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New inequalities related to entropy and relative entropy with respect to Hermite-Hadamard inequality

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Abstract. In the current paper, we obtain the new practical inequalities for a variety of parametric extended and relative entropies. In this method, we use the uniformly convex functions due to the Hermite-Hadamard inequality.

1. Introduction and preliminaries

In the new researches, there are many studies on the applications of the generalized entropies in many fields of basic sciences. For example, applications of Renyi and Tsallis entropies have been studied in the quantum and classical statistical physics and the other fields of science (see[4, 7, 10, 11, 13–16, 25]).

The researchers have also studied the divergence aspects of entropies. The divergence affects on the entropy, relative entropy and natural information (the conditional forms of the mutual information measures can be presented as a divergence) (see [8, 9, 22]).

In [6] the Hermite-Hadamard inequality with the integral relation $\ln_q(x)$ has been used and the new bounds have been obtained for the Tsallis quasilinear entropy and divergence. That the $\ln_q(x)$ is q-logarithmic and define as: $\ln_q : (0, +\infty) \to \mathbb{R}$,

$$\ln_q(x) = \int_0^1 x^{(1-q)t} \ln x dt = \begin{cases} \frac{x^{1-q}-1}{1-q} & \text{if } x \neq 1 \\ \ln x & \text{if } x = 1 \end{cases}.$$

for positive real number $q > 0$. Note that the q is a real number. Also, in [5, 12] the authors considered bounds for the biparametrical extended entropies and divergences. However, there are two important concepts in the study of divergence: the weighted quasilinear mean and the Tsallis quasilinear entropy. So, we have the weighted quasilinear mean for some continuous and strictly monotonic function $\psi \to \mathbb{R}$, as follows:

$$M_\psi(x_1, \cdots, x_n) = \psi^{-1}\left(\sum_{j=1}^n p_j \psi(x_j)\right), \quad \sum_{j=1}^n p_j = 1, p_j > 0, \ x_j \in I.$$

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We have the following Tsallis q-quasilinear entropy for a positive probability distribution $\mathbf{p} = \{p_i\}_{i=1}^n$, continuous and strictly monotonic function ψ on (0,+∞) and $q > 0$:

$$I_q^{\psi}(\mathbf{p}) = \ln_q \psi^{-1}\left(\sum_{j=1}^{n} p_j \psi\left(\frac{1}{p_j}\right)\right). \tag{1}$$

The author studied the mathematical attributes of the following biparametric extended entropy in [4]:

$$H_{r,q}(\mathbf{p}) := \sum_{i=1}^{n} p_i \ln_q \exp\left(\ln_r \frac{1}{p_i}\right) = \sum_{i=1}^{n} p_i \ln_{r,q} \frac{1}{p_i},$$

and $S_{r,q}(\mathbf{p}) := \sum_{i=1}^{n} \frac{p_i^q - p_i^r}{r-q}$. As a result, he obtained mathematical inequalities related to the some entropies and divergences and presented some biparametric extended divergences.

The theory of convexity and Hermite-Hadamard inequalities play a key role in information theory mathematics and optimization (see [3, 17–20, 23, 24]). In this paper, we obtain the new inequalities for a variety of parametric extended entropies and divergences, using the uniformly convex functions due to the Hermite-Hadamard inequality.

Theorem 1.1. *[2] Let $f : [a, b] \longrightarrow \mathbb{R}$ be an uniformly convex function with modulus $\phi : \mathbb{R}_+ \longrightarrow [0, +\infty]$ on $I := [a, b]$, then*

$$f\left(\frac{a+b}{2}\right) + \frac{1}{4(b-a)}\int_a^b \phi(t-a)dt \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2} - \frac{1}{6}\phi(b-a).$$

The following definition found in [8, 9, 12].

Let ψ be a continous and strictly monotonic function on $(0, \infty)$. For two positive probability distributions $\mathbf{p} = \{p_i\}_{i=1}^n$ and $\mathbf{q} = \{q_i\}_{i=1}^n$ the Tsallis quasilinear divergence for $\mathbf{p}$ and $\mathbf{q}$ is defined by

$$D_q^{\psi}(\mathbf{p} \parallel \mathbf{q}) = -\ln_q \psi^{-1}\left(\sum_{i=1}^{n} \psi\left(\frac{q_i}{p_i}\right)\right).$$

Definition 1.2. *[1] Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function. Then f is called uniformly convex with modulus $\phi : \mathbb{R}_{\geq 0} \longrightarrow [0, +\infty)$ if ϕ is increasing, vanishes only at 0, and*

$$f(\alpha x + (1-\alpha)y) + \alpha(1-\alpha)\phi(|x-y|) \leq \alpha f(x) + (1-\alpha)f(y)$$

for every $\alpha \in [0, 1]$ and $x, y \in [a, b]$.

In [2, 21] authors studied hermite-hadamard on uniformly convex, m-convex functions.

2. Main results

In this section, $\mathbf{p} = \{p_i\}_{i=1}^n$ and $\mathbf{q} = \{q_i\}_{i=1}^n$ are two positive probability distributions. We present the following lemma which is used in the next.

Lemma 2.1. *Let $a > 0$ and $f : [a, b] \longrightarrow \mathbb{R}$ defined by $f(t) = x^{(1-q)t}$,*

$$\phi_q(r) := \frac{(1-q)^2(\ln x)^2 x^{b(1-q)}}{2} r^2 \quad \text{and} \quad \phi_q'(r) := \frac{(1-q)^2(\ln x)^2 x^{a(1-q)}}{2} r^2.$$

1. *If $0 < x, q \leq 1$, then f is uniformly convex with modulus $\phi_q(r)$.*
2. *If $0 < x \leq 1 \leq q$, then f is uniformly convex with modulus $\phi_q'(r)$.*

3. *If $0 < q \le 1 \le x$, then f is uniformly convex with modulus $\phi_q'(r)$.*
4. *If $x, q \ge 1$, then f is uniformly convex with modulus $\phi_q(r)$.*

Proof. (1) Let $0 < x, q \le 1$. Consider fixed points $\alpha, \beta \in [a, b]$ and define

$$\begin{aligned} g_q(t) :&= x^{(1-q)(t\alpha+(1-t)\beta)} - tx^{(1-q)\alpha} - (1-t)x^{(1-q)\beta} \\ &+ t(1-t)\frac{(1-q)^2(\ln x)^2 x^{b(1-q)}}{2}(\beta-\alpha)^2 \end{aligned}$$

for every $t \in [0, 1]$. Since $g_q(0) = g_q(1) = 0$ and $g_q''(t) \ge 0$, $g_q(t) \le 0$ for every $\alpha, \beta \in [a, b]$ and $t \in [0, 1]$. Hence, f is uniformly convex with modulus $\phi_q(r)$. (2),(3) and (4) are proved similarly. □

Theorem 2.2. *Let q be a positive real number and $\gamma_q(x) := \frac{(1-q)^2(\ln x)^3}{24}$.*

1. *If $0 < x, q \le 1$, then*

$$\begin{aligned} &\ln x \le \frac{x^{1-q}+1}{2}\ln x \le \frac{x^{1-q}+1}{2}\ln x - 2\gamma_q(x)x^{1-q} \le \ln_q(x) \\ &\le x^{\frac{1-q}{2}}\ln x + \gamma_q(x)x^{1-q} \le x^{\frac{1-q}{2}}\ln x \le x^{1-q}\ln x. \end{aligned}$$

2. *If $0 < x \le 1$ and $q \ge 1$, then*

$$\begin{aligned} &x^{1-q}\ln x \le \frac{x^{1-q}+1}{2}\ln x \le \frac{x^{1-q}+1}{2}\ln x - 2\gamma_q(x) \le \ln_q(x) \\ &\le x^{\frac{1-q}{2}}\ln x + \gamma_q(x) \le x^{\frac{1-q}{2}}\ln x \le \ln x. \end{aligned}$$

3. *If $0 < q \le 1$ and $x \ge 1$, then*

$$\begin{aligned} &\ln x \le x^{\frac{1-q}{2}}\ln x \le x^{\frac{1-q}{2}}\ln x + \gamma_q(x) \le \ln_q(x) \\ &\le \frac{x^{1-q}+1}{2}\ln x - 2\gamma_q(x) \le \frac{x^{1-q}+1}{2}\ln x \le x^{1-q}\ln x. \end{aligned}$$

4. *If $x, q \ge 1$, then*

$$\begin{aligned} &x^{1-q}\ln x \le x^{\frac{1-q}{2}}\ln x \le x^{\frac{1-q}{2}}\ln x + \gamma_q(x)x^{1-q} \le \ln_q(x) \\ &\le \frac{x^{1-q}+1}{2}\ln x - 2\gamma_q(x)x^{1-q} \le \frac{x^{1-q}+1}{2}\ln x \le \ln x. \end{aligned}$$

Proof. Let $f_q(t) = x^{(1-q)t}$ be uniformly convex function with modulus $\phi_q(.)$ on $[0, 1]$. By the use of Lemma 2.8 we obtain

$$x^{\frac{1-q}{2}} + \frac{1}{4}\int_0^1 \phi_q(t)dt \le \frac{\ln_q(x)}{\ln x} \le \frac{x^{1-q}+1}{2} - \frac{1}{6}\phi_q(1) \tag{2}$$

1. Let $0 < x, q \le 1$. In this case by the use of Lemma 2.1, $f(t) = x^{(1-q)t}$ is uniformly convex with modulus $\phi_q(r) = \frac{(1-q)^2(\ln x)^2 x^{(1-q)}}{2}r^2$ on $[0, 1]$. Therefor, with the use of (2) and Theorem 1.1 we obtain

$$\frac{x^{1-q}+1}{2}\ln x - \frac{(1-q)^2(\ln x)^3 x^{1-q}}{12} \le \ln_q(x) \le x^{\frac{1-q}{2}}\ln x + \frac{(1-q)^2(\ln x)^3 x^{1-q}}{24}.$$

Since $(\ln x)^3 x^{1-q} \le 0$, the result follows from Theorem 2.0.1 in [6].

2. Let $0 < x \leq 1$ and $q \geq 1$. In this case by the use of Lemma 2.1, $f(t) = x^{(1-q)t}$ is uniformly convex with modulus $\frac{(1-q)^2(\ln x)^2}{2}r^2$ on $[0,1]$. Therefore, with the use of (2) and Theorem 1.1 we obtain

$$\frac{x^{1-q}+1}{2}\ln x - \frac{(1-q)^2(\ln x)^3}{12} \leq \ln_q(x) \leq x^{\frac{1-q}{2}}\ln x + \frac{(1-q)^2(\ln x)^3}{24}.$$

Since $(\ln x)^3 \leq 0$ for all $x \leq 1$, the result follows from Theorem 2.0.1 in [6].

3. Let $0 < q \leq 1$ and $x \geq 1$. In this case by the use of Lemma 2.1, $f(t) = x^{(1-q)t}$ is uniformly convex with modulus $\frac{(1-q)^2(\ln x)^2}{2}r^2$ on $[0,1]$. Therefor, with the use of (2) and Theorem 1.1 we obtain

$$x^{\frac{1-q}{2}}\ln x + \frac{(1-q)^2(\ln x)^3}{24} \leq \ln_q(x) \leq \frac{x^{1-q}+1}{2}\ln x - \frac{(1-q)^2(\ln x)^3}{12}.$$

Since $(\ln x)^3 \geq 0$ for all $x \geq 1$, the result follows from Theorem 2.0.1 in [6].

4. Let $x, q \geq 1$. In this case by the use of Lemma 2.1, $f(t) = x^{(1-q)t}$ is uniformly convex with modulus $\frac{(1-q)^2(\ln x)^2 x^{(1-q)}}{2}r^2$ on $[0,1]$. Therefor, with the use of (2) and Theorem 1.1 we obtain

$$x^{\frac{1-q}{2}}\ln x + \frac{(1-q)^2(\ln x)^3}{24}x^{1-q} \leq \ln_q(x) \leq \frac{x^{1-q}+1}{2}\ln x - \frac{(1-q)^2(\ln x)^3 x^{1-q}}{12}.$$

Since $(\ln x)^3 x^{1-q} \geq 0$, the result follows from Theorem 2.0.1 in [6]. □

The Renyi entropy [13] is defined by $R_q(\mathbf{p}) = \frac{1}{1-q}\ln(\sum_{i=1}^n p_i^q)$, $q \neq 1$.

The following theorem is a generalization of Theorem 2.0.3 in [6].

Theorem 2.3. *Let ψ be a continuous and strictly function on $(0,\infty)$, $q > 0$, and let $\mathbf{p} = \{p_i\}_{i=1}^n$ be a probability distribution with $p_j > 0$ for all $j = 1, ..., n$.*

1. *If $0 < q \leq 1$, then*

$$\begin{aligned} I_1^\psi(\mathbf{p}) &\leq I_1^\psi(\mathbf{p})\left(M_\psi\left(\frac{1}{\mathbf{p}}\right)\right)^{\frac{1-q}{2}} \leq I_1^\psi(\mathbf{p})\left(M_\psi\left(\frac{1}{\mathbf{p}}\right)\right)^{\frac{1-q}{2}} + \frac{(1-q)^2}{24}\left(I_1^\psi(\mathbf{p})\right)^3 \\ &\leq I_q^\psi(\mathbf{p}) \leq \frac{1}{2}I_1^\psi(\mathbf{p})\left[\left(M_\psi\left(\frac{1}{\mathbf{p}}\right)\right)^{1-q}+1\right] - \frac{(1-q)^2}{12}(I_1^\psi(\mathbf{p}))^3 \\ &\leq \frac{1}{2}I_1^\psi(\mathbf{p})\left[\left(M_\psi\left(\frac{1}{\mathbf{p}}\right)\right)^{1-q}+1\right] \leq I_1^\psi(\mathbf{p})\left(M_\psi\left(\frac{1}{\mathbf{p}}\right)\right)^{1-q}. \end{aligned} \tag{3}$$

2. *If $q \geq 1$, then*

$$\begin{aligned} I_1^\psi(\mathbf{p})\left(M_\psi\left(\frac{1}{\mathbf{p}}\right)\right)^{1-q} &\leq I_1^\psi(\mathbf{p})\left(M_\psi\left(\frac{1}{\mathbf{p}}\right)\right)^{\frac{1-q}{2}} \leq I_1^\psi(\mathbf{p})\left(M_\psi\left(\frac{1}{\mathbf{p}}\right)\right)^{\frac{1-q}{2}} + \frac{(1-q)^2}{24}(I_1^\psi(\mathbf{p}))^3\left(M_\psi\left(\frac{1}{\mathbf{p}}\right)\right)^{1-q} \\ &\leq I_q^\psi(\mathbf{p}) \leq \frac{1}{2}I_1^\psi(\mathbf{p})\left[\left(M_\psi\left(\frac{1}{\mathbf{p}}\right)\right)^{1-q}+1\right] \leq \frac{1}{2}I_1^\psi(\mathbf{p})\left[\left(M_\psi\left(\frac{1}{\mathbf{p}}\right)\right)^{1-q}+1\right] - \frac{(1-q)^2}{12}(I_1^\psi(\mathbf{p}))^3\left(M_\psi\left(\frac{1}{\mathbf{p}}\right)\right)^{1-q} \\ &\leq I_1^\psi(\mathbf{p}). \end{aligned} \tag{4}$$

Proof. Let ψ be strictly increasing on $(0,\infty)$ and $x = \psi^{-1}\left(\sum_{j=1}^n p_j\psi\left(\frac{1}{p_j}\right)\right)$. Since $\frac{1}{p_j} > 1$ for all $j = 1, ..., n$, ψ and ψ^{-1} are strictly increasing functions,

$$x \geq \psi^{-1}\left(\sum_{j=1}^n p_j\psi(1)\right) = \psi^{-1}\psi(1) = 1.$$

Similarly if ψ is strictly increasing on $(0,\infty)$, then $\psi(x) \leq \psi(1)$ and $x \geq 1$.

1. Let $0 < q \le 1$ and $x = \psi^{-1}\left(\sum_{j=1}^{n} p_j \psi\left(\frac{1}{p_j}\right)\right)$. Since $x \ge 1$,

$$\ln x = I_1^{\psi}(\mathbf{p}) \ge 0,\ x = M_{\psi}\left(\frac{1}{\mathbf{p}}\right) \text{ and } \ln_q^x = I_q^{\psi}(\mathbf{p}).$$

By the use of Theorem 2.2 (Part 3), we have

$$\begin{aligned}
&\left[\psi^{-1}\left(\sum_{j=1}^{n} p_j\psi\left(\frac{1}{p_j}\right)\right)\right]^{\frac{1-q}{2}} \ln\left(\psi^{-1}\left(\sum_{j=1}^{n} p_j\psi\left(\frac{1}{p_j}\right)\right)\right) + \frac{(1-q)^2\left(\ln\left(\psi^{-1}\left(\sum_{j=1}^{n} p_j\psi\left(\frac{1}{p_j}\right)\right)\right)\right)^3}{24}\\
&\le \ln_q\left(\psi^{-1}\left(\sum_{j=1}^{n} p_j\psi\left(\frac{1}{p_j}\right)\right)\right)\\
&\le \frac{\left(\psi^{-1}\left(\sum_{j=1}^{n} p_j\psi\left(\frac{1}{p_j}\right)\right)\right)^{1-q}+1}{2} \ln\left(\psi^{-1}\left(\sum_{j=1}^{n} p_j\psi\left(\frac{1}{p_j}\right)\right)\right) - \frac{(1-q)^2\left(\ln\left(\psi^{-1}\left(\sum_{j=1}^{n} p_j\psi\left(\frac{1}{p_j}\right)\right)\right)\right)^3}{12},
\end{aligned}$$

which shows the third and fourth inequalities in (3). The other inequalities in (3) follow from Theorem 2.0.3 in [6].

2. Let $q \ge 1$ and $x = \psi^{-1}\left(\sum_{j=1}^{n} p_j \psi\left(\frac{1}{p_j}\right)\right)$. Since $x \ge 1$,

$$\ln x = I_1^{\psi}(\mathbf{p}) \ge 0 \quad \ln_q^x = I_q^{\psi}(\mathbf{p}) \quad \text{and} \quad x^{1-q} = (1-q)I_q^{\psi}(\mathbf{p}) + 1 = \left(M_{\psi}\left(\frac{1}{\mathbf{p}}\right)\right)^{1-q}.$$

The desired inequalities follow from Theorem 2.2 (Part 4)and Theorem 2.0.3 in [6].

□

Corollary 2.4. *Let* $\mathbf{p} = \{p_i\}_{i=1}^n$ *be a positive probability distribution.*

1. *If* $0 < q < 1$, *then*

$$\begin{aligned}
H(\mathbf{p}) &\le H(\mathbf{p})\exp\left(\left(\frac{1-q}{2}\right)H(\mathbf{p})\right) \le H(\mathbf{p})\exp\left(\left(\frac{1-q}{2}\right)H(\mathbf{p})\right) + \frac{(1-q)^2}{24}(H(\mathbf{p}))^3\\
&\le \frac{1}{1-q}(\exp((1-q)H(\mathbf{p}))-1) \le \frac{1}{2}H(\mathbf{p})(\exp((1-q)H(\mathbf{p}))+1) - \frac{(1-q)^2}{12}(H(\mathbf{p}))^3\\
&\le \frac{1}{2}H(\mathbf{p})(\exp((1-q)H(\mathbf{p}))+1) \le H(\mathbf{p})\exp((1-q)H(\mathbf{p})).
\end{aligned}$$

2. *If* $q > 1$, *then*

$$\begin{aligned}
&H(\mathbf{p})\exp((1-q)H(\mathbf{p})) \le H(\mathbf{p})\exp\left(\left(\frac{1-q}{2}\right)H(\mathbf{p})\right)\\
&\le H(\mathbf{p})\exp\left(\left(\frac{1-q}{2}\right)H(\mathbf{p})\right) + \frac{(1-q)^2}{24}(H(\mathbf{p}))^3\exp((1-q)H(\mathbf{p}))\\
&\le \frac{1}{1-q}(\exp((1-q)H(\mathbf{p}))-1)\\
&\le \frac{1}{2}H(\mathbf{p})(\exp((1-q)H(\mathbf{p}))+1) - \frac{(1-q)^2}{12}(H(\mathbf{p}))^3\exp((1-q)H(\mathbf{p}))\\
&\le \frac{1}{2}H(\mathbf{p})(\exp((1-q)H(\mathbf{p}))+1) \le H(\mathbf{p}).
\end{aligned}$$

Proof. For $\psi(x) = \ln x$ Theorem 2.3 it follows that we have the inequalities of the statement. □

For any positive probability distribution $\mathbf{p}$ and any positive real number q, the Tsallis entropy is defined by

$$H_q(\mathbf{p}) = \sum_{i=1}^{n} \frac{p_i^q - p_i}{1-q} = -\sum_{i=1}^{n} p_i^q \ln_q(p_i) = \sum_{i=1}^{n} p_i \ln_q\left(\frac{1}{p_i}\right) \quad \text{and} \quad H_1(\mathbf{p}) := H(\mathbf{p}).$$

Corollary 2.5. *Under the notation of Corollary 2.4, we have*

1. *If* $0 < q < 1$, *then*

$$\begin{aligned} R_q(\mathbf{p}) &\le \exp\left(\left(\frac{1-q}{2}\right) R_q(\mathbf{p})\right) R_q(\mathbf{p}) \\ &\le R_q(\mathbf{p}) \exp\left(\left(\frac{1-q}{2}\right) R_q(\mathbf{p})\right) + \frac{(1-q)^2}{24}(R_q(\mathbf{p}))^3 \le H_q(\mathbf{p}) \\ &\le \frac{1}{2} R_q(\mathbf{p})[\exp((1-q)R_q(\mathbf{p}) + 1)] - \frac{(1-q)^2}{12}(R_q(\mathbf{p}))^3 \\ &\le \frac{1}{2} R_q(\mathbf{p})[\exp((1-q)R_q(\mathbf{p}) + 1)] \le \exp((1-q)R_q(\mathbf{p}))R_q(\mathbf{p}). \end{aligned}$$

2. *If* $q > 1$, *then*

$$\begin{aligned} &\exp((1-q)R_q(\mathbf{p}))R_q(\mathbf{p}) \le \exp\left(\left(\frac{1-q}{2}\right) R_q(\mathbf{p})\right) R_q(\mathbf{p}) \\ &\le \exp\left(\left(\frac{1-q}{2}\right) R_q(\mathbf{p})\right) R_q(\mathbf{p}) + \frac{(1-q)^2}{24}(R_q(\mathbf{p}))^3 \exp((1-q)R_q(\mathbf{p})) \le H_q(\mathbf{p}) \\ &\le \frac{1}{2} R_q(\mathbf{p})[\exp((1-q)R_q(\mathbf{p}) + 1)] - \frac{(1-q)^2}{12}(R_q(\mathbf{p}))^3 \exp((1-q)R_q(\mathbf{p})) \\ &\le \frac{1}{2} R_q(\mathbf{p})[\exp((1-q)R_q(\mathbf{p}) + 1)] \le R_q(\mathbf{p}). \end{aligned}$$

Proof. For $\psi(x) = x^{1-q}$ Theorem 2.3 it follows that we have the inequalities of the statement. □

For $q > 0$, the Tsallis quasilinear divergence for $\mathbf{p}$ and $\mathbf{q}$ is

$$D_q^{\psi}(\mathbf{p} \parallel \mathbf{q}) = \ln_q \psi^{-1}\left(\sum_{i=1}^{n} \psi\left(\frac{p_i}{q_i}\right)\right). \tag{5}$$

Theorem 2.6. *Let* $\psi : I \longrightarrow J$, $J \subseteq (0, \infty)$ *be a convex decreasing function or a concave increasing function, let* $\mathbf{p} = \{p_i\}_{i=1}^n$ *and* $\mathbf{q} = \{q_i\}_{i=1}^n$ *be two positive probability distributions.*

1. *If* $q > 1$, *then*

$$\begin{aligned} &D_1^{\psi}(\mathbf{p} \parallel \mathbf{q}) \le D_1^{\psi}(\mathbf{p} \parallel \mathbf{q}) \left(M_{\psi}\left(\frac{\mathbf{q}}{\mathbf{p}}\right)\right)^{\frac{1-q}{2}} \qquad (6) \\ &\le D_1^{\psi}(\mathbf{p} \parallel \mathbf{q}) \left(M_{\psi}\left(\frac{\mathbf{q}}{\mathbf{p}}\right)\right)^{\frac{1-q}{2}} + \frac{(1-q)^2}{24}(D_1^{\psi}(\mathbf{p} \parallel \mathbf{q}))^3 \le D_q^{\psi}(\mathbf{p} \parallel \mathbf{q}) \\ &\le \frac{1}{2} D_1^{\psi}(\mathbf{p} \parallel \mathbf{q}) \left(\left(M_{\psi}\left(\frac{\mathbf{q}}{\mathbf{p}}\right)\right)^{1-q} + 1\right) - \frac{(1-q)^2}{12}(D_1^{\psi}(\mathbf{p} \parallel \mathbf{q}))^3 \\ &\le \frac{1}{2} D_1^{\psi}(\mathbf{p} \parallel \mathbf{q}) \left(\left(M_{\psi}\left(\frac{\mathbf{q}}{\mathbf{p}}\right)\right)^{1-q} + 1\right) \le D_1^{\psi}(\mathbf{p} \parallel \mathbf{q}) \left(M_{\psi}\left(\frac{\mathbf{q}}{\mathbf{p}}\right)\right)^{1-q} \end{aligned}$$

2. *If* $q < 1$, *then*

$$D_1^{\psi}(\mathbf{p} \parallel \mathbf{q})\left(M_{\psi}\left(\frac{\mathbf{q}}{\mathbf{p}}\right)\right)^{1-q} \leq D_1^{\psi}(\mathbf{p} \parallel \mathbf{q})\left(M_{\psi}\left(\frac{\mathbf{q}}{\mathbf{p}}\right)\right)^{\frac{1-q}{2}} \tag{7}$$

$$\leq D_1^{\psi}(\mathbf{p} \parallel \mathbf{q})\left(M_{\psi}\left(\frac{\mathbf{q}}{\mathbf{p}}\right)\right)^{\frac{1-q}{2}} + \frac{(1-q)^2}{24}(D_1^{\psi}(\mathbf{p} \parallel \mathbf{q}))^3\left(M_{\psi}\left(\frac{\mathbf{q}}{\mathbf{p}}\right)\right)^{1-q} \leq D_q^{\psi}(\mathbf{p} \parallel \mathbf{q})$$

$$\leq \frac{1}{2}D_1^{\psi}(\mathbf{p} \parallel \mathbf{q})\left(\left(M_{\psi}\left(\frac{\mathbf{q}}{\mathbf{p}}\right)\right)^{1-q} + 1\right) - \frac{(1-q)^2}{12}(D_1^{\psi}(\mathbf{p} \parallel \mathbf{q}))^3\left(M_{\psi}\left(\frac{\mathbf{q}}{\mathbf{p}}\right)\right)^3$$

$$\leq \frac{1}{2}D_1^{\psi}(\mathbf{p} \parallel \mathbf{q})\left(\left(M_{\psi}\left(\frac{\mathbf{q}}{\mathbf{p}}\right)\right)^{1-q} + 1\right) \leq D_1^{\psi}(\mathbf{p} \parallel \mathbf{q}).$$

Proof. If ψ is a convex decreasing or concave increasing function, then

$$0 < x = \psi^{-1}\left(\sum_{i=1}^{n} p_i \psi\left(\frac{q_i}{p_i}\right)\right) = M_{\psi}\left(\frac{\mathbf{q}}{\mathbf{p}}\right) \leq 1.$$

Thus,

$$\ln(x) = -D_1^{\psi}(\mathbf{p} \parallel \mathbf{q}) \text{ and } \ln_q x = -D_q^{\psi}(\mathbf{p} \parallel \mathbf{q}).$$

1. Let $q > 1$ and $x = \psi^{-1}\left(\sum_{j=1}^{n} p_j \psi\left(\frac{q_j}{p_j}\right)\right) \leq 1$. The desired inequalities follow from Theorem 2.2 and Theorem 2.0.6 in [6].
2. Let $q < 1$ and $x = \psi^{-1}\left(\sum_{j=1}^{n} p_j \psi\left(\frac{q_j}{p_j}\right)\right) \leq 1$. The desired inequalities follow from Theorem 2.2 and Theorem 2.0.6 in [6].

□

Theorem 2.2 yields the following lemma.

Lemma 2.7. *Let* $q > 0$ *and* $\gamma_q(x) = \frac{(1-q)^2(\ln x)^3}{24}$.

1. *If* $0 < x, q \leq 1$, *then*

$$\ln x \leq \ln x - 2\gamma_q(x)x^{1-q} \leq \ln_q(x) \leq x^{1-q}\ln x + \gamma_q(x)x^{1-q} \leq x^{1-q}\ln x.$$

2. *If* $0 < x \leq 1$ *and* $q \geq 1$, *then*

$$x^{1-q}\ln x \leq x^{1-q}\ln x - 2\gamma_q(x) \leq \ln_q(x) \leq \ln x + \gamma_q(x) \leq \ln x.$$

3. *If* $0 < q \leq 1$ *and* $x \geq 1$, *then*

$$\ln x \leq \ln x + \gamma_q(x) \leq \ln_q(x) \leq x^{1-q}\ln x - 2\gamma_q(x) \leq x^{1-q}\ln x.$$

4. *If* $x, q \geq 1$, *then*

$$x^{1-q}\ln x \leq x^{1-q}\ln x + \gamma_q(x)x^{1-q} \leq \ln_q(x) \leq \ln x - 2\gamma_q(x)x^{1-q} \leq \ln x.$$

Proof. 1. Let $0 < x, q \leq 1$. Since $x^{1-q} \leq 1$, $x^{1-q} \leq x^{\frac{1-q}{2}}$ and $(\ln x)^3 x^{1-q} \leq 0$ for all $x \in (0, 1]$, the inequalities follow from Theorem 2.2.

2. Let $0 < x \leq 1$ and $q \geq 1$. Since $1 \leq x^{1-q}$, $1 \leq x^{\frac{1-q}{2}}$ and $(\ln x)^3 \leq 0$ for all $x \in (0, 1]$, the inequalities follow from Theorem 2.2.

3. Let $0 < q \leq 1$ and $x \geq 1$. Since $1 \leq x^{1-q}$, $1 \leq x^{\frac{1-q}{2}}$ and $(\ln x)^3 \geq 0$ for all $x \geq 1$, the inequalities follow from Theorem 2.2.

4. Let $x, q \geq 1$. Since $x^{1-q} \leq 1$, $x^{1-q} \leq x^{\frac{1-q}{2}}$ and $(\ln x)^3 x^{1-q} \geq 0$ for all $x \geq 1$, the inequalities follow from Theorem 2.2. □

Lemma 2.8. *Let $q > 0$ be a real number strictly positive.*

1. *If $0 < q \leq 1$, then*

$$\ln x \leq \ln x + \frac{(1-q)^2}{12}\left[\frac{1}{2}(\ln x)^3 - x^{1-q}(\ln x)^3 + \left|\frac{1}{2}(\ln x)^3 + x^{1-q}(\ln x)^3\right|\right] \leq \ln_q(x)$$
$$\leq x^{1-q}\ln x - \frac{(1-q)^2}{12}\left[(\ln x)^3 - \frac{1}{2}x^{1-q}(\ln x)^3 + \left|(\ln x)^3 + \frac{1}{2}x^{1-q}(\ln x)^3\right|\right] \leq x^{1-q}\ln x.$$

2. *If $q \geq 1$, then*

$$x^{1-q}\ln x \leq x^{1-q}\ln x + \frac{(1-q)^2}{12}\left[\frac{(\ln x)^3 x^{1-q}}{2} - (\ln x)^3 + \left|\frac{(\ln x)^3 x^{1-q}}{2} + (\ln x)^3\right|\right]$$
$$\leq \ln_q(x) \leq \ln x - \frac{(1-q)^2}{12}\left[(\ln x)^3 x^{1-q} - \frac{(\ln x)^3}{2} + \left|(\ln x)^3 x^{1-q} + \frac{(\ln x)^3}{2}\right|\right] \leq \ln x.$$

Proof. 1. Suppose that $x > 0$. Since

$$\max\left\{\frac{1}{2}(\ln x)^3, -x^{1-q}(\ln x)^3\right\} \geq 0, \ \min\left\{\frac{1}{2}(\ln x)^3, -x^{1-q}(\ln x)^3\right\} \leq 0,$$

and $a + b + |a - b| = 2max\{a, b\}$, the result follows from Lemma 2.7.

2. Suppose that $x > 0$. Since

$$\max\left\{\frac{(\ln x)^3 x^{1-q}}{2}, -(\ln x)^3\right\} \geq 0, \ \min\left\{\frac{(\ln x)^3 x^{1-q}}{2}, -(\ln x)^3\right\} \leq 0,$$

and $a + b + |a - b| = 2max\{a, b\}$, the result follows from Lemma 2.7.

□

In [5, 6], the Tsallis relative entropy between **p** and **q** is defined by

$$D_q^T(\mathbf{p} \parallel \mathbf{q}) := \sum_{i=1}^n \frac{p_i - p_i^q q_i^{1-q}}{1-q} = \sum_{i=1}^n p_i^q q_i^{1-q} \ln_q\left(\frac{p_i}{q_i}\right) = -\sum_{i=1}^n p_i \ln_q\left(\frac{q_i}{p_i}\right)$$

where $1 \neq q > 0$, if $q = 1$ then

$$D_1^T(\mathbf{p} \parallel \mathbf{q}) := \sum_{i=1}^n p_i \ln\left(\frac{p_i}{q_i}\right).$$

The biparametric extended divergence of order m is defined by

$$D_q^{T,m}(\mathbf{p} \parallel \mathbf{q}) := -\sum_{i=1}^n p_i^q q_i^{1-q}\left[\ln_q\left(\frac{q_i}{p_i}\right)\right]^m$$

where $1 \neq q > 0$, if $q = 1$ then

$$D_1^{T,m}(\mathbf{p} \parallel \mathbf{q}) := -\sum_{i=1}^n p_i\left(\ln\left(\frac{q_i}{p_i}\right)\right)^m, \ D_1^T(\mathbf{p} \parallel \mathbf{q}) := D_1^{T,1}(\mathbf{p} \parallel \mathbf{q}).$$

A convex combination between the biparametric extended divergence and the Tsallis divergence expressed by

$$\hat{D}_{q,r}(\mathbf{p} \parallel \mathbf{q}) = \left(\frac{q-1}{q-r}\right) D_q^T(\mathbf{p} \parallel \mathbf{q}) + \left(\frac{1-r}{q-r}\right) D_r^T(\mathbf{p} \parallel \mathbf{q}). \tag{8}$$

The quasi-divergence [6] is defined by $D_{(q)}(\mathbf{p} \parallel \mathbf{q}) := -\sum_{i=1}^n p_i^q q_i^{1-q} \ln\left(\frac{q_i}{p_i}\right)$.

Define $T_1^q(\mathbf{p} \parallel \mathbf{q}) := \sum_{i=1}^n \left(\frac{p_i}{2} + p_i^q q_i^{1-q}\right)|(\ln \frac{q_i}{p_i})^3|$ and $T_2^q(\mathbf{p} \parallel \mathbf{q}) := \sum_{i=1}^n \left(p_i + \frac{1}{2}p_i^q q_i^{1-q}\right)\left|\left(\ln \frac{q_i}{p_i}\right)^3\right|$.

Lemma 2.9. *Let $q > 0$ be a real number strictly positive*

1. *If $0 < q \leq 1$, then*

$$D_{(q)}(\mathbf{p} \parallel \mathbf{q}) \leq D_{(q)}(\mathbf{p} \parallel \mathbf{q}) + \frac{(1-q)^2}{12}[T_2^q(\mathbf{p} \parallel \mathbf{q}) - D_1^{T,3}(\mathbf{p} \parallel \mathbf{q}) + \frac{1}{2}D_q^{T,3}(\mathbf{p} \parallel \mathbf{q})]$$
$$\leq D_q^T(\mathbf{p} \parallel \mathbf{q}) \leq D_1(\mathbf{p} \parallel \mathbf{q}) - \frac{(1-q)^2}{12}[T_1^q(\mathbf{p} \parallel \mathbf{q}) + D_q^{T,3}(\mathbf{p} \parallel \mathbf{q}) - \frac{1}{2}D_1^{T,3}(\mathbf{p} \parallel \mathbf{q})] \leq D_1(\mathbf{p} \parallel \mathbf{q}).$$

2. *If $q \geq 1$, then*

$$D_1(\mathbf{p} \parallel \mathbf{q}) \leq D_1(\mathbf{p} \parallel \mathbf{q}) + \frac{(1-q)^2}{12}[T_1^q(\mathbf{p} \parallel \mathbf{q}) + \frac{1}{2}D_1^{T,3}(\mathbf{p} \parallel \mathbf{q}) - D_q^{T,3}(\mathbf{p} \parallel \mathbf{q})]$$
$$\leq D_q^T(\mathbf{p} \parallel \mathbf{q}) \leq D_{(q)}(\mathbf{p} \parallel \mathbf{q}) - \frac{(1-q)^2}{12}[T_2^q(\mathbf{p} \parallel \mathbf{q}) + D_1^{T,3}(\mathbf{p} \parallel \mathbf{q}) - \frac{1}{2}D_q^{T,3}(\mathbf{p} \parallel \mathbf{q})] \leq D_{(q)}(\mathbf{p} \parallel \mathbf{q}).$$

Proof. Let $q \leq 1$. By the use of Lemma 2.8 we have

$$\ln x + \frac{(1-q)^2}{12}\left[\frac{1}{2}(\ln x)^3 - x^{1-q}(\ln x)^3 + \left|\frac{1}{2}(\ln x)^3 + x^{1-q}(\ln x)^3\right|\right] \leq \ln_q(x)$$
$$\leq x^{1-q}\ln x - \frac{(1-q)^2}{12}\left[(\ln x)^3 - \frac{1}{2}x^{1-q}(\ln x)^3 + \left|(\ln x)^3 + \frac{1}{2}x^{1-q}(\ln x)^3\right|\right].$$

Consequently for $x = \frac{q_i}{p_i}$, we obtain

$$\ln\left(\frac{q_i}{p_i}\right) + \frac{(1-q)^2}{12}\Big[\frac{1}{2}\left(\ln\left(\frac{q_i}{p_i}\right)\right)^3 - \left(\frac{q_i}{p_i}\right)^{1-q}\left(\ln\frac{q_i}{p_i}\right)^3 + \left|\frac{1}{2}\left(\ln\frac{q_i}{p_i}\right)^3 + \left(\frac{q_i}{p_i}\right)^{1-q}\left(\ln\frac{q_i}{p_i}\right)^3\right|\Big]$$
$$\leq \ln_q\left(\frac{q_i}{p_i}\right) \leq \left(\frac{q_i}{p_i}\right)^{1-q}\ln\frac{q_i}{p_i} - \frac{(1-q)^2}{12}\Big[\left(\ln\frac{q_i}{p_i}\right)^3 - \frac{1}{2}\left(\frac{q_i}{p_i}\right)^{1-q}\left(\ln\frac{q_i}{p_i}\right)^3 + \left|\left(\ln\frac{q_i}{p_i}\right)^3 + \frac{1}{2}\left(\frac{q_i}{p_i}\right)^{1-q}\left(\ln\frac{q_i}{p_i}\right)^3\right|\Big].$$

Multiplying by p_i and passing to the sum, we conclude that the desired inequality holds. Similarly we prove for the case $q > 1$. □

For two positive probability distributions **p** and **q** define

$$\hat{T}_O^{q,r}(\mathbf{p} \parallel \mathbf{q}) := \frac{(q-1)^3}{q-r}\left[T_2^q(\mathbf{p} \parallel \mathbf{q}) - D_1^{T,3}(\mathbf{p} \parallel \mathbf{q}) + \frac{1}{2}D_q^{T,3}(\mathbf{p} \parallel \mathbf{q})\right]$$
$$+ \frac{(1-r)^3}{q-r}\left[T_1^r(\mathbf{p} \parallel \mathbf{q}) + \frac{1}{2}D_1^{T,3}(\mathbf{p} \parallel \mathbf{q}) - D_r^{T,3}(\mathbf{p} \parallel \mathbf{q})\right] \geq 0$$
$$\hat{T}_E^{q,r}(\mathbf{p} \parallel \mathbf{q}) := \frac{(q-1)^3}{q-r}\left[T_1^q(\mathbf{p} \parallel \mathbf{q}) + D_q^{T,3}(\mathbf{p} \parallel \mathbf{q}) - \frac{1}{2}D_1^{T,3}(\mathbf{p} \parallel \mathbf{q})\right]$$
$$+ \frac{(1-r)^3}{q-r}\left[T_2^r(\mathbf{p} \parallel \mathbf{q}) + D_1^{T,3}(\mathbf{p} \parallel \mathbf{q}) - \frac{1}{2}D_r^{T,3}(\mathbf{p} \parallel \mathbf{q})\right] \geq 0.$$

Lemma 2.9 yields the following proposition:

Proposition 2.10. *Let $\mathbf{p} = \{p_i\}_{i=1}^n$ and $\mathbf{q} = \{q_i\}_{i=1}^n$ are two positive probability distributions.*

1. *If $q < 1 < r$, then*

$$\frac{q-1}{q-r}D_{(q)}(\mathbf{p} \parallel \mathbf{q}) + \frac{1-r}{q-r}D_1(\mathbf{p} \parallel \mathbf{q}) + \frac{1}{12}\hat{T}_E^{q,r}(\mathbf{p} \parallel \mathbf{q}) \leq \hat{D}_{q,r}(\mathbf{p} \parallel \mathbf{q})$$
$$\leq \frac{q-1}{q-r}D_1(\mathbf{p} \parallel \mathbf{q}) + \frac{1-r}{q-r}D_{(r)}(\mathbf{p} \parallel \mathbf{q}) - \frac{1}{12}\hat{T}_O^{q,r}(\mathbf{p} \parallel \mathbf{q}).$$

2. *If $r < 1 < q$, then*

$$\frac{q-1}{q-r}D_1(\mathbf{p} \parallel \mathbf{q}) + \frac{1-r}{q-r}D_{(r)}(\mathbf{p} \parallel \mathbf{q}) + \frac{1}{12}\hat{T}_E^{r,q}(\mathbf{p} \parallel \mathbf{q}) \leq \hat{D}_{r,q}(\mathbf{p} \parallel \mathbf{q})$$
$$\leq \frac{q-1}{q-r}D_{(q)}(\mathbf{p} \parallel \mathbf{q}) + \frac{1-r}{q-r}D_1(\mathbf{p} \parallel \mathbf{q}) - \frac{1}{12}\hat{T}_O^{r,q}(\mathbf{p} \parallel \mathbf{q}).$$

Let $\psi : (0, \infty) \longrightarrow \mathbb{R}$ be a continuous and strictly monotonic function. For two positive real numbers $q, r \neq 1$, the (r, q)-quasilinear entropy is defined by $I_{r,q}^{\psi}(\mathbf{p}) := \ln_{r,q} \psi^{-1}\Big(\sum_{i=1}^{n} \psi\Big(\frac{1}{p_i}\Big)\Big)$ where the biparametric extended logarithmic function for $x > 0$ [12] is defined by $\ln_{r,q}(x) := \ln_q \exp(\ln_r x)$.

Lemma 2.11. *Let $q > 0$ be a real number strictly positive.*

1. *If $0 < x, q \leq 1$, then*

$$\ln_r(x) \leq \frac{\exp((1-q)\ln_r x) + 1}{2}\ln_r(x)$$
$$\leq \frac{\exp((1-q)\ln_r x) + 1}{2}\ln_r(x) - \frac{(1-q)^2}{12}(\ln_r(x))^3 \exp((1-q)\ln_r x) \leq \ln_{r,q}(x)$$
$$\leq \exp\left(\left(\frac{1-q}{2}\right)\ln_r x\right)\ln_r(x) + \frac{(1-q)^2(\ln_r(x))^3 \exp((1-q)\ln_r x)}{24}$$
$$\leq \exp\left(\left(\frac{1-q}{2}\right)\ln_r x\right)\ln_r(x) \leq \exp((1-q)\ln_r x)\ln_r(x).$$

2. *If $0 < x \leq 1$ and $q \geq 1$, then*

$$\exp((1-q)\ln_r x)\ln_r(x) \leq \frac{\exp((1-q)\ln_r x) + 1}{2}\ln_r(x)$$
$$\leq \frac{\exp((1-q)\ln_r x) + 1}{2}\ln_r(x) - \frac{(1-q)^2(\ln_r(x))^3}{12} \leq \ln_{r,q}(x)$$
$$\leq \exp\left(\frac{(1-q)}{2}\ln_r x\right)\ln_r(x) + \frac{(1-q)^2(\ln_r(x))^3}{24} \leq \exp\left(\frac{(1-q)}{2}\ln_r x\right)\ln_r(x) \leq \ln_r(x).$$

3. *If $0 < q \leq 1$ and $x \geq 1$, then*

$$\ln_r(x) \leq \exp\left(\left(\frac{1-q}{2}\right)\ln_r x\right)\ln_r(x) \leq \exp\left(\left(\frac{1-q}{2}\right)\ln_r x\right)\ln_r(x) + \frac{(1-q)^2(\ln_r(x))^3}{24}$$
$$\leq \ln_{r,q}(x) \leq \frac{\exp((1-q)\ln_r x) + 1}{2}\ln_r(x) - \frac{(1-q)^2(\ln_r(x))^3}{12}$$
$$\leq \frac{\exp((1-q)\ln_r x) + 1}{2}\ln_r(x) \leq \exp((1-q)\ln_r x)\ln_r(x).$$

4. *If $x, q \geq 1$, then*

$$\exp((1-q)\ln_r x)\ln_r(x) \leq \exp\left(\left(\frac{1-q}{2}\right)\ln_r x\right)\ln_r(x)$$
$$\leq \exp\left(\left(\frac{1-q}{2}\right)\ln_r x\right)\ln_r(x) + \frac{(1-q)^2(\ln_r(x))^3}{24}\exp((1-q)\ln_r x) \leq \ln_{r,q}(x)$$
$$\leq \frac{\exp((1-q)\ln_r x) + 1}{2}\ln_r(x) - \frac{(1-q)^2(\ln_r(x))^3 \exp((1-q)\ln_r x)}{12}$$
$$\leq \frac{\exp((1-q)\ln_r x) + 1}{2}\ln_r(x) \leq \ln_r(x).$$

Proof. Let $r \neq 1$, $0 < x, q < 1$, then $0 < \exp(\ln_r(x)) < 1$. By the use of Theorem 2.2 (replace x by $\exp(\ln_r(x))$ in (1)), we have

$$\begin{aligned}
&\ln(\exp(\ln_r(x))) \leq \frac{(\exp(\ln_r(x)))^{1-q}+1}{2}\ln(\exp(\ln_r(x)))\\
&\leq \frac{(\exp(\ln_r(x)))^{1-q}+1}{2}\ln(\exp(\ln_r(x))) - \frac{(1-q)^2(\ln(\exp(\ln_r(x))))^3(\exp(\ln_r(x)))^{1-q}}{12}\\
&\leq \ln_q(\exp(\ln_r(x))) \leq (\exp(\ln_r(x)))^{\frac{1-q}{2}}\ln(\exp(\ln_r(x))) + \frac{(1-q)^2(\ln(\exp(\ln_r(x))))^3(\exp(\ln_r(x)))^{1-q}}{24}\\
&\leq (\exp(\ln_r(x)))^{\frac{1-q}{2}}\ln(\exp(\ln_r(x))) \leq (\exp(\ln_r(x)))^{1-q}\ln(\exp(\ln_r(x))).
\end{aligned}$$

Since $\ln_r(x) = \ln(\exp(\ln_r(x)))$, $\exp((1-q)\ln_r x) = (\exp(\ln_r(x)))^{1-q}$ and $\ln_{r,q}(x) = \ln_q(\exp(\ln_r(x)))$, the inequalities in (1) hold. Similarly, we show the other cases. □

Theorem 2.12. *Let $\psi : (0,\infty) \longrightarrow \mathbb{R}$ be a continuous and strictly monotonic function. For two positive real numbers $q, r \neq 1$.*

1. *If $0 < q < 1$, then*

$$\begin{aligned}
&I_r^{\psi}(\mathbf{p}) \leq \exp\left(\left(\frac{1-q}{2}\right)I_r^{\psi}(\mathbf{p})\right)I_r^{\psi}(\mathbf{p}) \leq \exp\left(\left(\frac{1-q}{2}\right)I_r^{\psi}(\mathbf{p})\right)I_r^{\psi}(\mathbf{p}) + \frac{(1-q)^2(I_r^{\psi}(\mathbf{p}))^3}{24}\\
&\leq I_{r,q}^{\psi}(\mathbf{p}) \leq \frac{\exp((1-q)I_r^{\psi}(\mathbf{p}))+1}{2}I_r^{\psi}(\mathbf{p}) - \frac{(1-q)^2(I_r^{\psi}(\mathbf{p}))^3}{12}\\
&\leq \frac{\exp((1-q)I_r^{\psi}(\mathbf{p}))+1}{2}I_r^{\psi}(\mathbf{p}) \leq \exp((1-q)I_r^{\psi}(\mathbf{p}))I_r^{\psi}(\mathbf{p}).
\end{aligned}$$

2. *If $q > 1$, then*

$$\begin{aligned}
&\exp((1-q)I_r^{\psi}(\mathbf{p}))I_r^{\psi}(\mathbf{p}) \leq \exp\left(\left(\frac{1-q}{2}\right)I_r^{\psi}(\mathbf{p})\right)I_r^{\psi}(\mathbf{p})\\
&\leq \exp\left(\left(\frac{1-q}{2}\right)I_r^{\psi}(\mathbf{p})\right)I_r^{\psi}(\mathbf{p}) + \frac{(1-q)^2(I_r^{\psi}(\mathbf{p}))^3}{24}\exp((1-q)I_r^{\psi}(\mathbf{p})) \leq I_{r,q}^{\psi}(\mathbf{p})\\
&\leq \frac{\exp((1-q)I_r^{\psi}(\mathbf{p}))+1}{2}I_r^{\psi}(\mathbf{p}) - \frac{(1-q)^2(I_r^{\psi}(\mathbf{p}))^3\exp((1-q)I_r^{\psi}(\mathbf{p}))}{12}\\
&\leq \frac{\exp((1-q)I_r^{\psi}(\mathbf{p}))+1}{2}I_r^{\psi}(\mathbf{p}) \leq I_r^{\psi}(\mathbf{p}).
\end{aligned}$$

Proof. Let $x = \psi^{-1}(\sum_{i=1}^{n} p_i\psi(\frac{r_i}{p_j})) > 1$. In this case since $\ln_r x = I_r^{\psi}(\mathbf{p})$ and $\ln_{r,q} x = I_{r,q}^{\psi}(\mathbf{p})$, the results follow from Lemma 2.11. □

Define $I_q^{\psi}(\mathbf{p} \parallel \mathbf{q}) := -D_q^{\psi}(\mathbf{p} \parallel \mathbf{q}) = \ln_q \psi^{-1}\left(\sum_{i=1}^{n}\psi\left(\frac{q_i}{p_i}\right)\right)$ and $I_{r,q}^{\psi}(\mathbf{p} \parallel \mathbf{q}) := -D_{r,q}^{\psi}(\mathbf{p} \parallel \mathbf{q}) = \ln_{r,q}\psi^{-1}\left(\sum_{i=1}^{n}\psi(\frac{q_i}{p_i})\right)$.

Theorem 2.13. *Let $\psi : (0,\infty) \longrightarrow \mathbb{R}$ be a continuous and strictly monotonic function. For two positive real numbers $q, r \neq 1$. Then*

1. *If* $0 < q < 1$, *then*

$$\begin{aligned}
&I_r^{\psi}(\mathbf{p}\parallel\mathbf{q}) \le \frac{\exp((1-q)I_r^{\psi}(\mathbf{p}\parallel\mathbf{q}))+1}{2} I_r^{\psi}(\mathbf{p}\parallel\mathbf{q}) \\
&\le \frac{\exp((1-q)I_r^{\psi}(\mathbf{p}\parallel\mathbf{q}))+1}{2} I_r^{\psi}(\mathbf{p}\parallel\mathbf{q}) - \frac{(1-q)^2}{12}(I_r^{\psi}(\mathbf{p}\parallel\mathbf{q}))^3 \exp((1-q)I_r^{\psi}(\mathbf{p}\parallel\mathbf{q})) \\
&\le I_{r,q}^{\psi}(\mathbf{p}\parallel\mathbf{q}) \le \exp\left(\left(\frac{1-q}{2}\right) I_r^{\psi}(\mathbf{p}\parallel\mathbf{q})\right) I_r^{\psi}(\mathbf{p}\parallel\mathbf{q}) + \frac{(1-q)^2(I_r^{\psi}(\mathbf{p}\parallel\mathbf{q}))^3 \exp((1-q)I_r^{\psi}(\mathbf{p}\parallel\mathbf{q}))}{24} \\
&\le \exp\left(\left(\frac{1-q}{2}\right) I_r^{\psi}(\mathbf{p}\parallel\mathbf{q})\right) I_r^{\psi}(\mathbf{p}\parallel\mathbf{q}) \le \exp((1-q)I_r^{\psi}(\mathbf{p}\parallel\mathbf{q})) I_r^{\psi}(\mathbf{p}\parallel\mathbf{q}).
\end{aligned}$$

2. *If* $q > 1$, *then*

$$\begin{aligned}
&\exp((1-q)I_r^{\psi}(\mathbf{p}\parallel\mathbf{q})) I_r^{\psi}(\mathbf{p}\parallel\mathbf{q}) \le \frac{\exp((1-q)I_r^{\psi}(\mathbf{p}\parallel\mathbf{q}))+1}{2} I_r^{\psi}(\mathbf{p}\parallel\mathbf{q}) \\
&\le \frac{\exp((1-q)I_r^{\psi}(\mathbf{p}\parallel\mathbf{q}))+1}{2} I_r^{\psi}(\mathbf{p}\parallel\mathbf{q}) - \frac{(1-q)^2(I_r^{\psi}(\mathbf{p}\parallel\mathbf{q}))^3}{12} \\
&\le I_{r,q}^{\psi}(\mathbf{p}\parallel\mathbf{q}) \le \exp\left(\frac{(1-q)}{2} I_r^{\psi}(\mathbf{p}\parallel\mathbf{q})\right) I_r^{\psi}(\mathbf{p}\parallel\mathbf{q}) + \frac{(1-q)^2(I_r^{\psi}(\mathbf{p}\parallel\mathbf{q}))^3}{24} \\
&\le \exp\left(\frac{(1-q)}{2} I_r^{\psi}(\mathbf{p}\parallel\mathbf{q})\right) I_r^{\psi}(\mathbf{p}\parallel\mathbf{q}) \le I_r^{\psi}(\mathbf{p}\parallel\mathbf{q}).
\end{aligned}$$

Proof. Let $0 < x = \psi^{-1}\left(\sum_{i=1}^{n} p_i \psi\left(\frac{r_i}{p_j}\right)\right) < 1$. In this case since $\ln_r(x) = I_r^{\psi}(\mathbf{p}\parallel\mathbf{q})$ and $\ln_{r,q}(x) = I_{r,q}^{\psi}(\mathbf{p}\parallel\mathbf{q})$, the results follow from Lemma 2.11. □

Example 2.14. *Let* $\mathbf{q} = \{q_1, ..., q_{1000}\}$ *where*

$$q_i = \begin{cases} 5 \times 10^{-4} & \text{if } 1 \le i \le 500 \\ 15 \times 10^{-4} & \text{if } 500 < i \le 1000 \end{cases}.$$

Then by the use of Corollary 2.5, we have:

1. *If* $0 < q < 1$, *then*

$$\begin{aligned}
&2^{2(1-2q)}.3^{\frac{q}{2}}\left(\frac{4(1-2q)}{1-q}\ln 2 + \frac{q}{1-q}\ln 3\right) + \frac{(1-q)^2}{24}\left(\ln\left(2^{\frac{4(1-2q)}{1-q}}.3^{\frac{q}{1-q}}\right)\right)^3 \\
&\le H_q(\mathbf{q}) \le \frac{1}{2}\ln\left(2^{\frac{4(1-2q)}{1-q}}.3^{\frac{q}{1-q}}\right)[2^{4(1-2q)}.3^q+1)] - \frac{(1-q)^2}{12}\left(\ln\left(2^{\frac{4(1-2q)}{1-q}}.3^{\frac{q}{1-q}}\right)\right)^3.
\end{aligned}$$

2. *If* $q > 1$, *then*

$$\begin{aligned}
&2^{2(1-2q)}.3^{\frac{q}{2}}\ln\left(2^{\frac{4(1-2q)}{1-q}}.3^{\frac{q}{1-q}}\right) + \frac{(1-q)^2}{24}2^{4(1-2q)}.3^q\left(\ln\left(2^{\frac{4(1-2q)}{1-q}}.3^{\frac{q}{1-q}}\right)\right)^3 \le H_q(\mathbf{q}) \\
&\le \frac{1}{2}\ln\left(2^{\frac{4(1-2q)}{1-q}}.3^{\frac{q}{1-q}}\right)[2^{4(1-2q)}.3^q+1] - (1-q)^2\left(\ln\left(2^{\frac{4(1-2q)}{1-q}}.3^{\frac{q}{1-q}}\right)\right)^3.2^{3-8q}.3^{q-1}.
\end{aligned}$$

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