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Matrix transformation and application of Hausdorff measure of non-compactness on newly defined Fibo-Pascal sequence spaces

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Abstract. In this article, we introduce Fibo-Pascal sequence spaces $P_c^F$and P_0^F by utilizing a newly defined Fibo-Pascal matrix P^F. It is proved that $P_c^F$and P_0^F are BK-spaces that are linearly isomorphic to c and c_0, respectively. Furthermore, the Schauder basis and α-, β-, γ-duals of both the spaces are computed, and certain classes of matrix mappings are characterized. The final section is devoted to characterize compact operator on the space P_0^F via Hausdorff measure of non-compactness (shortly, HMNC).

1. Introduction

The one of the most interesting number sequence that attracted several mathematicians due to its fascinating properties is the Fibonacci number sequence. The Fibonacci numbers, whose terms are denoted by F_n, are defined by the recurrence relation $F_{n+2} = F_{n+1} + F_n$ with the initial conditions $F_0 = 0$ and $F_1 = 1$. Thus, $0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$ are the first few of Fibonacci numbers. Due to its intriguing nature, some authors developed Fibonacci (or F-) calculus or Golden calculus involving Fibonacci numbers in the literature. The readers may consult the studies [17, 22, 23] concerning the Golden calculus.

One of interesting notion in the Golden calculus is the development of fibonomial coefficients. For $0 \le k \le n$, the fibonomial coefficient (see [22]) is defined by

$$\binom{n}{k}_F = \frac{F_n!}{F_k! F_{n-k}!},$$

where $F_n!$ is the F-factorial (or Fibonomial) given as

$$F_n! = F_n F_{n-1} \ldots F_1, \quad F_0! = 1$$

with $\binom{n}{0}_F = \binom{n}{n}_F = 1$ and $\binom{n}{k}_F = 0$ for $n < k$.

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The followings are some properties sufficed by fibonomial coefficients:

$$\binom{n}{k}_F = \binom{n}{n-k}_F,$$
$$\binom{n}{k}_F \binom{k}{i}_F = \binom{n}{i}_F \binom{n-i}{k-i}_F,$$
$$(x+y)_F^n = \sum_{k=0}^{n} \binom{n}{k}_F x^k y^{n-k}. \text{ (Fibonomial Theorem)}$$

Let the space of all real-valued sequences be denoted by ω and recall that any vector subspace of ω is known as a sequence space. Some of the examples of classical seqeunce spaces can be given as ℓ_p, ℓ_∞, c, and c_0 defined as the set of all $p-$absolutely summable sequences, bounded sequences, convergent sequences, and null sequences, respectively.

Let X be a Banach space. Then, it is called as a BK-space if each map $p_k : X \to \mathbb{R}$ defined by $p_k(z) = z_k$ is continuous for all $k \in \mathbb{N}$. We recall that the spaces c and c_0 are BK-spaces due to the bounded norm $\|x\|_c = \|x\|_{c_0} = \sup_{n\in\mathbb{N}_0} |x_k|$ for $x = (x_k) \in \omega$.

Let $T = (t_{nk})$ be an infinite matrix with real entries t_{nk} for all $n, k \in \mathbb{N}_0$ and T_n be the sequence in the nth row of T for each $n \in \mathbb{N}$. Then, the sequence $Tz = ((Tz)_n) = \left\{ \sum_k t_{nk} z_k \right\}$ is said to be the $T-$transform of $z = (z_k) \in \omega$ under the assumption of the convergence of series for each $n \in \mathbb{N}$. Besides, we say that T is a matrix mapping from a sequence space Λ to a sequence space Ξ whenever Tz exists and belongs to Ξ for all $z \in \Lambda$. By (Λ, Ξ), we denote the class of all matrices T such that $T : \Lambda \to \Xi$.

In this study, $\mathbb{N} = \{0, 1, 2, ...\}$ and $\mathbb{R}$ denotes the set of all real numbers. For simplicity in notation, in the sequel, the summation without limits runs from 0 to ∞.

Recall that the following set is called the domain of the infinite matrix T in the space Λ:

$$\Lambda_T = \{x \in \omega : Tx \in \Lambda\}.$$

In recent years, creating new sequence spaces by using a special limitation method with the help of matrix domain and studying their topological structures, algebraic features and matrix transformations have been intensively studied. One may refer to these nice articles [1–3, 11, 13–16, 19] and the textbook [5] for relevant studies.

The construction of new sequence spaces by employing Pascal matrix via the matrix summability method has been considered in [4, 24]. Later on, Yaying et al. [27] introduced $q-$Pascal sequence spaces and studied their certain topological properties. Also, Schauder bases and Köthe duals as well as characterization of certain matrix classes were derived.

By B_Λ, we mean a unit sphere in a normed space Λ. We use the following notation involving a BK-space $\Lambda \supset \psi$ and $f = (f_k) \in \omega$:

$$\|f\|_\Lambda^* = \sup_{v\in B_\Lambda} \left| \sum_k f_k v_k \right|.$$

We note that $f \in \Lambda^\beta$.

Lemma 1.1. *[18, Theorem 1.29 (c)] For $\Lambda \in \{\ell_\infty, c, c_0\}$, we have $\Lambda^\beta = \ell_1$ and $\|f\|_\Lambda^* = \|f\|_{\ell_1}$.*

Further, we use the notation $B(\Lambda, \Xi)$ to denote the family of all bounded (continuous) linear operators from Λ to Ξ.

Lemma 1.2. *[18, Theorem 1.23 (a)] Assume that Λ and Ξ are any two BK-spaces. Then, corresponding to each $H \in (\Lambda, \Xi)$, there exists a linear operator $\mathcal{T}_H \in B(\Lambda, \Xi)$ with $\mathcal{T}_H u = Hu$ for all $u \in \Lambda$.*

Lemma 1.3. *[18] Assume that $\Lambda \supset \psi$ is any BK-space and $\Xi \in \{c_0, c, \ell_\infty\}$. If $H \in (\Lambda, \Xi)$, then*

$$\|\mathcal{T}_H\| = \|H\|_{(\Lambda,\Xi)} = \sup_{n\in\mathbb{N}_0} \|H_n\|^*_\Lambda < \infty.$$

Choose a bounded subset G of a metric space Λ. Then, the Hausdorff measure of noncompactness (HMNC) of G is denoted by $\chi(G)$ and is defined by

$$\chi(G) = \inf\{\delta > 0 : G \subset \cup_{j=0}^{n} B(u_j, a_j), u_j \in \Lambda, a_j < \delta, j \in \mathbb{N}_0\},$$

where $B(u_j, a_j)$ is an open ball centred at u_j and radius a_j. One may consult [18] and references therein for getting indepth idea about HMNC.

Theorem 1.4. *Let $L_k : c_0 \to c_0$ be an operator defined by*

$$L_k(u) = (u_0, u_1, u_2, ..., u_k, 0, 0, ...)$$

for all $u = (u_k) \in c_0$ and $k \in \mathbb{N}_0$. Then, for any bounded set $G \subset c_0$, we have

$$\chi(G) = \lim_{k\to\infty} \left(\sup_{u\in G} \|(I - L_k)(u)\|_{c_0} \right),$$

where I is the identity operator on c_0.

Let Λ and Ξ be any two Banach spaces. Then, a linear operator $L : \Lambda \to \Xi$ is called a compact operator if the domain of L is all of Λ and for every bounded sequence $u = (u_k) \in \Lambda$, the sequence $(L(u_k))$ has a convergent subsequence in Ξ.

It is evident from the relationship

$$\|L\|_\chi = \chi(L(B_\Lambda)) = 0$$

that a linear operator is compact iff its HMNC is zero. Thus, HMNC of a linear operator has an important role in characterizing compact operator between BK spaces. We refer to [7–10, 20, 21] for interesting papers involving compactness and the applications of HMNC between BK-spaces.

In this paper, we define the Fibo-Pascal matrix $P^F = \left(p^F_{nk}\right)$ involving Fibonomial coefficient by

$$p^F_{nk} = \begin{cases} \binom{n}{n-k}_F, & (0 \le k \le n), \\ 0, & (k > n), \end{cases}$$

and its inverse $\left[P^F\right]^{-1} = \left(\left(p^F\right)^{-1}_{nk}\right)$ by

$$\left(p^F\right)^{-1}_{nk} = \begin{cases} b_{n-k+1}\binom{n}{n-k}_F, & (0 \le k \le n), \\ 0, & (k > n), \end{cases}$$

where $b_n = -\sum\limits_{i=1}^{n-1} b_i \binom{n-1}{i-1}_F$ for $n \ge 2$ with $b_1 = 1$.

Also, we introduce Fibo-Pascal sequence spaces P^F_0 and P^F_c by utilizing Fibo-Pascal matrix P^F. It is proved that Fibo-Pascal sequence spaces P^F_0 and P^F_c are BK-spaces that are linearly isomorphic to c_0 and c, respectively. Besides, after obtaining Schauder basis and α-, β-, and γ-duals, certain matrix transformations related to the spaces P^F_0 and P^F_c are established. Moreover, the compactness of certain matrix operators are characterized helped by the concept of Hausdorff measure of non-compactness.

2. Fibo-Pascal sequence spaces

Let us introduce the Fibo-Pascal sequence spaces P_0^F and P_c^F as follows:

$$P_0^F = \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \left| \sum_{k=0}^{n} \binom{n}{n-k}_F x_k \right| = 0 \right\},$$
$$P_c^F = \left\{ x = (x_k) \in \omega : \lim_{n\to\infty} \left| \sum_{k=0}^{n} \binom{n}{n-k}_F x_k \right| < \infty \right\}.$$

That is to say that

$$P_0^F = (c_0)_{P^F} \text{ and } P_c^F = c_{P^F}. \tag{1}$$

Let us consider the sequence $y = (y_n)$ as the P^F-transform of the sequence $x = (x_k)$. Namely,

$$y_n = \left(P^F x\right)_n = \sum_{k=0}^{n} \binom{n}{n-k}_F x_k. \tag{2}$$

Equivalently,

$$x_k = \sum_{i=0}^{k} b_{k-i+1} \binom{k}{k-i}_F y_i. \tag{3}$$

We are known that the sequence spaces c_0 and c are BK-spaces due to the bounded norm and Fibo-Pascal matrix P^F is a triangle. Also, the relation (1) is valid. In the light of these facts and Wilansky [26, Theorem 4.3.2], the Fibo-Pascal sequence spaces P_0^F and P_c^F are BK-spaces normed by

$$\|x\|_{P_c^F} = \|x\|_{P_0^F} = \left\|P^F x\right\|_{\ell_\infty} = \sup_{n\in\mathbb{N}} \left|\left(P^F x\right)_n\right|.$$

Theorem 2.1. *The Fibo-Pascal sequence spaces P_0^F and P_c^F are linearly isomorphic to c_0 and c, respectively.*

Proof. To prove this, we shall establish a linear bijection $L : P_0^F \to c_0$. The linearity is clear. The injectiveness of L is clear from the realization that $z = 0$ whenever $L(z) = 0$. Consider a sequence $y = (y_n) \in c_0$. By using (2) and (3), we obtain that

$$\begin{aligned} \lim_{n\to\infty} \left(P^F x\right)_n &= \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{n-k}_F x_k \\ &= \lim_{n\to\infty} \sum_{k=0}^{n} \binom{n}{n-k}_F \sum_{i=0}^{k} b_{k-i+1} \binom{k}{k-i}_F y_i \\ &= \lim_{n\to\infty} y_n = 0. \end{aligned}$$

Thus, $x \in P_0^F$. Also,

$$\|x\|_{P_0^F} = \sup_{n\in\mathbb{N}} \left| \sum_{k=0}^{n} \binom{n}{n-k}_F \sum_{i=0}^{k} b_{k-i+1} \binom{k}{k-i}_F y_i \right| = \sup_{n\in\mathbb{N}} \left|y_n\right| = \left\|y\right\|_{c_0} < \infty,$$

which yields that L is surjective and norm-preserving. The other case of the theorem can be verified analogously. Hence, the proof is completed. □

Next, we develop Schauder basis of the spaces P_0^F and P_C^F. If a normed space $(\Lambda, \|.\|)$ contains a sequence (δ_n) such that for every $x \in \Lambda$, there exists a unique sequence of scalars (τ_n) for which

$$\left\| x - \sum_{k=0}^{n} \tau_k \delta_k \right\| \to 0, \quad \text{as } n \to \infty.$$

Then, we say that (δ_n) is a Schauder basis for the space Λ, and we write

$$x = \sum_{k=0}^{\infty} \tau_k \delta_k.$$

Combining Theorem 2.1 and the fact that the domain Λ_T of an infinite matrix T in Λ has a basis iff Λ has a basis allows us to present the following theorem.

Theorem 2.2. *Let $\psi^{(k)} = \left\{\psi_n^{(k)}\right\}_{n\in\mathbb{N}} \in P_0^F$ for each $k \in \mathbb{N}$ be defined by*

$$\psi_n^{(k)} = \begin{cases} b_{n-k+1}\binom{n}{n-k}_F, & (0 \le k \le n), \\ 0, & (k > n). \end{cases}$$

Then,

(1) The set $\left\{\psi^{(0)}, \psi^{(1)}, ...\right\}$ is a basis for the space P_0^F and any x in P_0^F is uniquely determined as $x = \sum_k t_k s^{(k)}$.

(2) For $\mu = \lim_{k\to\infty} t_k = \lim_{k\to\infty} P^F x$ and $e = (1^k)$, the set $\left\{e, \psi^{(0)}, \psi^{(1)}, ...\right\}$ is a basis for the space P_c^F and any x in P_c^F is uniquely determined as $x = \mu e + \sum_k (t_k - \mu)\, \psi^{(k)}$.

3. The α-, β-, and γ-duals

We devote this section in determining $\alpha-$dual, $\beta-$dual and $\gamma-$dual of the spaces P_0^F and P_c^F.
By $S(\Lambda, \Xi)$, we denote the multiplier space of Λ and Ξ, defined by

$$S(\Lambda, \Xi) = \{u \in \omega : zu \in \Xi \text{ for all } z \in \Lambda\}.$$

Let the sequence spaces of all convergent and bounded series be denoted by cs and bs, respectively. Then, α-dual, β-dual and γ-dual of a sequence space Λ are given by

$$\Lambda^\alpha = S(\Lambda, \ell_1), \quad \Lambda^\beta = S(\Lambda, cs) \text{ and } \Lambda^\gamma = S(\Lambda, bs), \text{ respectively.}$$

We state the following lemma, which is an effective tool in obtaining α-dual, β-dual and γ-dual of Fibo-Pascal sequence spaces P_0^F and P_c^F. Also, by F, we denote the family of all finite subsets of $\mathbb{N}$.
Before this, we list some conditions, needed in our theorems.

$$\sup_{K\in F} \sum_n \left| \sum_{k\in K} t_{nk} \right| < \infty, \tag{4}$$

$$\sup_{n\in\mathbb{N}} \sum_k |t_{nk}| < \infty, \tag{5}$$

$$\lim_{n\to\infty} t_{nk} = \varsigma_k, \quad \text{for each } k \in \mathbb{N}, \tag{6}$$

$$\lim_{n\to\infty} \sum_k t_{nk} = \varsigma. \tag{7}$$

Lemma 3.1. *([25]) Let $T = (t_{nk})$ be an infinite matrix. Then, each of the following assertions hold:*
1) $T = (t_{nk}) \in (c_0, \ell_1) = (c, \ell_1)$ iff (4) holds.
2) $T = (t_{nk}) \in (c_0, c)$ iff (5) and (6) hold.
3) $T = (t_{nk}) \in (c, c)$ iff (5), (6) and (7) hold.
4) $T = (t_{nk}) \in (c_0, \ell_\infty) = (c, \ell_\infty)$ iff (5) holds.

Theorem 3.2. *Define the set φ_1^F by*

$$\varphi_1^F = \left\{ q = (q_n) \in \omega : \sup_{K \in F} \sum_n \left| \sum_{k \in K} b_{n-k+1} \binom{n}{n-k}_F q_n \right| < \infty \right\}.$$

Then $\left(P_0^F\right)^\alpha = \left(P_c^F\right)^\alpha = \varphi_1^F$.

Proof. For any $q = (q_n) \in \omega$, one can write from (3) that

$$q_n x_n = \sum_{k=0}^{n} b_{n-k+1} \binom{n}{n-k}_F q_n y_k = \left(G^F y\right)_n$$

for all $n \in \mathbb{N}$. So, we have $qx = (q_n x_n) \in \ell_1$ whenever $x = (x_k) \in P_0^F$ or $x = (x_k) \in P_c^F$ iff $G^F y \in \ell_1$ whenever $y = (y_k) \in c_0$ or $y = (y_k) \in c$. This implies that $q = (q_n) \in \left\{P_0^F\right\}^\alpha$ or $q = (q_n) \in \left\{P_c^F\right\}^\alpha$ iff $G^F \in (c_0, \ell_1)$ or $G^F \in (c, \ell_1)$. So, by combining these facts and 1) of Lemma 3.1, we deduce that

$$q = (q_n) \in \left\{P_0^F\right\}^\alpha = \left\{P_c^F\right\}^\alpha$$

iff

$$\sup_{K \in F} \sum_n \left| \sum_{k \in K} b_{n-k+1} \binom{n}{n-k}_F q_n \right| < \infty.$$

This completes the proof. □

Theorem 3.3. *Define the sets φ_2^F, φ_3^F and φ_4^F by*

$$\varphi_2^F = \left\{ q = (q_n) \in \omega : \sup_{n \in \mathbb{N}} \sum_{k=0}^{n} \left| \sum_{i=k}^{n} b_{i-k+1} \binom{i}{i-k}_F q_i \right| < \infty \right\},$$

$$\varphi_3^F = \left\{ q = (q_n) \in \omega : \sum_{i=k}^{\infty} b_{i-k+1} \binom{i}{i-k}_F q_i \text{ exists for each } k \in \mathbb{N} \right\}$$

and

$$\varphi_4^F = \left\{ q = (q_n) \in \omega : \lim_{n \to \infty} \sum_{k=0}^{n} \sum_{i=k}^{n} b_{i-k+1} \binom{i}{i-k}_F q_i \text{ exists} \right\}.$$

Then, $\left\{P_0^F\right\}^\beta = \varphi_2^F \cap \varphi_3^F$, $\left\{P_c^F\right\}^\beta = \varphi_2^F \cap \varphi_3^F \cap \varphi_4^F$ and $\left\{P_0^F\right\}^\gamma = \left\{P_c^F\right\}^\gamma = \varphi_2^F$.

Proof. For any $q = (q_n) \in \omega$, by (3), one has

$$\begin{aligned}
\sum_{k=0}^{n} q_k x_k &= \sum_{k=0}^{n} \left(\sum_{i=0}^{k} b_{k-i+1} \binom{k}{k-i}_F y_i \right) q_k \\
&= \sum_{k=0}^{n} \left(\sum_{i=k}^{n} b_{i-k+1} \binom{i}{i-k}_F q_i \right) y_k \\
&= \left(M^F y\right)_n,
\end{aligned}$$

for all $n \in \mathbb{N}$. Here, $M^F y = \left(m^F_{nk}\right)$ is a triangle defined by

$$m^F_{nk} = \begin{cases} \sum\limits_{i=k}^{n} b_{i-k+1}\binom{i}{i-k}_F q_i, & (0 \le k \le n),\\ 0, & (k > n), \end{cases}$$

for all $n, k \in \mathbb{N}$. So, $qx = (q_n x_n) \in cs$ whenever $x = (x_k) \in P^F_0$ iff $M^F y \in c$ whenever $y = (y_k) \in c_0$, from which one concludes that $q = (q_k) \in \left\{P^F_0\right\}^\beta$ iff $M^F \in (c_0, c)$. Considering these facts and 2) of Lemma 3.1, we obtain that

$$\sup_{n\in\mathbb{N}} \sum_{k=0}^{n} \left|m^F_{nk}\right| < \infty \tag{8}$$

and

$$\lim_{n\to\infty} m^F_{nk} \text{ exists for each } k \in \mathbb{N}. \tag{9}$$

Thus, we have $\left\{P^F_0\right\}^\beta = \varphi^F_2 \cap \varphi^F_3$. By using the similar argument, one readily obtains that $q = (q_k) \in \left\{P^F_c\right\}^\beta$ iff $M^F \in (c, c)$. In this case, by applying 3) of Lemma 3.1, we obtain that (8) and (9) hold and

$$\lim_{n\to\infty} \sum_{k=0}^{n} m^F_{nk} \text{ exists,}$$

which concludes that $\left\{P^F_c\right\}^\beta = \varphi^F_2 \cap \varphi^F_3 \cap \varphi^F_4$. Finally, the assertion $\left\{P^F_0\right\}^\gamma = \left\{P^F_c\right\}^\gamma = \varphi^F_2$ can be proved in a similar way. □

4. Matrix transformation

Here, we characterize some classes of matrix transformation related to the spaces P^F_c and P^F_0. We state a theorem that characterizes matrix transformation from P^F_c or P^F_0 to any arbitrary sequence space Ξ.

Theorem 4.1. *Let $\Xi \in \omega$. Then $T = (t_{nk}) \in (P^F_0, \Xi)$ (or respectively (P^F_c, Ξ)) iff for each $n \in \mathbb{N}_0$, $G^{(n)} = (g^{(n)}_{mk}) \in (c_0, c)$ (or respectively (c, c)) and $G = (g_{nk}) \in (c_0, \Xi)$ (or respectively (c, Ξ)) where*

$$g^{(n)}_{mk} = \begin{cases} 0, & (k > m),\\ \sum\limits_{j=k}^{m} (-1)^{j-k} b_{j-k+1}\binom{j}{k}_F t_{nj}, & (0 \le k \le m), \end{cases}$$

and

$$g_{nk} = \sum_{j=k}^{\infty} (-1)^{j-k} b_{j-k+1} \binom{j}{k}_F t_{nj} \tag{10}$$

for all $k, m \in \mathbb{N}_0$.

Proof. This being analogous to the proof of Theorem 4.1 of [16], is left out. □

Each of the following conditions for each $n, k \in \mathbb{N}_0$ are necessary for the next result:

$$\lim_{m\to\infty} g^{(n)}_{mk} \text{ exists;} \tag{11}$$

$$\sup_{m\in\mathbb{N}_0} \sum_{k=0}^{\infty} \left|g^{(n)}_{mk}\right| < \infty; \tag{12}$$

$$\lim_{m\to\infty} \sum_{k=0}^{\infty} g^{(n)}_{mk} \text{ exists.} \tag{13}$$

As a consequence of Theorem 4.1 and by using Lemma 3.1, we give the following corollaries:

Corollary 4.2. *Each of the following statements hold true:*

1. $T \in (P_0^F, \ell_\infty)$ *iff (11) and (12) hold, and (5) also holds by substituting* g_{nk} *instead of* t_{nk}.
2. $T \in (P_0^F, c)$ *iff (11) and (12) hold, and (5) and (6) also hold by substituting* g_{nk} *instead of* t_{nk}.
3. $T \in (P_0^F, c_0)$ *iff (11) and (12) hold, and (5) and (6) with* $\varsigma_k = 0$ *hold by substituting* g_{nk} *instead of* t_{nk}.
4. $T \in (P_0^F, \ell_1)$ *iff (11) and (12) hold, and (4) also holds by substituting* g_{nk} *instead of* t_{nk}.

Corollary 4.3. *Each of the following statements hold true:*

1. $T \in (P_c^F, \ell_\infty)$ *iff (11), (12) and (13) hold, and (5) also holds by substituting* g_{nk} *instead of* t_{nk}.
2. $T \in (P_c^F, c)$ *iff (11), (12) and (13) hold, and (5), (6) and (7) also hold by substituting* g_{nk} *instead of* t_{nk}.
3. $T \in (P_c^F, c_0)$ *iff (11), (12) and (13) hold, and (5), (6) with* $\varsigma_k = 0$ *and (7) with* $\varsigma = 0$ *also hold by substituting* g_{nk} *instead of* t_{nk}.
4. $T \in (P_c^F, \ell_1)$ *iff (11), (12) and (13) hold, and (4) also holds by substituting* g_{nk} *instead of* t_{nk}.

Lemma 4.4. *[6] Let* $\Lambda, \Xi \subset \omega$, *T be an infinite matrix and G be a triangle. Then,* $T \in (\Lambda, \Xi_G)$ *iff* $GT \in (\Lambda, \Xi)$.

Let $T = (t_{nk})$ be an infinite matrix. Then, as a consequence of Lemma 4.4 with Corollaries 4.2 and 4.3, one obtains the following results:

Corollary 4.5. *Choose the matrix* $\Sigma = (s_{nk})$ *defined by*

$$s_{nk} = \sum_{j=0}^{n} t_{jk}$$

for all $n, k \in \mathbb{N}_0$. *Then, the necessary and sufficient conditions that* $T \in (\Lambda, \Xi)$, *where* $\Lambda \in \{P_0^F, P_c^F\}$ *and* $\Xi \in \{cs_0, cs, bs\}$ *are obtained from Corollaries 4.2 and 4.3, by replacing the elements of T by those of* Σ.

Corollary 4.6. *Choose the matrix* $C(q) = (c_{nk}^q)$ *defined by*

$$c_{nk}^q = \sum_{j=0}^{n} q^j \frac{c_j(q)c_{n-j}(q)}{c_{n+1}(q)} t_{jk}, \ (0 < q < 1)$$

for all $n, k \in \mathbb{N}_0$, *where* $(c_k(q))$ *is a sequence of q-Catalan numbers. Then, the necessary and sufficient conditions that* $T \in (\Lambda, \Xi)$, *where* $\Lambda \in \{P_0^F, P_c^F\}$ *and* $\Xi \in \{c_0(C(q)), c(C(q))\}$ *are obtained from Corollaries 4.2 and 4.3, by replacing the elements of T by those* $C(q)$, *where* $c_0(C(q))$ *and* $c(C(q))$ *are q-Catalan sequence spaces developed by Yaying et al. [28].*

Corollary 4.7. *Choose the matrix* $\mathcal{F}(r,s) = (f_{nk}(r,s))$ *defined by*

$$f_{nk}(r,s) = \sum_{j=0}^{n} \frac{1}{(r+s)_F^n} \binom{n}{j}_F r^j s^{n-j} t_{jk}$$

for all $n, k \in \mathbb{N}_0$. *Then, the necessary and sufficient conditions that* $T \in (\Lambda, \Xi)$, *where* $\Lambda \in \{P_0^F, P_c^F\}$ *and* $\Xi \in \{b_0^{r,s,F}, b_c^{r,s,F}\}$ *are obtained from Corollaries 4.2 and 4.3, by replacing the elements of T by those of* $\mathcal{F}(r,s)$, *where* $b_0^{r,s,F}$ *and* $b_c^{r,s,F}$ *are Fibonomial sequence spaces developed by Dağlı and Yaying [12].*

5. Compactness on P_0^F

Consider a sequence $g = (g_k)$ defined via the sequence $f = (f_k)$ by

$$g_k = \sum_{j=k}^{\infty} (-1)^{j-k} b_{j-k+1} \binom{j}{k}_F f_j$$

for all $k \in \mathbb{N}_0$.

Lemma 5.1. *If $f = (f_k) \in \left(P_0^F\right)^\beta$, then $g = (g_k) \in \ell_1$ and*

$$\sum_k f_k x_k = \sum_k g_k y_k \tag{14}$$

for all $x = (x_k) \in P_0^F$.

Lemma 5.2. $\|f\|^*_{P_0^F} = \sum_k |g_k| < \infty$ *for all $f = (f_k) \in \left(P_0^F\right)^\beta$.*

Proof. Consider $f = (f_k) \in \left[P_0^F\right]^\beta$. Then, $g = (g_k) \in \ell_1$ by Lemma 5.1, and the equality (14) holds. Further $\|x\|_{P_0^F} = \|y\|_{c_0}$ holds true which implies that $x \in B_{P_0^F}$ iff $y \in B_{c_0}$. Thus, we obtain that $\|f\|^*_{P_0^F} = \sup_{x \in B_{P_0^F}} |\sum_k f_k x_k| = \sup_{y \in B_{c_0}} |\sum_k g_k y_k| = \|g\|^*_{c_0}$. Consequently, by using Lemma 1.1, we get that $\|f\|^*_{P_0^F} = \|g\|^*_{c_0} = \|g\|_{\ell_1} = \sum_k |g_k|$. □

Let us define a matrix $\tilde{T} = (\tilde{t}_{nk})$ via an infinite matrix $T = (t_{nk})$ by

$$\tilde{t}_{nk} = \sum_{j=k}^{\infty} (-1)^{j-k} b_{j-k+1} \binom{j}{k}_F t_{nj}$$

for all $n, k \in \mathbb{N}_0$, where we assume that the infinite sum converges.

Lemma 5.3. *Let $\Lambda \subset \omega$ and $T = (t_{nk})$ be an infinite matrix. If $T \in (P_0^F, \Lambda)$, then $\tilde{T} \in (c_0, \Lambda)$ and $Tx = \tilde{T}y$ for all $x \in P_0^F$.*

Proof. This is obtained easily from Lemma 5.1. □

Lemma 5.4. *The expression*

$$\|\mathcal{T}_T\| = \|T\|_{(P_0^F, \Xi)} = \sup_{n \in \mathbb{N}_0} \left(\sum_k |\tilde{t}_{nk}| \right) < \infty$$

holds true for any $T \in (P_0^F, \Xi)$ and $\Xi \in \{c_0, c, \ell_\infty\}$.

Lemma 5.5. *[20, Theorem 3.7] Assume that $\Lambda \supset \psi$ is any BK-space. Then, each of the following expressions hold true:*

(1) If $T \in (\Lambda, \ell_\infty)$, then $0 \le \|\mathcal{T}_T\|_\chi \le \limsup_n \|T_n\|^*_\Lambda$.
(2) $T \in (\Lambda, c_0)$, then $\|\mathcal{T}_T\|_\chi = \limsup_n \|T_n\|^*_\Lambda$.
(3) If Λ has AK or $\Lambda = \ell_\infty$ and $T \in (\Lambda, c)$, then

$$\frac{1}{2} \limsup_n \|T_n - t\|^*_\Lambda \le \|\mathcal{T}_T\|_\chi \le \limsup_n \|T_n - t\|^*_\Lambda,$$

where $t = (t_k)$ and $t_k = \lim_n t_{nk}$ for each $k \in \mathbb{N}_0$.

Lemma 5.6. *[20, Theorem 3.11] Assume that $\Lambda \supset \psi$ is any BK-space. If $T \in (\Lambda, \ell_1)$, then*

$$\lim_r \left(\sup_{N \in F_r} \left\| \sum_{n \in N} T_n \right\|_{\Lambda}^{*} \right) \leq \|\mathcal{T}_T\|_{\chi} \leq 4 \lim_r \left(\sup_{N \in F_r} \left\| \sum_{n \in N} T_n \right\|_{\Lambda}^{*} \right).$$

In addition, $\mathcal{T}_T$ is compact iff $\lim_r \left(\sup_{N \in F_r} \left\| \sum_{n \in N} T_n \right\|_{\Lambda}^{*} \right) = 0$, where F_r is a sub-family of F consisting of subsets of $\mathbb{N}_0$ with elements that are greater than r.

Theorem 5.7.

1. *If $T \in (P_0^F, \ell_\infty)$, then $0 \leq \|\mathcal{T}_T\|_{\chi} \leq \limsup_n \sum_k |\tilde{t}_{nk}|$ holds.*
2. *If $T \in (P_0^F, c)$, then*

$$\frac{1}{2} \limsup_n \sum_k |\tilde{t}_{nk} - \tilde{t}_k| \leq \|\mathcal{T}_T\|_{\chi} \leq \limsup_n \sum_k |\tilde{t}_{nk} - \tilde{t}_k|$$

   holds.
3. *If $T \in (P_0^F, c_0)$, then $\|\mathcal{T}_T\|_{\chi} = \limsup_n \sum_k |\tilde{t}_{nk}|$ holds.*
4. *If $T \in (P_0^F, \ell_1)$, then $\lim_r \|T\|_{(P_0^F, \ell_1)}^{(r)} \leq \|\mathcal{T}_T\|_{\chi} \leq 4 \lim_r \|T\|_{(P_0^F, \ell_1)}^{(r)}$ holds, where*

$$\|T\|_{(P_0^F, \ell_1)}^{(r)} = \sup_{N \in F_r} \left(\sum_k \Big| \sum_{n \in N} \tilde{t}_{nk} \Big| \right) \ (r \in \mathbb{N}_0).$$

Proof. (1) Let $T \in (P_0^F, \ell_\infty)$. Clearly, the infinite sum $\sum_{k=0}^{\infty} t_{nk} x_k$ converges for each $n \in \mathbb{N}_0$ which implies that $T_n \in \left(P_0^F\right)^{\beta}$. As a result of Lemma 5.2, it follows that

$$\|T_n\|_{P_0^F}^{*} = \|\tilde{T}_n\|_{c_0}^{*} = \|\tilde{T}_n\|_{\ell_1} = \left(\sum_k |\tilde{t}_{nk}| \right)$$

for each $n \in \mathbb{N}_0$. Thus by utilizing Lemma 5.5 (a), we conclude that

$$0 \leq \|\mathcal{T}_T\|_{\chi} \leq \limsup_n \left(\sum_k |\tilde{t}_{nk}| \right).$$

(2) Let $T \in (P_0^F, c)$. One obtains from Lemma 5.3 that $\tilde{T} \in (c_0, c)$. Thus, it follows from Lemma 5.5 (c) that

$$\frac{1}{2} \limsup_n \|\tilde{T}_n - \tilde{t}\|_{c_0}^{*} \leq \|\mathcal{T}_T\|_{\chi} \leq \limsup_n \|\tilde{T}_n - \tilde{t}\|_{c_0}^{*},$$

with $\tilde{t} = (\tilde{t}_k)$ and $\tilde{t}_k = \lim_n \tilde{t}_{nk}$ for each $k \in \mathbb{N}_0$. Consequently, by using Lemma 1.1, one obtains that

$$\|\tilde{T}_n - \tilde{t}\|_{c_0}^{*} = \|\tilde{T}_n - \tilde{t}\|_{\ell_1} = \left(\sum_k |\tilde{t}_{nk} - \tilde{t}_k| \right)$$

for each $n \in \mathbb{N}_0$.

(3) Let $T \in (P_0^F, c_0)$. Since $\|T_n\|^*_{P_0^F} = \|\tilde{T}_n\|^*_{c_0} = \|\tilde{T}_n\|_{\ell_1} = \left(\sum_k |\tilde{t}_{nk}|\right)$ for each $n \in \mathbb{N}_0$, we conclude from Lemma 5.5 (b) that

$$\|\mathcal{T}_T\|_\chi = \limsup_n \left(\sum_k |\tilde{t}_{nk}|\right).$$

(4) Let $T \in (P_0^F, \ell_1)$. Then, $\tilde{T} \in (c_0, \ell_1)$ from Lemma 5.3. Again by using Lemma 5.6, we get that

$$\lim_r \left(\sup_{N \in F_r} \left\|\sum_{n \in N} \tilde{T}_n\right\|^*_{c_0}\right) \leq \|\mathcal{T}_T\|_\chi \leq 4 \lim_r \left(\sup_{N \in F_r} \left\|\sum_{n \in N} \tilde{T}_n\right\|^*_{c_0}\right).$$

Moreover, it follows from Lemma 1.1 that $\|\sum_{n \in N} \tilde{T}_n\|^*_{c_0} = \|\sum_{n \in N} \tilde{T}_n\|_{\ell_1} = \left(\sum_k |\sum_{n \in N} \tilde{t}_{nk}|\right)$. So, the proofs of the assertions in the theorem are complete. □

The following Corollary is an immediate consequence of the theorem above.

Corollary 5.8.

1 *For $T \in (P_0^F, \ell_\infty)$, $\mathcal{T}_T$ is compact if* $\lim_n \sum_k |\tilde{t}_{nk}| = 0$.

2 *For $T \in (P_0^F, c)$, $\mathcal{T}_T$ is compact iff* $\lim_n \sum_k |\tilde{t}_{nk} - \tilde{t}_k| = 0$.

3 *For $T \in (P_0^F, c_0)$, $\mathcal{T}_T$ is compact iff* $\lim_n \sum_k |\tilde{t}_{nk}| = 0$.

4 *For $T \in (P_0^F, \ell_1)$, $\mathcal{T}_T$ is compact iff* $\lim_r \|T\|^{(r)}_{(P_0^F, \ell_1)} = 0$, *where* $\|T\|^{(r)}_{(P_0^F, \ell_1)} = \sup_{N \in F_r} \left(\sum_k |\sum_{n \in N} \tilde{t}_{nk}|\right)$.

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