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Statistical order limit points in Riesz spaces

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Abstract. The concept of statistical order convergence of sequences in Riesz spaces was introduced and studied. In the present paper, we define the statistical order limit points of a sequence (x_n) as a vector x that is the order limit of a subsequence $(x_k)_{k\in K}$ of (x_n) such that the set K does not have density zero. Moreover, we introduce the statistical order cluster points of sequences in Riesz spaces, and also, we give some relations between them.

1. Introduction

Statistical convergence first emerged in the works of Fast in [11] and Steinhaus in [21], where it was proposed as a generalization of real number convergence. Building upon this idea, Fridy [12] later introduced the notion of statistical limits and cluster points for real sequences, and investigated their properties concerning closed sets. Subsequently, these results were extended to other spaces, such as topological spaces, probabilistic normed spaces, and metric lattices, as seen in [8, 14, 18].

In the context of Riesz spaces, Ercan [10] pioneered the study of statistical convergence in Riesz spaces. Following this, Şençimen and Pehlivan [20] introduced the concept of statistical order convergence. Further research on various types of statistical convergence in Riesz spaces was conducted, as evidenced by Aydın [4–7]. The objective of this paper is to introduce the concepts of statistical order limits and statistical order cluster points for sequences in Riesz spaces. Analogous to Fridy's earlier findings, we establish fundamental results for these concepts. It is worth noting that the notion of statistical order limits presented in this paper is more general than the one in [20].

2. Preliminaries

An ordered vector space denoted by E possesses a real-valued vector space with an order relation "$\leq$" that adheres to the properties of antisymmetry, reflexivity, and transitivity. It is defined as an *ordered vector space* if, for any elements x and y in E the following conditions hold:

(i) $x \leq y$ implies $x + z \leq y + z$ for all z in E,

(ii) $x \leq y$ implies $\lambda x \leq \lambda y$ for every λ with $0 \leq \lambda \in \mathbb{R}$.

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An ordered vector space E is categorized as a *Riesz space* or *vector lattice* when, for any two vectors x and y in E, the infimum and supremum, denoted as $x \wedge y$ and $x \vee y$, respectively, exist in E. To express certain aspects of the vector lattice E, we introduce the *positive part*, *negative part*, and *module* of an element x as $x^+ := x \vee 0$, $x^- := (-x) \vee 0$, and $|x| := x \vee (-x)$, respectively. In this paper, the vertical bar $|\cdot|$ is used to represent the module of elements in the vector lattices.

In a Riesz space E, a sequence (x_n) is considered increasing if the elements satisfy the condition $x_1 \leq x_2 \leq \cdots$, and it is termed decreasing if the elements satisfy $x_1 \geq x_2 \geq \cdots$. We represent the increasing sequence as $x_n \uparrow$ and the decreasing sequence as $x_n \downarrow$. Furthermore, if an increasing sequence $x_n \uparrow$ has a supremum (i.e., a least upper bound) denoted as x, we write it as $x_n \uparrow x$. Similarly, if a decreasing sequence $x_n \downarrow$ has an infimum (i.e., a greatest lower bound) denoted as x, we write it as $x_n \downarrow x$. Such sequences that are either increasing or decreasing are referred to as monotonic. One of the fundamental and crucial concepts in the study of Riesz spaces is order convergence, as stated in Theorem 16.2 [17].

Definition 2.1. *A sequence (x_n) in a Riesz space E is said to be* order convergent *to $x \in E$ (denoted as $x_n \xrightarrow{o} x$) when there exists a sequence $q_n \downarrow 0$ in E such that $|x_n - x| \leq q_n$ for all $n \in \mathbb{N}$.*

Moving on to statistical convergence, we define the *natural density* of a subset K of positive integers as

$$\delta(K) := \lim_{n\to\infty} \frac{1}{n}\Big|\{k \in K : k \leq n\}\Big|,$$

where $|\{k \in K : k \leq n\}|$ represents the number of elements in K that do not exceed n. When considering a vector lattice E and a sequence (x_n) in E, the sequence (x_n) is labeled:

- *statistical monotone convergent* to $x \in E$ (denoted as $x_n \downarrow^{st} x$) if there exists a subset J in $\mathbb{N}$ with $\delta(J) = 1$ and $x_n \downarrow x$ on J,
- *statistical order convergent* to $x \in E$ (denoted as $x_n \xrightarrow{st_o} x$) if there are a sequence $q_n \downarrow^{st} 0$ and a subset J of $\mathbb{N}$ with $\delta(J) = 1$ such that $|x_n - x| \leq q_n$ for all $n \in J$.

We recall the notions of *statistical limit point* and *statistical cluster point* of a real sequence (x_n).

Definition 2.2. *Let (x_n) be a real sequence.*

(1) A real number x is considered a statistical limit point *of (x_n) if there exists a subsequence $(x_k)_{k\in K}$ of (x_n) with $\delta(K) > 0$ and $x_k \to x$.*

(2) A real number x is identified as a statistical cluster point *of (x_n) if, for every $\varepsilon > 0$, $\delta(\{k \in \mathbb{N} : |x_k - x| < \varepsilon\}) \neq 0$.*

Definition 2.3. *A sequence (x_n) satisfying the property that there exists a positive element $u \in E_+$ such that $\delta(\{n \in \mathbb{N} : |x_n| \nleq u\}) = 0$ is calle* statistical order bounded sequence.

It is obvious that every order bounded sequence is statistical order bounded, and every statistical order convergent sequence is statistical order bounded. Now, we present two theorems and a corollary related to statistical order convergence of sequences in σ-Dedekind complete Riesz spaces (i.e. in which every countable subset that is bounded above has a supremum and every countable subset that is bounded below has a infimum), along with their respective proofs.

Theorem 2.4. *Let (x_n) be a statistical order bounded sequence in a σ-Dedekind complete Riesz space E for which the density of $\{n \in \mathbb{N} : x_n \leq x_{n+1}\}$ is one. Then, (x_n) is a statistical order convergent sequence.*

Proof. Consider the set $J := \{j_n : n \in \mathbb{N}\}$. It follows from the statistical order boundedness of (x_n) that there exists a positive element $u \in E_+$ such that $\delta(\{n \in \mathbb{N} : |x_n| \nleq u\}) = 0$. Thus, we have $\delta(\{j_n \in J : |x_{j_n}| \nleq u\}) = 0$. Take $M = \{j_n \in J : |x_{j_n}| \leq u\}$. Then we have $\delta(M) = 1$ and (x_n) is increasing on M since $\delta(\{n \in \mathbb{N} : x_n \leq x_{n+1}\}) = 1$. Therefore, since E is σ-Dedekind complete and (x_n) is order bounded on M, $\sup\limits_{m_n\in M}(x_{m_n})$ exists. Assume that $\sup\limits_{m_n\in M}(x_{m_n}) = w \in E$. Thus, we have $x_{m_n} \xrightarrow{o} w$, and so, we obtain $x_{m_n} \xrightarrow{st_o} w$. Therefore, (x_n) is statistical order convergent to w. $\square$

Theorem 2.5. *Let (x_n) be a statistical order bounded sequence in a σ-Dedekind complete Riesz space E with $\delta(\{n \in \mathbb{N} : x_{n+1} \leq x_n\}) = 1$. Then, (x_n) is a statistical order convergent sequence.*

Proof. The proof can be obtained using similar arguments as in the proof of Theorem 2.4. □

Corollary 2.6. *Every monotone and statistical order bounded sequence is statistical order convergent.*

This corollary follows from Theorem 2.5 by observing that a sequence with $\delta(\{n \in \mathbb{N} : x_{n+1} \leq x_n\}) = 1$ is a monotone non-increasing sequence, and hence, every monotone and statistical order bounded sequence is statistical order convergent.

3. Statistical order limit and cluster points

We begin the section with the following basic notion of order convergence.

Definition 3.1. *An element x is referred to as an* order limit point *of a sequence (x_n) in a Riesz space if there exists a subsequence $(x_k)_{k\in K}$ of (x_n) such that (x_k) is order-convergent to x.*

Remind that in a Riesz space E, a positive vector e is referred to as an *atom* when the conditions $x \wedge y = 0$ and $x, y \in [0, e]$ imply that either $x = 0$ or $y = 0$. Also, a Banach lattice E is classified as a Kantorovich–Banach space, denoted as a *KB*-space, if every increasing norm bounded sequence from E_+ converges in norm.

Remark 3.2.

(i) *Every sequence that is order-convergent has at least one order limit point.*
(ii) *Every sequence that is norm-convergent in a Banach lattice has an order limit point (see for example Theorem.VII.2.1 [23]).*
(iii) *Every order-bounded sequence in atomic KB-spaces has an order limit point.*

Not all sequences are required to possess a limit point. To illustrate this, let us examine Exercise 105.8 [24].

Example 3.3. *Consider Lebesgue measure on Riesz space $E := L_1 \cup L_\infty$ with the interval $[0, 1]$. For any natural number $n \in \mathbb{N}$, define the function u_n as n on the interval $[0, n^{-2}]$ and zero elsewhere. Thus, there does not exist any subsequence $(u_{j_k})_{k=1}^{\infty}$ that converges order to zero. This is due to the fact that $0 \leq u_{j_k} \leq p_k \downarrow 0$ implies $p_k \geq u_{j_m}$ for all $m \geq k$. So, p_k would be an unbounded function, which is impossible. Thus, (u_n) does not have any order limit point.*

Definition 3.4. *Let E be a Riesz space, and (x_n) be a sequence in E. We define the following notions:*

(i) Statistical order limit point *of (x_n): An element $x \in E$ is termed a statistical order limit point of (x_n) if there exists a subsequence $(x_k)_{k\in K}$ of (x_n) with the property that (x_k) is order-convergent to x (i.e., $x_k \xrightarrow{o} x$), and the set K has positive lower density $\delta(K) = w > 0$.*
(ii) Statistical order cluster point *of (x_n): An element $x \in E$ is called a statistical order cluster point of (x_n) if there exists another sequence q_n decreasing to zero, such that the set $\{n \in \mathbb{N} : |x_n - x| \leq q_n\}$ has positive lower density.*

In simpler terms, a statistical order limit point of (x_n) is an element x to which a certain subsequence of (x_n) converges in order, and the subsequence is *sufficiently dense* in the sense of positive lower density. On the other hand, a statistical order cluster point of (x_n) is an element x around which the terms of the sequence (x_n) cluster in order, with the clustering controlled by the decreasing sequence q_n of positive numbers. For a sequence $x = (x_n)$, we denote the sets of all order limit points, statistical order limit points, and statistical order cluster points of x as $OL(x)$, $SOL(x)$, and $SOC(x)$, respectively.

Remark 3.5. *Consider E as the Riesz space $\mathbb{R}$ equipped with the order relation "$\leq$". In this case:*

(i) *Every order limit point of a sequence is also a limit point of the sequence.*
(ii) *Every statistical order limit point of a sequence is also a statistical limit point of the sequence.*

(iii) *Every statistical order cluster point is a statistical cluster point of the sequence.*

In other words, when we consider the Riesz space $\mathbb{R}$ with the order relation "$\leq$", the notions of order limit points, statistical order limit points, and statistical order cluster points coincide with their respective limit point counterparts.

It is obvious that $SOL(x) \subseteq OL(x)$. However, the reverse inclusion is not valid. This can be illustrated with the following instance.

Example 3.6. *Consider the Riesz space $E := \mathbb{R}^2$, which is equipped with the coordinatewise ordering. Let (x_k) be a sequence in E defined as follows:*

$$x_k = \begin{cases} (0, 1 + \dfrac{1}{k}), & k = n^2 \\ (0,0), & \text{otherwise} \end{cases}$$

where k, n belong to the set of natural numbers, $\mathbb{N}$. Now, we will demonstrate that $(0,1) \in OL(x)$. To do this, we consider the indexes $k_n = n^2$, and we form the subsequence $x_{k_n} = (0, 1 + \frac{1}{k_n^2})$ of (x_k). As a result, we observe that $|x_{k_n} - (0,1)| = |(0, \frac{1}{k_n^2})| = (0, \frac{1}{k_n^2}) \downarrow (0,0)$. Therefore, we conclude that $(0,1) \in OL(x)$. On the other hand, it is essential to note that $(0,1) \notin SOL(x)$ due to the reason that $\delta(K) = 0$.

In the next study, we explore the relationship between $SOL(x)$ and $SOC(x)$.

Theorem 3.7. *$SOL(x) \subseteq SOC(x)$ holds for any sequence x in Riesz spaces.*

Proof. Let's assume that $x = (x_k)$ is a sequence in a Riesz space E, and e belongs to $SOL(x)$. Thus, there exists a subsequence $(x_{k_n})_{k_n \in K}$ of x, where $x_{k_n} \xrightarrow{o} e$ and $\delta(K) > 0$. From the fact that $x_{k_n} \xrightarrow{o} e$, we can find a sequence $y_{k_n} \downarrow 0$ in E such that for every $k_n \in K$, the inequality $|x_{k_n} - e| \leq y_{k_n}$ holds. Additionally, the given condition $\delta(K) > 0$ implies that

$$\lim_{n\to\infty} \frac{1}{n} |\{k_n \in K : \ k_n \leq n\} = w > 0.$$

Thus, we observe the following inclusion

$$\{k_n \in K : \ k_n \leq n\} \subseteq \{k \in \mathbb{N} : \ |x_k - e| < y_k\}.$$

Therefore, we get

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k_n \in K : \ k_n \leq n\}\right| < \lim_{n\to\infty} \frac{1}{n} \left|\{k \in \mathbb{N} : \ |x_k - e| < y_k\}\right|.$$

Hence, it follows that $\delta(\{k \in \mathbb{N} : \ |x_k - e| < y_k\}) > 0$, which ultimately implies that e belongs to $SOC(x)$. This concludes the proof. □

It should be noted that the converse statement of Theorem 3.7 may not necessarily be true in all cases. To demonstrate this, an example presented in Example 3 [12] can be considered.

Theorem 3.8. *Let $x = (x_n)$ and $y = (y_n)$ be two sequences in a Riesz space E. If the density of $\delta(\{n \in \mathbb{N} : x_n \neq y_n\})$ equals zero, then $SOL(x) = SOL(y)$ and $SOC(x) = SOC(y)$.*

Proof. Let $e \in SOL(x)$. This implies that there exists a subsequence $(x_{k_n})_{k_n \in K}$ of sequence x such that $x_{k_n} \xrightarrow{o} e$ and the density $\delta(K) > 0$. Since $x_{k_n} \xrightarrow{o} e$, we can find another sequence $a_{k_n} \downarrow 0$ in E such that $|x_{k_n} - e| < a_{k_n}$ holds for every $k_n \in K$. It is important to note that the set $\{k_n \in K : x_{k_n} \neq y_{k_n}\}$ is a subset of $\{k \in \mathbb{N} : x_k \neq y_k\}$. Thus, we have

$$\delta(\{k \in \mathbb{N} : x_k \neq y_k\}) \geq \delta(\{k_n \in K : x_{k_n} \neq y_{k_n}\}),$$

which implies that $\delta(\{k_n \in K : x_{k_n} \neq y_{k_n}\}) = 0$. Given that $\delta(K) > 0$, it follows that $\delta(M) > 0$, where $M := \{k_n \in K : x_{k_n} = y_{k_n}\}) > 0$. Consequently, for the sequence y, we can select the subsequence $(y_{k_n})_{k_n \in M}$ such that $y_{k_n} \xrightarrow{o} e$ and the measure $\delta(M) > 0$. This implies $e \in SOL(y)$. By employing similar arguments, we can demonstrate that $SOL(y) \subseteq SOL(x)$.

Next, let's consider an element $e \in SOC(x)$. This means there exists a sequence $q_n \downarrow 0$ such that the measure $\delta(\{k \in \mathbb{N} : |x_k - e| < q_k\}) > 0$. Given that $\delta(\{k_n \in \mathbb{N} : x_{k_n} = y_{k_n}\}) = 0$, we can deduce that $\delta(\{k_n \in \mathbb{N} : |y_{k_n} - e| < q_{k_n}\}) > 0$. This implies that $e \in SOC(y)$. Similarly, it can be shown that $SOC(y) \subseteq SOC(x)$. □

Theorem 3.9. *Let $x = (x_n)$ be a sequence in a Riesz space E. Then, $SOL(x)$ is closed under the statistical order convergence.*

Proof. Let (p_n) be a sequence in $SOL(x)$, and let (p_n) statistically order converge to $p \in E$. We aim to show that p also belongs to $SOL(x)$. Since $p_n \xrightarrow{st_o} p \in E$, there is a sequence $q_n \downarrow^{st} 0$ with a subset $\delta(J) = 1$ of $\mathbb{N}$ such that $|p_j - p| \leq q_j$ for all $j \in J$. Now, consider an arbitrary index $j \in J$. There exists a subsequence $(x_{k_n})_{k_n \in K}$ of (x_n) such that $x_{k_n} \xrightarrow{o} p_j$, and $\delta(K) > 0$ due to the fact that $p_j \in SOL(x)$. Therefore, there exists another sequence $y_n \downarrow 0$ in E such that $|x_{k_n} - p_j| \leq y_{k_n}$ holds for all $k_n \in K$. We can then deduce that:

$$|x_{k_n} - p| \leq |x_{k_n} - p_j| + |p_j - p| \leq y_{k_n} + q_j$$

for each $k_n \in K$. Therefore, we obtain that $|x_m - p| \leq y_m + q_m$ for all $m \in M$, where M is the set $K \cap J$. Since $\delta(M) = 1$ and $(y_m + q_m)_{m \in M} \downarrow 0$, we get $x_m \xrightarrow{o} p$, i.e., we have $p \in SOL(x)$. □

Theorem 3.10. *The set $SOC(x)$ is closed under the statistical order convergence for any sequence $x = (x_n)$ in a Riesz space.*

Proof. Suppose that (w_n) is a sequence in $SOC(x)$ and $w_n \xrightarrow{st_o} w$ for a sequence $x = (x_n)$ in a Riesz space E. Then, we have a sequence $q_n \downarrow^{st} 0$ with a subset J of $\mathbb{N}$ such that $\delta(J) = 1$ and $|w_j - w| \leq q_j$ for all $j \in J$. For any fixed index $j \in J$, there exists a sequence $t_n \downarrow 0$ in E such that $\delta\Big(\{k \in \mathbb{N} : |x_k - w_j| \leq t_k\}\Big) > 0$ because of $w_j \in SOC(x)$. It follows from the inequality

$$|x_k - w| \leq |x_k - w_j| + |w_j - w| \leq t_k + q_j$$

that we have the following inclusion $\{k \in \mathbb{N} : |x_k - w_j| \leq t_k\} \cap \{j \in J : |w_j - w| \leq q_j\} \subseteq \{k \in \mathbb{N} : |x_k - w| \leq t_k + q_k\}$. Therefore, we obtain that $\delta\Big(\{k \in \mathbb{N} : |x_k - w| \leq t_k + q_k\}\Big) > 0$. Since $(t_k + q_j)_{(k,j) \in K \times J} \downarrow 0$, we have $w \in SOC(x)$. □

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