + d(\rho^*, f\rho_{n-1})}{2}\Big\}\Big) - \tau \\
&= F\Big(\max\Big\{d(\rho_{n-1}, \rho^*), d(\rho_{n-1}, \rho_n), d(\rho^*, \rho^*), \frac{d(\rho_{n-1}, \rho^*) + d(\rho^*, \rho_n)}{2}\Big\}\Big) - \tau \\
&= F\Big(\max\Big\{d(\rho_{n-1}, \rho^*), d(\rho_{n-1}, \rho_n), d(\rho_n, \rho^*)\Big\}\Big) - \tau.
\end{aligned}$$

We have following cases:

Case (i) : If $\max\{d(\rho_{n-1}, \rho^*), d(\rho_{n-1}, \rho_n), d(\rho_n, \rho^*)\} = d(\rho_n, \rho^*)$ then $\forall\ n \in \mathbb{N}$, we have

$$F(d(\rho_n, \rho^*)) \leq F(d(\rho_n, \rho^*)) - \tau,$$

which is a contradiction for any $\tau > 0$.

Case (ii) : If $\max\{d(\rho_{n-1}, \rho^*), d(\rho_{n-1}, \rho_n), d(\rho_n, \rho^*)\} = d(\rho_{n-1}, \rho_n)$ then $\forall\ n \in \mathbb{N}$, we have

$$F(d(\rho_n, \rho^*)) \leq F(d(\rho_{n-1}, \rho_n)) - \tau = F(\vartheta_{n-1}) - \tau.$$

Using (3), we get

$$F(d(\rho_n, \rho^*)) \leq F(\vartheta_{n-1}) - \tau \leq \cdots \leq F(\vartheta_0) - n\tau.$$

Taking $n \to +\infty$, we get

$$\lim_{n\to+\infty} F(d(\rho_n, \rho^*)) = -\infty.$$

By $(F2)$ property, $\lim\limits_{n\to+\infty} d(\rho_n, \rho^*) = 0$ implies $\rho = \rho^*$.

Case (iii) : If $\max\{d(\rho_{n-1}, \rho^*), d(\rho_{n-1}, \rho_n), d(\rho_n, \rho^*)\} = d(\rho_{n-1}, \rho^*)$ then $\forall\ n \in \mathbb{N}$, we have

$$\begin{aligned}
F(d(\rho_n, \rho^*)) &\leq F(d(\rho_{n-1}, \rho^*)) - \tau \\
&\leq F(d(\rho_{n-2}, \rho^*)) - 2\tau \leq \cdots \leq F(d(\rho_0, \rho^*)) - n\tau.
\end{aligned}$$

Taking $n \to +\infty$ and using $(F2)$ property, we obtain $\rho = \rho^*$. Thus, we conclude that f has a unique fixed point in X. □

Remark 3.5. *Theorem 3.4 proved above provides a proper extension of Theorem 3.10 and Theorem 3.3 of [2] and [12] respectively. The following example further substantiates the claim.*

Example 3.6. *Consider the orthogonal metric space discussed in Example 3.3. Then, the function defined in it can be verified for $\perp$-continuous and $\perp$-preserving. Also, X is an O-complete metric space since for any arbitrary Cauchy O-sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in X, $\exists$ a subsequence $\{\rho_{n_k}\}_{n_k\in\mathbb{N}}$ of $\{\rho_n\}_{n\in\mathbb{N}}$ such that $\rho_{n_k} = 0\ \forall\ k \geq k_1$ or $\rho_{n_k} = 4\ \forall\ k \geq k_2$ for some $k_1, k_2 \in \mathbb{N}$. Thus, $\{\rho_{n_k}\}_{n_k\in\mathbb{N}}$ converges to 0 or 4. Therefore, $\{\rho_n\}_{n\in\mathbb{N}}$ is convergent. Since all the conditions of Theorem 3.4 hold. Thus, f has a unique fixed point which is $\rho = 0$ even though f is not $\perp_F$-contraction.*

Example 3.7. *Let $X = \mathbb{R}^+ \cup \{0\}$ along with metric d on X defined as $d(\rho,\sigma) = \max\{\rho,\sigma\}$. Define a map $f : X \to X$ as*

$$f(\rho) = \begin{cases} 0 & for\ \rho \in [0,1); \\ 9/10 & otherwise. \end{cases}$$

Consider $F(\alpha) = \ln(\alpha)$ and let $\rho \perp \sigma$ if and only if either $\rho = 0$ or $\sigma = 0$. For $d(f\rho, f\sigma) > 0$ where $\rho \perp \sigma$ we must have either $\rho = 0$ and $\sigma \in [1,+\infty)$ or $\rho \in [1,+\infty)$ and $\sigma = 0$. Let $\rho = 0$ and $\sigma \in [1,+\infty)$ then, $f\rho = 0$ and $f\sigma = 9/10$ so that

$$\tau + F(d(f\rho, f\sigma)) = \tau + F(d(0, 9/10)) = \tau + \ln(9/10), \tag{8}$$

and,

$$\begin{aligned} F\Big(\max\Big\{d(\rho,\sigma), d(\rho,f\rho), d(\sigma,f\sigma), \frac{d(\rho,f\sigma)+d(\sigma,f\rho)}{2}\Big\}\Big) &= \ln\Big(\max\Big\{\sigma, \frac{9/10+\sigma}{2}\Big\}\Big) \\ &= \ln\Big(\sigma\Big). \end{aligned} \tag{9}$$

From (8), (9) and for any value of τ where $0 < \tau \leqslant -\ln(9/10)$, f satisfies $\perp_F$-weak contraction condition. For $\sigma = 0$ and $\rho \in [1,+\infty)$, the contraction condition holds on similar lines as above. Also, f is $\perp$-preserving and $\perp$-continuous, thus satisfying conditions of Theorem 3.4 and hence f possesses a unique fixed point viz. $\rho = 0$.

Theorem 3.8. *Let $(X, \perp, d)$ be an O-complete metric space and $F \in \mathfrak{F}$. If a map $f : X \to X$ is $\perp_F$-weak contraction and $\perp$-preserving such that*

(I) F is continuous;

(II) If $\exists$ an O-sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in X is such that for $\rho_n \to \rho$ as $n \to +\infty$ we have $\rho_n \perp \rho\ \forall\ n \in \mathbb{N}$ or $\rho \perp \rho_n\ \forall\ n \in \mathbb{N}$.

Then, f has a unique fixed point in X.

Proof. Working on the lines of Theorem 3.4, it can be shown that $\exists$ an O-sequence $\{\rho_n\}_{n\in\mathbb{N}}$ such that $\rho_n \to \rho$ as $n \to +\infty$. We claim that ρ is the desired fixed point. However, once the existence of fixed point is established then the uniqueness follows similar to Theorem 3.4. Suppose on the contrary that $d(\rho, f\rho) > 0$.

Case(i) : If $\{n \in \mathbb{N} : f\rho_n = f\rho\}$ is infinite. Then, $\exists$ subsequence $\{\rho_{n_i}\}_{n_i\in\mathbb{N}}$ of $\{\rho_n\}_{n\in\mathbb{N}}$ such that

$$f\rho_{n_i} = f\rho \text{ implies } \rho_{n_i+1} = f\rho.$$

Taking limit $n \to +\infty$, we get $\rho = f\rho$, which is a contradiction.

Case(ii) : If $\{n \in \mathbb{N} : f\rho_n = f\rho\}$ is finite, that is, for some $n_0 \in \mathbb{N}$, $d(f\rho_n, f\rho) > 0\ \forall\ n > n_0$. By given condition we have $[\rho_n \perp \rho\ \forall\ n \in \mathbb{N}]$ or $[\rho \perp \rho_n\ \forall\ n \in \mathbb{N}]$. Since f is $\perp$-preserving, therefore

$$[f\rho_n \perp f\rho\ \forall\ n \in \mathbb{N}] \quad \text{or} \quad [f\rho \perp f\rho_n\ \forall\ n \in \mathbb{N}].$$

As f is $\perp_F$-weak contraction, we have

$$\begin{aligned} \tau + F(d(f\rho_n, f\rho)) &\leq F\Big(\max\Big\{d(\rho_n,\rho), d(\rho_n, f\rho_n), d(\rho, f\rho), \frac{d(\rho_n, f\rho) + d(\rho, f\rho_n)}{2}\Big\}\Big) \\ &\leq F\Big(\max\Big\{d(\rho_n,\rho), d(\rho_n, \rho_{n+1}), d(\rho, f\rho), \frac{d(\rho_n, \rho) + d(\rho, f\rho) + d(\rho, \rho_{n+1})}{2}\Big\}\Big). \end{aligned}$$

Since $\rho_n \to \rho$ as $n \to +\infty$. Therefore, $\exists\ n_1 \in \mathbb{N}$, such that

$$d(\rho_n, \rho) = 0 \quad \forall\ n > n_1.$$

Hence, $\forall\ n > \max\{n_0, n_1\}$, we obtain

$$\max\left\{d(\rho_n, \rho), d(\rho_n, \rho_{n+1}), d(\rho, f\rho), \frac{d(\rho_n, \rho) + d(\rho, f\rho) + d(\rho, \rho_{n+1})}{2}\right\} = d(\rho, f\rho).$$

Since F is continuous, on taking limit $n \to +\infty$ in (10), we get

$$\tau + F(d(f\rho, f\rho)) \le F(d(f\rho, f\rho)),$$

which is again a contradiction. Thus, we conclude that f has a fixed point ρ in X. $\square$

Remark 3.9. *In Theorem 3.8, we have dropped the condition of $\perp$-continuity of f and instead we consider F to be continuous function along with orthogonal F-weak contraction of f, which gives a more generalized result in this setting.*

Corollary 3.10. *Let $(X, \perp, d)$ be an O-complete metric space and $F \in \mathfrak{F}$. If a map $f : X \to X$ is $\perp_F$-weak contraction and $\perp$-preserving such that*

(I) F is continuous;

(II) If $\exists$ an O-sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in X is such that for $\rho_n \to \rho$ as $n \to +\infty$ we have $\rho_n \perp \rho\ \forall\ n \in \mathbb{N}$ or $\rho \perp \rho_n\ \forall\ n \in \mathbb{N}$.

Then, f in X has a unique fixed point. Further, for each $\rho^* \in X$ the Picard sequence $\{f^n(\rho^*)\}_{n\in\mathbb{N}}$ converges to fixed point ρ of f.

Proof. The existence and uniqueness of fixed point can be proved on the steps of Theorem 3.8. We show that Picard sequence $\{f^n(\rho^*)\}_{n\in\mathbb{N}}$ converges to fixed point ρ, that is

$$\lim_{n\to+\infty} f^n(\rho^*) = \rho.$$

Since $\rho^* \in X$ is any arbitrary point and X is an orthogonal set, therefore

$$[\rho^* \perp \rho_0] \quad \text{or} \quad [\rho_0 \perp \rho^*],$$

and as f is $\perp$-preserving, thus

$$[f^n(\rho^*) \perp f^n(\rho_0)\ \forall\ n \in \mathbb{N}] \quad \text{or} \quad [f^n(\rho_0) \perp f^n(\rho^*)\ \forall\ n \in \mathbb{N}].$$

Using $\perp_F$-weak contraction of f, we obtain

$$\begin{aligned}\tau + F(d(f^n(\rho^*), \rho_n)) &= \tau + F(d(f^n(\rho^*), f^n(\rho_0))\\ &= \tau + F(d(f(f^{n-1}(\rho^*)), f(f^{n-1}(\rho_0)))\\ &\le F\Big(\max\Big\{d(f^{n-1}(\rho^*), \rho_{n-1}), d(f^{n-1}(\rho^*), f^n(\rho^*)), d(\rho_{n-1}, \rho_n),\\ &\quad \frac{d(f^{n-1}(\rho^*), \rho_n) + d(\rho_{n-1}, f^n(\rho^*))}{2}\Big\}\Big). \end{aligned} \tag{10}$$

Now, as $n \to +\infty$, $\rho_n \to \rho$. Therefore, we have

$$\max\left\{d(f^{n-1}(\rho^*), \rho_{n-1}), d(f^{n-1}(\rho^*), \rho_n) + d(\rho_n, f^n(\rho^*)), d(\rho_{n-1}, \rho_n), \frac{d(f^{n-1}(\rho^*), \rho_n) + d(\rho_{n-1}, f^n(\rho^*))}{2}\right\}$$
$$= d(\lim_{n\to+\infty} f^n\rho^*, \rho).$$

Taking limit as $n \to +\infty$ in (10) and using continuity of F, we get

$$\tau + F(d(\lim_{n\to+\infty} f^n\rho^*, \rho)) \le F(d(\lim_{n\to+\infty} f^n\rho^*, \rho)),$$

which holds if and only if

$$\lim_{n\to+\infty} f^n(\rho^*) = \rho.$$

Thus, f is a Picard operator. $\square$

4. Application

The existence of solution to an ordinary differential equation arising in a mathematical model occurring from real world problem can be verified using a suitable fixed point technique. The purpose of this section is to apply the result obtained in Theorem 3.4 and to prove existence and uniqueness of the solution of an ordinary differential equation:

$$\begin{aligned} \theta^{'}(u) - f(u,\theta(u)) = 0 \quad \text{a.e} \quad u \in I = [0,T]; \\ \theta(0) = a \quad \text{for} \quad a \geq 1, \end{aligned} \tag{11}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ is an integrable function which satisfies the following:

(I) $f(u,v) \geq 0 \ \forall \ u \in I$ and $v \geq 0$;

(II) $\exists \ \alpha(u) \in \mathcal{L}^1(I)$ and $\tau > 0$ such that

$$|f(u,p(u)) - f(u,q(u))| \leq \frac{\alpha(u)}{e^{\tau}}|p(u) - q(u)|,$$

$\forall \ p,q \in \mathcal{L}^1(I)$ such that $p(u)q(u) \geq p(u)$ or $p(u)q(u) \geq q(u)$.

Theorem 4.1. *The differential equation given in (11) along with condition (I) and (II) has a unique solution.*

Proof. Let $X = \{p \in C(I,\mathbb{R}) : p(u) > 0 \ \forall \ u \in I\}$ and define a binary relation on X as

$$p \perp q \text{ implies } p(u)q(u) \geq p(u) \quad \text{or} \quad p(u)q(u) \geq q(u) \ \forall \ u \in I.$$

Then $(X,\perp)$ is an orthogonal set. Let $A(u) = \int_0^u |\alpha(u)|du$. Then we have $A^{'} = |\alpha(u)|$ a.e $u \in I$. Define a mapping $d : X \times X \to \mathbb{R}^+$ by

$$d(p,q) = ||p - q|| = \sup_{u \in I} e^{-A(u)}|\ p(u) - q(u)\ | \ \forall \ p,q \in X.$$

Now, we claim that $(X,\perp,d)$ is an O-complete metric space. Let $\{p_n\}_{n\in\mathbb{N}}$ be a Cauchy O-sequence in X then, $\{p_n\}_{n\in\mathbb{N}}$ converges to a point p in $C(I)$. It is enough to show that $p \in X$. Let $u \in I$ then

$$p_n(u)p_{n+1}(u) \geq p_n(u) \quad \text{or} \quad p_n(u)p_{n+1}(u) \geq p_{n+1}(u).$$

As $p_n(u) > 0 \ \forall \ n \in \mathbb{N}$, then $\exists$ a subsequence $\{p_{n_k}\}_{n_k\in\mathbb{N}}$ of $\{p_n\}_{n\in\mathbb{N}}$ for which $p_{n_k} \geq 1$ and since $p_n \to p$ as $n \to +\infty$ so $p_{n_k} \to p$ as $n_k \to +\infty$ implies $p(u) \geq 1$. Thus $p \in X$. Define a map $\mathcal{Y} : X \to X$ as:

$$(\mathcal{Y}p)(u) = \beta + \int_0^u f(t,p(t))dt.$$

Then:

(1) $\mathcal{Y}$ **is $\perp$-preserving:** Let $p \perp q$, then

$$(\mathcal{Y}p)(u) = \beta + \int_0^u f(t,p(t))dt \geq 1,$$

which shows that $(\mathcal{Y}p)(u)(\mathcal{Y}q)(u) \geq (\mathcal{Y}q)(u)$ or $(\mathcal{Y}p)(u)(\mathcal{Y}q)(u) \geq (\mathcal{Y}p)(u)$. Therefore, $\mathcal{Y}p \perp \mathcal{Y}q$.

(2) $\mathcal{Y}$ **is $\perp$-continuous:** Let $\{p_n\}_{n\in\mathbb{N}}$ be an O-sequence in X which converges to $p \in X$. Then it is well evident from previous working that $p(u) \geq 1$ implies $p_n(u) \perp p(u) \ \forall \ n \in \mathbb{N}$ and $u \in I$. Also,

$$\begin{aligned} e^{-A(u)}|(\mathcal{Y}p_n)(u) - (\mathcal{Y}p)(u)| &\leq e^{-A(u)} \int_0^u |f(t,p_n(t)) - f(t,p(t))|dt \\ &\leq e^{-A(u)} \int_0^u |p_n(t)) - p(t))|\frac{|\alpha(t)|}{e^{\tau}} e^{-A(t)}e^{A(t)}dt \\ &\leq e^{-A(u)}e^{-\tau}d(p_n,p) \int_0^u |\alpha(t)|e^{A(t)}dt \\ &\leq e^{-A(u)}e^{-\tau}d(p_n,p)(e^{A(u)-1}). \end{aligned}$$

Since above inequality hold for any arbitrary $u \in I$ and $n \in \mathbb{N}$. So, we have

$$d(\mathcal{Y}p_n, \mathcal{Y}p) \leq e^{-\tau}(1 - e^{-\|\alpha\|_1})d(p_n, p) \ \forall \ n \in \mathbb{N}.$$

Thus $\mathcal{Y}p_n \to \mathcal{Y}p$ as $n \to +\infty$.

(3) $\mathcal{Y}$ **is** $\perp_F$**-weak contraction:** Let $p, q \in X$ such that $p \perp q$ and $d(\mathcal{Y}p, \mathcal{Y}q) > 0$, then for each $u \in I$, we obtain

$$\begin{aligned}
|(\mathcal{Y}p)(u) - (\mathcal{Y}q)(u)| &\leq \int_0^u |f(t, p(t)), f(t, q(t))| dt \\
&\leq \int_0^u e^{-\tau}|\alpha(t)||p(t) - q(t)|e^{-A(t)}e^{A(t)}dt \\
&\leq e^{-\tau}d(p, q)\int_0^u |\alpha(t)|e^{A(t)} \\
&\leq e^{-\tau}d(p, q)(e^{A(u)} - 1), \\
\text{and,} \quad e^{-A(u)}|(\mathcal{Y}p)(u) - (\mathcal{Y}q)(u)| &\leq e^{-A(u)}(e^{A(u)} - 1)e^{-\tau}d(p, q) \\
&\leq (1 - e^{-A(u)})e^{-\tau}d(p, q) \\
&\leq (1 - e^{-\|\alpha\|_1})e^{-\tau}d(p, q).
\end{aligned}$$

Thus, it follows that

$$d(\mathcal{Y}p, \mathcal{Y}q) \leq e^{-\tau}d(p, q).$$

Taking logarithm, we get

$$\tau + \ln(d(\mathcal{Y}p, \mathcal{Y}q)) \leq \ln\Big(\max\Big\{d(p, q), d(p, \mathcal{Y}p), d(q, \mathcal{Y}q), \frac{d(p, \mathcal{Y}q) + d(q, \mathcal{Y}p)}{2}\Big\}\Big).$$

On defining $F : \mathbb{R}^+ \to \mathbb{R}$ by $F(\beta) = \ln(\beta)$ we conclude that $\mathcal{Y}$ is an $\perp_F$-weak contraction. Therefore, using Theorem 3.4, $\mathcal{Y}$ has a unique fixed point and hence differential equation has a unique positive solution. □

Conclusion

In the main result of this paper we have introduced an orthogonal F-weak contractive condition in two settings when *(i)* f is $\perp$-continuous, and *(ii)* F is continuous, to establish some fixed point results in an O-complete metric space. The application of this theory plays a crucial role in finding the solution of an ordinary differential equation.

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