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Some remarks for subclasses of bi-univalent functions defined by Ruscheweyh derivative operator

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Abstract. This paper presents two subclasses of analytic and bi-univalent functions associated with the Ruscheweyh derivative operator to investigate the bounds for $|a_2|$ and $|a_3|$, where a_2 and a_3 are the initial Tayler-Maclaurin coefficients. The current results would generalize and improve some corresponding recent works. Additionally, in certain cases, our estimates correct some of the existing coefficient bounds.

1. Introduction

Let $\mathcal{A}$ represent the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\},$$

together with a normalization given by

$$f(0) = f'(0) - 1 = 0.$$

The Hadamard product (or convolution) $f(z) * l(z)$ of $f(z)$ and $l(z)$ is defined by

$$(f * l)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (l * f)(z) \quad (z \in \mathcal{U}),$$

where the function $l(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is also analytic in $\mathcal{U}$.

For a function $f \in \mathcal{A}$ defined by (1), the Ruscheweyh derivative operator $\mathcal{R}^{\ell} : \mathcal{A} \to \mathcal{A}$ (see [17]) is defined as follows:

$$\mathcal{R}^{\ell} f(z) = \frac{z\left(z^{\ell-1} f(z)\right)^{(\ell)}}{\ell!} = \frac{z}{(1-z)^{\ell+1}} * f(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\ell+n)}{\Gamma(n)\Gamma(\ell+1)} a_n z^n,$$
$$\left(\ell \in \mathbb{N}_0 = \{0, 1, 2, \ldots\} = \mathbb{N} \cup \{0\},\ z \in \mathcal{U}\right).$$

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A function f is said to be univalent in $\mathcal{U}$ if it is one-to-one (injective) in $\mathcal{U}$. We denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions that are univalent in $\mathcal{U}$. One of the most important examples of a function in $\mathcal{S}$ is the Koebe function

$$k(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n \quad (z \in \mathcal{U}).$$

This function maps the unit disc $\mathcal{U}$ conformally onto the complex plane except for a slit along the negative real axis from $-\infty$ to $-1/4$, and plays an extremal role in many problems in the theory of univalent functions.

The class $\mathcal{S}^*(\gamma)$ of starlike functions of order $\gamma\,(0 \le \gamma < 1)$ in U and the class $\mathcal{K}(\gamma)$ of convex functions of order $\gamma\,(0 \le \gamma < 1)$ in U are two of the most important and well-investigated subclasses of the analytic and univalent function class $\mathcal{S}$.

According to the Koebe 1/4-Theorem (see [8]) the image of the open unit disk $\mathcal{U}$ under any univalent function contains a disk of radius 4 . As a consequence, every function $f \in \mathcal{S}$ has an inverse f^{-1} such that

$$f^{-1}(f(z)) = z \quad (z \in \mathcal{U})$$

and

$$f\left(f^{-1}(w)\right) = w\left(|w| < r_0(f), r_0(f) \ge \frac{1}{4}\right).$$

The inverse function $g = f^{-1}$ has the form

$$g(w) = f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots . \tag{2}$$

If f and f^{-1} both are univalent in $\mathcal{U}$, then $f \in \mathcal{A}$ is said to be bi-univalent function. The family of all bi-univalent functions in $\mathcal{U}$ given by (1) is denoted by Σ.

Lewin [11] constructed a study on the class Σ of bi-univalent functions and discovered that $|a_2| < 1.51$ for the functions belonging to the class Σ. Later, Brannan and Clunie [4] proposed the conjecture that $|a_2| \le \sqrt{2}$. Subsequently, Netanyahu [15] demonstrated that max $|a_2| = \frac{4}{3}$ for $f \in \Sigma$. To explore various interesting examples of functions in the class Σ, refer to the pioneering work on this subject by Srivastava et al. [19], which actually revitalized the study of analytic and bi-univalent functions in recent years.

Based on the findings of Srivastava et al. [19], we would like to mention the following examples of functions in the class Σ :

$$\frac{z}{1-z} = \sum_{n=1}^{\infty} z^n, \quad -\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \quad \text{and} \quad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right) = \sum_{n=0}^{\infty} (-1)^{n-1}\frac{z^{2n+1}}{2n+1}.$$

It is evident that the class Σ is not empty. However, the Koebe function does not belong to the class Σ.

Brannan and Taha [5] introduced specific subclasses within the bi-univalent class Σ, analogous to the well-known subclasses $\mathcal{S}^*(\gamma)$ and $\mathcal{K}(\gamma)$ of starlike and convex functions of order $\gamma\,(0 \le \gamma < 1)$, respectively. Thus, for $0 \le \gamma < 1$, a function $f \in \Sigma$ falls into the class $\mathcal{S}^*_\Sigma(\gamma)$ of bi-starlike functions of order γ if both f and f^{-1} are starlike functions of order γ, or into the class $\mathcal{K}_\Sigma(\gamma)$ of bi-convex functions of order γ if both f and f^{-1} are convex functions of order γ. Moreover, A function $f \in \mathcal{A}$ is classified as a strongly bi-starlike functions of order $\gamma\,(0 < \gamma \le 1)$, denoted by $\mathcal{S}^*_\Sigma[\gamma]$ (see [5, 30]), if it satisfies each of the following conditions:

$$\left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\gamma\pi}{2} \quad \text{and} \quad \left|\arg\left(\frac{wg'(w)}{g(w)}\right)\right| < \frac{\gamma\pi}{2},$$

where, g is the univalent extension of f^{-1} to $\mathcal{U}$.

There have been numerous recent works dedicated to studying the class Σ of bi-univalent functions and obtaining non-sharp bounds on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Notably, the pioneering work by Srivastava et al. [19] has significantly advanced the study of various subclasses within the bi-univalent function class Σ and discovered bounds on $|a_2|$ and $|a_3|$. Remarkably, a substantial number of subsequent works have been published in the literature, building upon the groundbreaking research by Srivastava et al. [19] and focusing on coefficient problems for different subclasses of the analytic and bi-univalent function class Σ (see, for example, [1, 3, 7, 13, 18, 20–23, 27–29, 33]). However, the general coefficient estimate bounds on $|a_n|$ $(n \in \{4, 5, 6, \ldots\})$ for a function $f \in \Sigma$ defined by (1) remain an unsolved problem. In fact, for coefficients greater than three, there is no natural method to obtain an upper bound. A few articles have utilized Faber polynomial techniques to derive upper bounds for higherorder coefficients (see, for example [6, 24–26]).

The determination of estimates for the Tayler-Maclaurin coefficients a_n is an important concern problem in geometric function theory as it provides information about the geometric properties of these functions. For instance, the bounds for the second and third coefficients a_2 and a_3 of functions $f \in \Sigma$ yield growth and distortion bounds, as well as covering theorems. Motivated by the aforementioned works and making use of Ruscheweyh derivative operator, we investigate two subclasses of analytic and bi-univalent functions using the techniques employed by Srivastava et al. [19] and Frasin and Aouf [9]. The obtained results improve and generalize some recent works and rectify remarkable mistakes in existing coefficient estimates.

To derive the results, we need to the following lemma in proving the theorems of sections 2 and 3.

Lemma 1.1. *[8] If $h \in \mathcal{P}$, then $|h_k| \leq 2$ for each $k \in \mathbb{N}$, where $\mathcal{P}$ is the subclass of functions $h(z)$ of the form*

$$h(z) = 1 + h_1 z + h_2 z^2 + h_3 z^3 + \cdots \tag{3}$$

which is analytic in $\mathcal{U}$ and the real part, $\Re(h(z))$, is positive.

2. Bounds for the Coefficient Functions in the Class $\mathcal{T}_\Sigma(\eta, \omega, \ell; \alpha)$

Let $h \in \mathcal{P}$ be given by (3) and $\mathcal{K}(z)$ be any complex-valued function such that $\mathcal{K}(z) = [h(z)]^\alpha, 0 < \alpha \leq 1$. Then

$$|\arg(\mathcal{K}(z))| = \alpha |\arg(h(z))| < \frac{\alpha\pi}{2}.$$

Therefore, if $|\arg(\mathcal{K}(z))| < \frac{\alpha\pi}{2}$, then it can be said that there exists $h \in \mathcal{P}$ such that $\mathcal{K}(z)$ can be written in terms of h and α as follows

$$\mathcal{K}(z) = [h(z)]^\alpha.$$

Definition 2.1. *A function $f \in \Sigma$ given by (1) is called in the class*

$$\mathcal{T}_\Sigma(\eta, \omega, \ell; \alpha) \ (z, w \in \mathcal{U},\ \eta \geq 0,\ \omega \in \mathbb{C}\backslash\{0\},\ \ell \in \mathbb{N}_0,\ 0 < \alpha \leq 1)$$

if it meets the following requirements

$$\left|\arg\left(1 + \frac{1}{\omega}\left[(1-\eta)\frac{z\left(\mathcal{R}^\ell f(z)\right)'}{\mathcal{R}^\ell f(z)} + \eta\frac{\left(\mathcal{R}^\ell f(z)\right)' + z\left(\mathcal{R}^\ell f(z)\right)''}{(\mathcal{R}^\ell f(z))'} - 1\right]\right)\right| < \frac{\alpha\pi}{2} \tag{4}$$

and

$$\left|\arg\left(1 + \frac{1}{\omega}\left[(1-\eta)\frac{w\left(\mathcal{R}^\ell g(w)\right)'}{\mathcal{R}^\ell g(w)} + \eta\frac{\left(\mathcal{R}^\ell g(w)\right)' + w\left(\mathcal{R}^\ell g(w)\right)''}{(\mathcal{R}^\ell g(w))'} - 1\right]\right)\right| < \frac{\alpha\pi}{2}, \tag{5}$$

where the function $g = f^{-1}$ is defined by (2).

Remark 2.2. *It is evident that by specializing η, ω and ℓ in the class $\mathcal{T}_\Sigma(\eta, \omega, \ell; \alpha)$, several known subclasses can be obtained, as recently investigated by various authors. We provide some examples:*

1. *For $\omega = 1$, the class $\mathcal{T}_\Sigma(\eta, \omega, \ell; \alpha)$ reduces to the class $F_\Sigma(\alpha, \eta)$ which was studied by Juma and Aziz [10].*
2. *For $\ell = 0$, the class $\mathcal{T}_\Sigma(\eta, \omega, \ell; \alpha)$ reduces to the class $\mathcal{M}_{\Sigma_1}(\alpha, \eta, \omega)$ which was examined by Motamednezhad et al. [14].*
3. *For $\ell = 0$ and $\omega = 1$, the class $\mathcal{T}_\Sigma(\eta, \omega, \ell; \alpha)$ reduces to the class $M_\Sigma(\alpha, \eta)$ which was investigated by Liu and Wang [12].*
4. *For $\ell = \eta = 0$ and $\omega = 1$, the class $\mathcal{T}_\Sigma(\eta, \omega, \ell; \alpha)$ reduces to the class $S^*_\Sigma(\alpha)$ which was considered by Brannan and Taha [5].*

Theorem 2.3. *Let $f \in \mathcal{T}_\Sigma(\eta, \omega, \ell; \alpha)$ be given by (1). Then*

$$|a_2| \leq \frac{2\alpha|\omega|}{\sqrt{\left|2\alpha\omega\left[(1+2\eta)(\ell+2) - (1+3\eta)(\ell+1)^2\right] - (\alpha-1)(1+\eta)^2(\ell+1)^2\right|}} \tag{6}$$

and

$$|a_3| \leq \frac{4\alpha^2|\omega|^2}{(1+\eta)^2(\ell+1)^2} + \frac{2\alpha|\omega|}{(1+2\eta)(\ell+2)}. \tag{7}$$

Proof. It follows from (4) and (5) that

$$1 + \frac{1}{\omega}\left[(1-\eta)\frac{z\left(\mathcal{R}^\ell f(z)\right)'}{\mathcal{R}^\ell f(z)} + \eta\frac{\left(\mathcal{R}^\ell f(z)\right)' + z\left(\mathcal{R}^\ell f(z)\right)''}{(\mathcal{R}^\ell f(z))'} - 1\right] = [\mathfrak{p}(z)]^\alpha \tag{8}$$

and

$$1 + \frac{1}{\omega}\left[(1-\eta)\frac{w\left(\mathcal{R}^\ell g(w)\right)'}{\mathcal{R}^\ell g(w)} + \eta\frac{\left(\mathcal{R}^\ell g(w)\right)' + w\left(\mathcal{R}^\ell g(w)\right)''}{(\mathcal{R}^\ell g(w))'} - 1\right] = [\mathfrak{q}(w)]^\alpha, \tag{9}$$

where $\mathfrak{p}, \mathfrak{q} \in \mathcal{P}$ have the following representations

$$\mathfrak{p}(z) = 1 + \mathfrak{p}_1 z + \mathfrak{p}_2 z^2 + \mathfrak{p}_3 z^3 + \cdots \tag{10}$$

and

$$\mathfrak{q}(w) = 1 + \mathfrak{q}_1 w + \mathfrak{q}_2 w^2 + \mathfrak{q}_3 w^3 + \cdots . \tag{11}$$

Clearly, we have

$$\begin{aligned}[\mathfrak{p}(z)]^\alpha = 1 &+ \alpha\mathfrak{p}_1 z + \left(\frac{1}{2}\alpha(\alpha-1)\mathfrak{p}_1^2 + \alpha\mathfrak{p}_2\right)z^2 \\ &+ \left(\frac{1}{6}\alpha(\alpha-1)(\alpha-2)\mathfrak{p}_1^3 + \alpha(1-\alpha)\mathfrak{p}_1\mathfrak{p}_2 + \alpha\mathfrak{p}_3\right)z^3 + \cdots\end{aligned} \tag{12}$$

and

$$\begin{aligned}[\mathfrak{q}(w)]^\alpha = 1 &+ \alpha\mathfrak{q}_1 w + \left(\frac{1}{2}\alpha(\alpha-1)\mathfrak{q}_1^2 + \alpha\mathfrak{q}_2\right)w^2 \\ &+ \left(\frac{1}{6}\alpha(\alpha-1)(\alpha-2)\mathfrak{q}_1^3 + \alpha(1-\alpha)\mathfrak{q}_1\mathfrak{q}_2 + \alpha\mathfrak{q}_3\right)w^3 + \cdots .\end{aligned} \tag{13}$$

We also find that

$$\begin{aligned}&1+\frac{1}{\omega}\left[(1-\eta)\frac{z\left(\mathcal{R}^{\ell}f(z)\right)'}{\mathcal{R}^{\ell}f(z)}+\eta\frac{\left(\mathcal{R}^{\ell}f(z)\right)'+z\left(\mathcal{R}^{\ell}f(z)\right)''}{(\mathcal{R}^{\ell}f(z))'}-1\right]\\&\quad=1+\frac{(1+\eta)(\ell+1)}{\omega}a_2z+\left(\frac{(1+2\eta)(\ell+2)}{\omega}a_3-\frac{(1+3\eta)(\ell+1)^2}{\omega}a_2^2\right)z^2+\cdots\end{aligned}\tag{14}$$

and

$$\begin{aligned}&1+\frac{1}{\omega}\left[(1-\eta)\frac{w\left(\mathcal{R}^{\ell}g(w)\right)'}{\mathcal{R}^{\ell}g(w)}+\eta\frac{\left(\mathcal{R}^{\ell}g(w)\right)'+w\left(\mathcal{R}^{\ell}g(w)\right)''}{(\mathcal{R}^{\ell}g(w))'}-1\right]\\&\quad=1-\frac{(1+\eta)(\ell+1)}{\omega}a_2w+\left(\frac{2(1+2\eta)(\ell+2)-(1+3\eta)(\ell+1)^2}{\omega}a_2^2-\frac{(1+2\eta)(\ell+2)}{\omega}a_3\right)w^2+\cdots.\end{aligned}\tag{15}$$

Now, by using (12), (13), (14) and (15), together with comparing the coefficients of (8) and (9), we get

$$\frac{(1+\eta)(\ell+1)}{\omega}a_2=\alpha p_1,\tag{16}$$

$$\frac{(1+2\eta)(\ell+2)}{\omega}a_3-\frac{(1+3\eta)(\ell+1)^2}{\omega}a_2^2=\frac{1}{2}\alpha(\alpha-1)\mathfrak{p}_1^2+\alpha\mathfrak{p}_2,\tag{17}$$

$$-\frac{(1+\eta)(\ell+1)}{\omega}a_2=\alpha\mathfrak{q}_1,\tag{18}$$

and

$$\frac{2(1+2\eta)(\ell+2)-(1+3\eta)(\ell+1)^2}{\omega}a_2^2-\frac{(1+2\eta)(\ell+2)}{\omega}a_3=\frac{1}{2}\alpha(\alpha-1)\mathfrak{q}_1^2+\alpha\mathfrak{q}_2.\tag{19}$$

In view of (16) and (18), we conclude that

$$\mathfrak{p}_1=-\mathfrak{q}_1\tag{20}$$

and

$$\frac{2(1+\eta)^2(\ell+1)^2}{\omega^2}a_2^2=\alpha^2\left(\mathfrak{p}_1^2+\mathfrak{q}_1^2\right).\tag{21}$$

Adding (17) to (19), we obtain

$$\frac{2(1+2\eta)(\ell+2)-2(1+3\eta)(\ell+1)^2}{\omega}a_2^2=\frac{1}{2}\alpha(\alpha-1)\left(\mathfrak{p}_1^2+\mathfrak{q}_1^2\right)+\alpha\left(\mathfrak{p}_2+\mathfrak{q}_2\right).\tag{22}$$

Substituting the value of $\mathfrak{p}_1^2+\mathfrak{q}_1^2$ form (21) into (22) and further computations imply that

$$a_2^2=\frac{\alpha^2\omega^2\left(\mathfrak{p}_2+\mathfrak{q}_2\right)}{2\alpha\omega\left[(1+2\eta)(\ell+2)-(1+3\eta)(\ell+1)^2\right]-(\alpha-1)(1+\eta)^2(\ell+1)^2}.\tag{23}$$

Applying Lemma 1.1 for the coefficients $\mathfrak{p}_2$ and $\mathfrak{q}_2$ on (23) imply that

$$|a_2|\le\frac{2\alpha|\omega|}{\sqrt{\left|2\alpha\omega\left[(1+2\eta)(\ell+2)-(1+3\eta)(\ell+1)^2\right]-(\alpha-1)(1+\eta)^2(\ell+1)^2\right|}}.$$

Next, in order to derive the bound on $|a_3|$, by subtracting (19) from (17), we obtain

$$\frac{2(1+2\eta)(\ell+2)}{\omega}\left(a_3-a_2^2\right)=\frac{1}{2}\alpha(\alpha-1)\left(\mathfrak{p}_1^2-\mathfrak{q}_1^2\right)+\alpha\left(\mathfrak{p}_2-\mathfrak{q}_2\right). \tag{24}$$

Now, substituting the value of a_2^2 from (21) into (24) and using (20), we conclude that

$$a_3=\frac{\alpha^2\omega^2\left(\mathfrak{p}_1^2+\mathfrak{q}_1^2\right)}{2(1+\eta)^2(\ell+1)^2}+\frac{\alpha\omega\left(\mathfrak{p}_2-\mathfrak{q}_2\right)}{2(1+2\eta)(\ell+2)}. \tag{25}$$

Finally, applying Lemma 1.1 once again for the coefficients $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}_1$ and $\mathfrak{q}_2$ on (25), we deduce that

$$|a_3|\le\frac{4\alpha^2|\omega|^2}{(1+\eta)^2(\ell+1)^2}+\frac{2\alpha|\omega|}{(1+2\eta)(\ell+2)}.$$

This completes the proof. $\square$

Remark 2.4. *By specializing the parameters in Theorem 2.3, it is observed that, several estimate bounds for known subclasses of Σ can be obtained as special cases. For example, if we set*

1. $\omega = 1$, *then we have correctness of the estimates given by Juma and Aziz [10, Theorem 2.2].*
2. $\ell = 0$ *and* $\omega = 1$, *then we obtain the results derived by Liu and Wang [12, Theorem 2.2]; the estimates derived by Ramadhan and Al-Ziadi [16, Corollary 4.2] in the class* $\mathcal{AR}_\Sigma(\delta, 0; \alpha)$; *the results given by Wanas and Páll-Szabó [31, Corollary 2.4] in the class* $AS_\Sigma(v, 1; \alpha)$; *and the estimates obtained by Wanas and Raadhi [32, Corollary 2.1] in the class* $\eta_\Sigma(0, \eta; \alpha)$.
3. $\ell = \eta = 0$ *and* $\omega = 1$, *then we retrieve the results derived by Brannan and Taha [5, Theorem 2.1], as well as the estimates obtained by Altinkaya and Yalçin [2, Theorem 3] in the class* $S_{\Sigma_1}(\alpha)$.

3. Bounds for the Coefficient Functions in the Class $\mathcal{T}_\Sigma^\star(\eta, \omega, \ell; \lambda)$

If $h \in \mathcal{P}$ be given by (3) and $\mathcal{L}(z)$ be any complex-valued function such that $\mathcal{L}(z) = \lambda + (1-\lambda)h(z)$, $0 \le \lambda < 1$, then

$$\Re\{\mathcal{L}(z)\} = \lambda + (1-\lambda)\Re\{h(z)\} > \lambda.$$

Therefore, if $\Re\{\mathcal{L}(z)\} > \lambda$, it can be said that there exists $h \in \mathcal{P}$ such that $\mathcal{L}(z)$ can be written in terms of h and λ as follows

$$\mathcal{L}(z) = \lambda + (1-\lambda)h(z).$$

Definition 3.1. *A function $f \in \sum$ given by (1) is called in the class*

$$\mathcal{T}_\Sigma^\star(\eta,\omega,\ell;\lambda)\ \ (z,w\in\mathcal{U},\ \eta\ge 0,\ \omega\in\mathbb{C}\backslash\{0\},\ \ell\in\mathbb{N}_0,\ 0\le\lambda<1)$$

if it meets the following requirements

$$\Re\left\{1+\frac{1}{\omega}\left[(1-\eta)\frac{z\left(\mathcal{R}^\ell f(z)\right)'}{\mathcal{R}^\ell f(z)}+\eta\frac{\left(\mathcal{R}^\ell f(z)\right)'+z\left(\mathcal{R}^\ell f(z)\right)''}{(\mathcal{R}^\ell f(z))'}-1\right]\right\}>\lambda \tag{26}$$

and

$$\Re\left\{1+\frac{1}{\omega}\left[(1-\eta)\frac{w\left(\mathcal{R}^\ell g(w)\right)'}{\mathcal{R}^\ell g(w)}+\eta\frac{\left(\mathcal{R}^\ell g(w)\right)'+w\left(\mathcal{R}^\ell g(w)\right)''}{(\mathcal{R}^\ell g(w))'}-1\right]\right\}>\lambda, \tag{27}$$

where the function $g = f^{-1}$ is defined by (2).

Remark 3.2. *It can be seen that, by specializing η, ω and ℓ in the class $\mathcal{T}^{\star}_{\Sigma}(\eta,\omega,\ell;\lambda)$, we get several known subclasses of Σ recently investigated by such authors. Let us present some examples:*

1. *For $\omega = 1$, the class $\mathcal{T}^{\star}_{\Sigma}(\eta,\omega,\ell;\lambda)$ reduces to the class $F_{\Sigma}(\lambda,\eta)$ which was examined by Juma and Aziz [10].*
2. *For $\ell = 0$, the class $\mathcal{T}^{\star}_{\Sigma}(\eta,\omega,\ell;\lambda)$ reduces to the class $\mathcal{M}_{\Sigma_1}(\lambda,\eta,\omega)$ which was investigated by Motamednezhad et al. [14].*
3. *For $\ell = 0$ and $\omega = 1$, the class $\mathcal{T}^{\star}_{\Sigma}(\eta,\omega,\ell;\lambda)$ reduces to the class $M_{\Sigma}(\lambda,\eta)$ which was studied by Liu and Wang [12].*
4. *For $\ell = \eta = 0$ and $\omega = 1$, the class $\mathcal{T}^{\star}_{\Sigma}(\eta,\omega,\ell;\lambda)$ reduces to the class $S^{*}_{\Sigma}(\lambda)$ which was considered by Brannan and Taha [5].*

Theorem 3.3. *Let $f \in \mathcal{T}^{\star}_{\Sigma}(\eta,\omega,\ell;\lambda)$ be given by (1). Then*

$$|a_2| \leq \begin{cases} \left(\frac{2|\omega|(1-\lambda)}{\left|(1+2\eta)(\ell+2)-(1+3\eta)(\ell+1)^2\right|}\right)^{1/2}, & 0 \leq \lambda \leq 1 - \frac{(1+\eta)^2(\ell+1)^2}{2|\omega|\left|(1+2\eta)(\ell+2)-(1+3\eta)(\ell+1)^2\right|} \\ \frac{2|\omega|(1-\lambda)}{(1+\eta)(\ell+1)}, & 1 - \frac{(1+\eta)^2(\ell+1)^2}{2|\omega|\left|(1+2\eta)(\ell+2)-(1+3\eta)(\ell+1)^2\right|} \leq \lambda < 1 \end{cases}$$

and

$$|a_3| \leq \begin{cases} \frac{2|\omega|(1-\lambda)}{\left|(1+2\eta)(\ell+2)-(1+3\eta)(\ell+1)^2\right|} + \frac{2|\omega|(1-\lambda)}{(1+2\eta)(\ell+2)}, & 0 \leq \lambda \leq 1 - \frac{(1+\eta)^2(\ell+1)^2}{2|\omega|\left|(1+2\eta)(\ell+2)-(1+3\eta)(\ell+1)^2\right|} \\ \frac{4|\omega|^2(1-\lambda)^2}{(1+\eta)^2(\ell+1)^2} + \frac{2|\omega|(1-\lambda)}{(1+2\eta)(\ell+2)}, & 1 - \frac{(1+\eta)^2(\ell+1)^2}{2|\omega|\left|(1+2\eta)(\ell+2)-(1+3\eta)(\ell+1)^2\right|} \leq \lambda < 1 \end{cases}$$

Proof. It follows from (26) and (27) that

$$1 + \frac{1}{\omega}\left[(1-\eta)\frac{z\left(\mathcal{R}^{\ell}f(z)\right)'}{\mathcal{R}^{\ell}f(z)} + \eta\frac{\left(\mathcal{R}^{\ell}f(z)\right)' + z\left(\mathcal{R}^{\ell}f(z)\right)''}{(\mathcal{R}^{\ell}f(z))'} - 1\right] = \lambda + (1-\lambda)\mathfrak{p}(z) \tag{28}$$

and

$$1 + \frac{1}{\omega}\left[(1-\eta)\frac{w\left(\mathcal{R}^{\ell}g(w)\right)'}{\mathcal{R}^{\ell}g(w)} + \eta\frac{\left(\mathcal{R}^{\ell}g(w)\right)' + w\left(\mathcal{R}^{\ell}g(w)\right)''}{(\mathcal{R}^{\ell}g(w))'} - 1\right] = \lambda + (1-\lambda)\mathfrak{q}(w), \tag{29}$$

where $\mathfrak{p}, \mathfrak{q} \in \mathcal{P}$ have the representations (10) and (11), respectively.

Clearly, we have

$$\lambda + (1-\lambda)\mathfrak{p}(z) = 1 + (1-\lambda)\mathfrak{p}_1 z + (1-\lambda)\mathfrak{p}_2 z^2 + (1-\lambda)\mathfrak{p}_3 z^3 + \cdots \tag{30}$$

and

$$\lambda + (1-\lambda)\mathfrak{q}(w) = 1 + (1-\lambda)\mathfrak{q}_1 w + (1-\lambda)\mathfrak{q}_2 w^2 + (1-\lambda)\mathfrak{q}_3 w^3 + \cdots . \tag{31}$$

Now, by using (30), (31), (14) and (15), together with comparing the coefficients of (28) and (29), yields

$$\frac{(1+\eta)(\ell+1)}{\omega}a_2 = (1-\lambda)\mathfrak{p}_1, \tag{32}$$

$$\frac{(1+2\eta)(\ell+2)}{\omega}a_3 - \frac{(1+3\eta)(\ell+1)^2}{\omega}a_2^2 = (1-\lambda)\mathfrak{p}_2, \tag{33}$$

$$-\frac{(1+\eta)(\ell+1)}{\omega}a_2=(1-\lambda)\mathfrak{q}_1, \tag{34}$$

and

$$\frac{2(1+2\eta)(\ell+2)-(1+3\eta)(\ell+1)^2}{\omega}a_2^2-\frac{(1+2\eta)(\ell+2)}{\omega}a_3=(1-\lambda)\mathfrak{q}_2. \tag{35}$$

From (32) and (34), we get

$$\mathfrak{p}_1=-\mathfrak{q}_1 \tag{36}$$

and

$$\frac{2(1+\eta)^2(\ell+1)^2}{\omega^2}a_2^2=(1-\lambda)^2\left(\mathfrak{p}_1^2+\mathfrak{q}_1^2\right). \tag{37}$$

Adding (33) to (35), we obtain

$$\frac{2(1+2\eta)(\ell+2)-2(1+3\eta)(\ell+1)^2}{\omega}a_2^2=(1-\lambda)\left(\mathfrak{p}_2+\mathfrak{q}_2\right). \tag{38}$$

From (37) and (38), we find

$$a_2^2=\frac{\omega^2(1-\lambda)^2\left(\mathfrak{p}_1^2+\mathfrak{q}_1^2\right)}{2(1+\eta)^2(\ell+1)^2} \tag{39}$$

and

$$a_2^2=\frac{\omega(1-\lambda)\left(\mathfrak{p}_2+\mathfrak{q}_2\right)}{2(1+2\eta)(\ell+2)-2(1+3\eta)(\ell+1)^2}, \tag{40}$$

respectively.

The equations (39) and (40) together with applying Lemma 1.1 for the coefficients $\mathfrak{p}_1, \mathfrak{q}_1, \mathfrak{p}_2$ and $\mathfrak{q}_2$, we find that

$$|a_2|\le\frac{2|\omega|(1-\lambda)}{(1+\eta)(\ell+1)}$$

and

$$|a_2|\le\sqrt{\frac{2|\omega|(1-\lambda)}{\left|(1+2\eta)(\ell+2)-(1+3\eta)(\ell+1)^2\right|}},$$

respectively.

To determine the estimates on $|a_3|$, by subtracting (35) from (33), we obtain

$$\frac{2(1+2\eta)(\ell+2)}{\omega}\left(a_3-a_2^2\right)=(1-\lambda)\left(\mathfrak{p}_2-\mathfrak{q}_2\right)$$

or, equivalently,

$$a_3=a_2^2+\frac{\omega(1-\lambda)\left(\mathfrak{p}_2-\mathfrak{q}_2\right)}{2(1+2\eta)(\ell+2)}. \tag{41}$$

Substituting the value of a_2^2 from (39) and (40) into (41), imply that

$$a_3=\frac{\omega^2(1-\lambda)^2\left(\mathfrak{p}_1^2+\mathfrak{q}_1^2\right)}{2(1+\eta)^2(\ell+1)^2}+\frac{\omega(1-\lambda)\left(\mathfrak{p}_2-\mathfrak{q}_2\right)}{2(1+2\eta)(\ell+2)} \tag{42}$$

and

$$a_3 = \frac{\omega(1-\lambda)\,(\mathfrak{p}_2+\mathfrak{q}_2)}{2(1+2\eta)(\ell+2)-2(1+3\eta)(\ell+1)^2} + \frac{\omega(1-\lambda)\,(\mathfrak{p}_2-\mathfrak{q}_2)}{2(1+2\eta)(\ell+2)}, \tag{43}$$

respectively.

Finally, applying Lemma 1.1 once again for the coefficients $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{q}_1$ and $\mathfrak{q}_2$ on equations (42) and (43) together, we conclude that

$$|a_3| \leq \frac{4|\omega|^2(1-\lambda)^2}{(1+\eta)^2(\ell+1)^2} + \frac{2|\omega|(1-\lambda)}{(1+2\eta)(\ell+2)}$$

and

$$|a_3| \leq \frac{2|\omega|(1-\lambda)}{\left|(1+2\eta)(\ell+2)-(1+3\eta)(\ell+1)^2\right|} + \frac{2|\omega|(1-\lambda)}{(1+2\eta)(\ell+2)'}$$

respectively. This completes the proof. □

Remark 3.4. *By specializing the parameters in Theorem 3.3, it can be seen that, several bound estimates for known subclasses of Σ can be attend as special cases. For example, if we put*

1. $\omega = 1$, *then we have correctness of the estimates obtained by Juma and Aziz [10, Theorem 3.2].*
2. $\ell = 0$ *and* $\omega = 1$, *then we have improvements of the estimates derived by Liu and Wang [12, Theorem 3.2]; improvements of the results derived by Wanas and Páll-Szabó [31, Corollary 3.4] in the class* $AS^*_\Sigma(v, 1; \lambda)$; *improvements of the estimates given by Wanas and Raadhi [32, Corollary 3.1] in the class* $\eta^*_\Sigma(0, \eta; \lambda)$; *and improvements of the results obtained by Ramadhan and Al-Ziadi [16, Corollary 4.3] in the class* $\mathcal{AR}_\Sigma(\delta, 0; \lambda)$.
3. $\ell = \eta = 0$ *and* $\omega = 1$, *then we have improvements of the estimates obtained by Brannan and Taha [5, Theorem 3.1], as well as the improvements of the results given by Altinkaya and Yalçin [2, Theorem 5] in the class* $S_{\Sigma_1}(\lambda)$.

4. Conclusions

In this study, we have introduced and examined two specific subclasses of analytic bi-univalent functions associated with the Ruscheweyh derivative. Our investigation focused on deriving initial coefficient bounds for functions belonging to these subclasses. The outcomes of our research demonstrate significant improvements, generalizations, and corrections in relation to previous studies. Moreover, we have highlighted certain implications of these subclasses by considering specific parameter specifications.

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