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Some properties of tensorial perspective for convex functions of selfadjoint operators in Hilbert spaces

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Abstract.
Let H be a Hilbert space. Assume that $f : [0, \infty) \to \mathbb{R}$ is continuous and $A, B > 0$. We define the *tensorial perspective* for the function f and the pair of operators (A, B) by

$$\mathcal{P}_{f,\otimes}(A, B) := (1 \otimes B) f\left(A \otimes B^{-1}\right).$$

In this paper we show among others that, if f is differentiable convex, then

$$\mathcal{P}_{f,\otimes}(A, B) \geq \left[f(u) - f'(u)u\right](1 \otimes B) + f'(u)(A \otimes 1),$$

for $A, B > 0$ and $u > 0$. Moreover, if $\operatorname{Sp}(A) \subset I$, $\operatorname{Sp}(B) \subset J$ and such that $0 < \gamma \leq \frac{t}{s} \leq \Gamma$ for $t \in I$ and $s \in J$, then

$$\mathcal{P}_{f,\otimes}(A, B) \leq \left[f(u) - f'(u)u\right](1 \otimes B) + f'(u)(A \otimes 1) + \left[f'_-(\Gamma) - f'_+(\gamma)\right]|A \otimes 1 - u(1 \otimes B)|$$

for $u \in [\gamma, \Gamma]$.

1. Introduction

Let $I_1, ..., I_k$ be intervals from $\mathbb{R}$ and let $f : I_1 \times ... \times I_k \to \mathbb{R}$ be an essentially bounded real function defined on the product of the intervals. Let $A = (A_1, ..., A_n)$ be a k-tuple of bounded selfadjoint operators on Hilbert spaces $H_1, ..., H_k$ such that the spectrum of A_i is contained in I_i for $i = 1, ..., k$. We say that such a k-tuple is in the domain of f. If

$$A_i = \int_{I_i} \lambda_i dE_i(\lambda_i)$$

is the spectral resolution of A_i for $i = 1, ..., k$; by following [1], we define

$$f(A_1, ..., A_k) := \int_{I_1} ... \int_{I_k} f(\lambda_1, ..., \lambda_k)\, dE_1(\lambda_1) \otimes ... \otimes dE_k(\lambda_k) \quad (1)$$

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as a bounded selfadjoint operator on the tensorial product $H_1 \otimes ... \otimes H_k$.

If the Hilbert spaces are of finite dimension, then the above integrals become finite sums, and we may consider the functional calculus for arbitrary real functions. This construction [1] extends the definition of Korányi [3] for functions of two variables and have the property that

$$f(A_1, ..., A_k) = f_1(A_1) \otimes ... \otimes f_k(A_k),$$

whenever f can be separated as a product $f(t_1, ..., t_k) = f_1(t_1)...f_k(t_k)$ of k functions each depending on only one variable.

It is know that, *if f is super-multiplicative (sub-multiplicative)* on $[0, \infty)$, namely

$$f(st) \geq (\leq) f(s) f(t) \text{ for all } s, t \in [0, \infty)$$

and if f is continuous on $[0, \infty)$, then [5, p. 173]

$$f(A \otimes B) \geq (\leq) f(A) \otimes f(B) \text{ for all } A,\ B \geq 0. \tag{2}$$

This follows by observing that, if

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s)$$

are the spectral resolutions of A and B, then

$$f(A \otimes B) = \int_{[0,\infty)} \int_{[0,\infty)} f(st) dE(t) \otimes dF(s) \tag{3}$$

for the continuous function f on $[0, \infty)$.

Recall the *geometric operator mean* for the positive operators $A, B > 0$

$$A \#_t B := A^{1/2}(A^{-1/2} B A^{-1/2})^t A^{1/2},$$

where $t \in [0, 1]$ and

$$A \# B := A^{1/2}(A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}.$$

By the definitions of # and $\otimes$ we have

$$A \# B = B \# A \text{ and } (A \# B) \otimes (B \# A) = (A \otimes B) \# (B \otimes A).$$

In 2007, S. Wada [7] obtained the following *Callebaut type inequalities* for tensorial product

$$\begin{aligned} (A \# B) \otimes (A \# B) &\leq \frac{1}{2} [(A \#_\alpha B) \otimes (A \#_{1-\alpha} B) + (A \#_{1-\alpha} B) \otimes (A \#_\alpha B)] \\ &\leq \frac{1}{2} (A \otimes B + B \otimes A) \end{aligned} \tag{4}$$

for $A, B > 0$ and $\alpha \in [0, 1]$.

Assume that $f : [0, \infty) \to \mathbb{R}$ is continuous and $A, B > 0$. We define the *tensorial perspective* for the function f and the pair of operators (A, B)

$$\mathcal{P}_{f,\otimes}(A, B) := (1 \otimes B) f\left(A \otimes B^{-1}\right).$$

Motivated by the above results, in this paper we show among others that, if f is differentiable convex, then

$$\mathcal{P}_{f,\otimes}(A, B) \geq [f(u) - f'(u) u] (1 \otimes B) + f'(u) (A \otimes 1),$$

for $A, B > 0$ and $u > 0$. Moreover, if $\operatorname{Sp}(A) \subset I$, $\operatorname{Sp}(B) \subset J$ and such that $0 < \gamma \leq \frac{t}{s} \leq \Gamma$ for $t \in I$ and $s \in J$, then

$$\begin{aligned} \mathcal{P}_{f,\otimes}(A, B) &\leq [f(u) - f'(u) u] (1 \otimes B) + f'(u) (A \otimes 1) \\ &\quad + [f'_-(\Gamma) - f'_+(\gamma)] |A \otimes 1 - u (1 \otimes B)| \end{aligned}$$

for $u \in [\gamma, \Gamma]$.

2. Some Preliminary Facts

Recall the following property of the tensorial product

$$(AC) \otimes (BD) = (A \otimes B)(C \otimes D) \tag{5}$$

that holds for any $A, B, C, D \in B(H)$.

If we take $C = A$ and $D = B$, then we get

$$A^2 \otimes B^2 = (A \otimes B)^2 .$$

By induction and using (5) we derive that

$$A^n \otimes B^n = (A \otimes B)^n \text{ for natural } n \geq 0. \tag{6}$$

In particular

$$A^n \otimes 1 = (A \otimes 1)^n \text{ and } 1 \otimes B^n = (1 \otimes B)^n \tag{7}$$

for all $n \geq 0$.

We also observe that, by (5), the operators $A \otimes 1$ and $1 \otimes B$ are commutative and

$$(A \otimes 1)(1 \otimes B) = (1 \otimes B)(A \otimes 1) = A \otimes B. \tag{8}$$

Moreover, for two natural numbers m, n we have

$$(A \otimes 1)^m (1 \otimes B)^n = (1 \otimes B)^n (A \otimes 1)^m = A^m \otimes B^n. \tag{9}$$

According with the properties of tensorial products and functional calculus for continuous functions of selfadjoint operators, we have

$$\begin{aligned}\mathcal{P}_{f,\otimes}(A, B) &= (1 \otimes B) f\left((A \otimes 1)(1 \otimes B)^{-1}\right) \\ &= f\left((A \otimes 1)(1 \otimes B)^{-1}\right)(1 \otimes B) \\ &= f\left((1 \otimes B)^{-1}(A \otimes 1)\right)(1 \otimes B),\end{aligned}$$

due to the commutativity of $A \otimes 1$ and $1 \otimes B$.

In the following, we consider the spectral resolutions of A and B given by

$$A = \int_{[0,\infty)} t dE(t) \text{ and } B = \int_{[0,\infty)} s dF(s). \tag{10}$$

We have the following representation result for continuous functions:

Lemma 2.1. *Assume that $f : [0, \infty) \to \mathbb{R}$ is continuous and $A, B > 0$, then*

$$\mathcal{P}_{f,\otimes}(A, B) = \int_{[0,\infty)} \int_{[0,\infty)} s f\left(\frac{t}{s}\right) dE(t) \otimes dF(s). \tag{11}$$

Proof. By Stone-Weierstrass theorem, any continuous function can be approximated by a sequence of polynomials, therefore it suffices to prove the equality for the power function $\varphi(t) = t^n$ with n any natural number.

We have that

$$\begin{aligned}
&\int_{[0,\infty)}\int_{[0,\infty)} s\varphi\left(\frac{t}{s}\right) dE(t)\otimes dF(s)\\
&=\int_{[0,\infty)}\int_{[0,\infty)} s\left(\frac{t}{s}\right)^{n} dE(t)\otimes dF(s)\\
&=\int_{[0,\infty)}\int_{[0,\infty)} t^{n}s^{1-n} dE(t)\otimes dF(s)\\
&=A^{n}\otimes B^{1-n}=A^{n}\otimes BB^{-n}=(1\otimes B)\left(A^{n}\otimes B^{-n}\right)\\
&=(1\otimes B)\left(A\otimes B^{-1}\right)^{n}=\mathcal{P}_{\varphi,\otimes}(A,B),
\end{aligned}$$

which shows that (11) holds for the power function.

This proves the lemma. □

We assume in the following that $A, B > 0$.

If we consider the function $\Pi_r(u) = u^r - 1$, $u \geq 0$, $r > 0$, then we have

$$\begin{aligned}
\mathcal{P}_{\Pi_{r,\otimes}}(A,B) &:= (1\otimes B)\,\Pi_r\left(A\otimes B^{-1}\right)\\
&= (1\otimes B)\left[\left(A\otimes B^{-1}\right)^{r}-1\right]\\
&= (A\otimes 1)^{r}(1\otimes B)^{1-r}-1\otimes B.
\end{aligned}$$

If we take $f = -\ln(\cdot)$, then we get

$$\begin{aligned}
\mathcal{P}_{-\ln(\cdot),\otimes}(A,B) &:= -(1\otimes B)\ln\left(A\otimes B^{-1}\right)\\
&= -\ln\left((1\otimes B)^{-1}(A\otimes 1)\right)(1\otimes B)\\
&= (1\otimes B)\left[\ln(1\otimes B)-\ln(A\otimes 1)\right].
\end{aligned}$$

If we take $f = (\cdot)\ln(\cdot)$, then we get

$$\begin{aligned}
\mathcal{P}_{(\cdot)\ln(\cdot),\otimes}(A,B) &:= (1\otimes B)\left(A\otimes B^{-1}\right)\ln\left(A\otimes B^{-1}\right)\\
&= (A\otimes 1)\left[\ln(A\otimes 1)-\ln(1\otimes B)\right].
\end{aligned}$$

If we take $f = |\cdot - \alpha|$, $\alpha \in \mathbb{R}$, then

$$\begin{aligned}
\mathcal{P}_{|\cdot-\alpha|,\otimes}(A,B) &= \int_{[0,\infty)}\int_{[0,\infty)} s\left|\frac{t}{s}-\alpha\right| dE(t)\otimes dF(s)\\
&= \int_{[0,\infty)}\int_{[0,\infty)} |t-\alpha s|\, dE(t)\otimes dF(s)\\
&= |A\otimes 1-\alpha 1\otimes B|,
\end{aligned}$$

where for the last equality we used the result obtained in [2],

$$\psi(h(A)\otimes 1+1\otimes k(B))=\int_I\int_J \psi(h(t)+k(s))\,dE(t)\otimes dF(s), \tag{12}$$

here A and B are selfadjoint operators with $\mathrm{Sp}(A) \subset I$ and $\mathrm{Sp}(B) \subset J$, h is continuous on I, k is continuous on J and ψ is continuous on an interval U that contains the sum of the intervals $h(I) + k(J)$, while A and B have the spectral resolutions

$$A=\int_I t\,dE(t) \text{ and } B=\int_J s\,dF(s).$$

For $f = |\cdot - 1|$ we get

$$\mathcal{P}_{|\cdot-1|,\otimes}(A,B) = |A \otimes 1 - 1 \otimes B|.$$

Consider the q-logarithm defined by

$$\ln_q u = \begin{cases} \frac{u^{1-q}-1}{1-q} \text{ if } q \neq 1, \\ \\ \ln u \text{ if } q = 1. \end{cases}$$

For $q \neq 1$ we define

$$\mathcal{P}_{\ln_q,\otimes}(A,B) := (1 \otimes B) \ln_q \left((A \otimes 1)(1 \otimes B)^{-1} \right) = \frac{(A \otimes 1)^{1-q} (1 \otimes B)^q - 1 \otimes B}{1-q}. \tag{13}$$

Let f be a continuous function defined on the interval I of real numbers, B a selfadjoint operator on the Hilbert space H and A a positive invertible operator on H. Assume that the spectrum $\operatorname{Sp}\left(A^{-1/2}BA^{-1/2}\right) \subset \mathring{I}$. Then by using the continuous functional calculus, we can define the *perspective* $\mathcal{P}_f(B,A)$ by setting

$$\mathcal{P}_f(B,A) := A^{1/2} f\left(A^{-1/2}BA^{-1/2}\right) A^{1/2}.$$

If A and B are commutative, then

$$\mathcal{P}_f(B,A) = Af\left(BA^{-1}\right)$$

provided $\operatorname{Sp}\left(BA^{-1}\right) \subset \mathring{I}$.

It is well known that (see for instance [4]), if f is an *operator convex function* defined in the positive half-line, then the mapping

$$(B,A) \mapsto \mathcal{P}_f(B,A)$$

defined in pairs of positive definite operators, is operator convex.

The following inequality is also of interest, see [6]:

Theorem 2.2. *Assume that f is nonnegative and operator monotone on $[0,\infty)$. If $A \geq C > 0$ and $B \geq D > 0$, then*

$$\mathcal{P}_f(A,B) \geq \mathcal{P}_f(C,D). \tag{14}$$

We can state the following result for the tensorial perspective:

Theorem 2.3. *If f is an operator convex function defined in the positive half-line, then $\mathcal{P}_{f,\otimes}$ is operator convex in pairs of positive definite operators as well. If $A \geq C > 0$ and $B \geq D > 0$, then also*

$$\mathcal{P}_{f,\otimes}(A,B) \geq \mathcal{P}_{f,\otimes}(C,D). \tag{15}$$

Proof. Assume f is an *operator convex function* in the positive half-line. Since $A \otimes 1$ and $1 \otimes B$ are commutative, hence

$$\mathcal{P}_{f,\otimes}(A,B) = (1 \otimes B) f\left((A \otimes 1)(1 \otimes B)^{-1}\right) = \mathcal{P}_f(A \otimes 1, 1 \otimes B) \tag{16}$$

for $A, B > 0$.

If $A, B, C, D > 0$ and $\lambda \in [0,1]$, then we have

$$\begin{aligned}
&\mathcal{P}_{f,\otimes}((1-\lambda)(A,B) + \lambda(C,D)) \\
&= \mathcal{P}_{f,\otimes}(((1-\lambda)A + \lambda C, (1-\lambda)B + \lambda D)) \\
&= \mathcal{P}_{f}(((1-\lambda)A + \lambda C) \otimes 1, 1 \otimes ((1-\lambda)B + \lambda D)) \\
&= \mathcal{P}_{f}((1-\lambda)A \otimes 1 + \lambda C \otimes 1, (1-\lambda)1 \otimes B + \lambda 1 \otimes D) \\
&= \mathcal{P}_{f}((1-\lambda)(A \otimes 1, 1 \otimes B) + \lambda(C \otimes 1, 1 \otimes D)) \\
&\le (1-\lambda)\mathcal{P}_{f}(A \otimes 1, 1 \otimes B) + \lambda \mathcal{P}_{f}(C \otimes 1, 1 \otimes D) \\
&= (1-\lambda)\mathcal{P}_{f,\otimes}(A,B) + \lambda \mathcal{P}_{f,\otimes}(C,D),
\end{aligned}$$

which shows that $\mathcal{P}_{f,\otimes}$ is operator convex in pairs of positive definite operators.

If $A \ge C > 0$ and $B \ge D > 0$, then $A \otimes 1 \ge C \otimes 1 > 0$ and $1 \otimes B \ge 1 \otimes D > 0$. By utilizing Theorem 2.2 we derive that

$$\mathcal{P}_{f}(A \otimes 1, 1 \otimes B) \ge \mathcal{P}_{f}(C \otimes 1, 1 \otimes D).$$

By utilizing the representation (16) we derive the desired result (15). □

3. Main Results

Suppose that I is an interval of real numbers with interior $\mathring{I}$ and $f : I \to \mathbb{R}$ is a convex function on I. Then f is continuous on $\mathring{I}$ and has finite left and right derivatives at each point of $\mathring{I}$. Moreover, if $x, y \in \mathring{I}$ and $x < y$, then $f'_-(x) \le f'_+(x) \le f'_-(y) \le f'_+(y)$ which shows that both f'_- and f'_+ are nondecreasing function on $\mathring{I}$. It is also known that a convex function must be differentiable except for at most countably many points.

For a convex function $f : I \to \mathbb{R}$, the subdifferential of f denoted by ∂f is the set of all functions $\varphi : I \to [-\infty, \infty]$ such that $\varphi\left(\mathring{I}\right) \subset \mathbb{R}$ and

$$f(x) \ge f(a) + (x-a)\varphi(a) \text{ for any } x, a \in I. \tag{17}$$

It is also well known that if f is convex on I, then ∂f is nonempty, $f'_-, f'_+ \in \partial f$ and if $\varphi \in \partial f$, then

$$f'_-(x) \le \varphi(x) \le f'_+(x) \text{ for any } x \in \mathring{I}.$$

In particular, φ is a nondecreasing function.

If f is differentiable and convex on $\mathring{I}$, then $\partial f = \{f'\}$.

Theorem 3.1. *Assume that f is convex on $(0,\infty)$, $A, B > 0$ and $u \in (0,\infty)$ while $\varphi \in \partial f$, then*

$$\mathcal{P}_{f,\otimes}(A,B) \ge [f(u) - \varphi(u)u](1 \otimes B) + \varphi(u)(A \otimes 1). \tag{18}$$

Moreover, if f is differentiable, then

$$\mathcal{P}_{f,\otimes}(A,B) \ge [f(u) - f'(u)u](1 \otimes B) + f'(u)(A \otimes 1), \tag{19}$$

for all $A, B > 0$ and $u \in (0,\infty)$.

Proof. By the gradient inequality we have

$$f(x) \ge f(u) + (x-u)\varphi(u) \tag{20}$$

for all $x, u \in (0,\infty)$.

If we take $x=\frac{t}{s}$ in (20), then we get

$$f\left(\frac{t}{s}\right)\geq f(u)+\left(\frac{t}{s}-u\right)\varphi(u) \tag{21}$$

for all $t,s>0$.

If we multiply (21) by $s>0$, then we get

$$sf\left(\frac{t}{s}\right)\geq sf(u)+\varphi(u)(t-us) \tag{22}$$

for all $t,s>0$.

We consider the spectral resolutions of A and B given by

$$A=\int_{[0,\infty)}tdE(t) \text{ and } B=\int_{[0,\infty)}sdF(s).$$

If we take in (22) the integral $\int_{[0,\infty)}\int_{[0,\infty)}$ over $dE(t)\otimes dF(s)$, then we get

$$\begin{aligned}
&\int_{[0,\infty)}\int_{[0,\infty)}sf\left(\frac{t}{s}\right)dE(t)\otimes dF(s)\\
&\geq\int_{[0,\infty)}\int_{[0,\infty)}\left[sf(u)+\varphi(u)(t-us)\right]dE(t)\otimes dF(s)\\
&=f(u)\int_{[0,\infty)}\int_{[0,\infty)}sdE(t)\otimes dF(s)\\
&+\varphi(u)\left[\int_{[0,\infty)}\int_{[0,\infty)}tdE(t)\otimes dF(s)-u\int_{[0,\infty)}\int_{[0,\infty)}sdE(t)\otimes dF(s)\right]\\
&=f(u)(1\otimes B)+\varphi(u)(A\otimes 1-u1\otimes B)
\end{aligned}$$

and by the representation (11) we get the desired inequality (18). □

Corollary 3.2. *With the assumptions of Theorem 3.1 and for $x,y\in H$ with $\|x\|=\|y\|=1$, we have*

$$\left\langle\mathcal{P}_{f,\otimes}(A,B)(x\otimes y),x\otimes y\right\rangle\geq\left[f(u)-\varphi(u)u\right]\langle By,y\rangle+\varphi(u)\langle Ax,x\rangle, \tag{23}$$

for all $u>0$.

If f is differentiable, then

$$\left\langle\mathcal{P}_{f,\otimes}(A,B)(x\otimes y),x\otimes y\right\rangle\geq\left[f(u)-f'(u)u\right]\langle By,y\rangle+f'(u)\langle Ax,x\rangle. \tag{24}$$

In particular, if we take $u=\frac{\langle Ax,x\rangle}{\langle By,y\rangle}$ in (23) then we get the Jensen's type inequality of interest

$$\frac{\left\langle\mathcal{P}_{f,\otimes}(A,B)(x\otimes y),x\otimes y\right\rangle}{\langle By,y\rangle}\geq f\left(\frac{\langle Ax,x\rangle}{\langle By,y\rangle}\right). \tag{25}$$

Proof. If we take the tensorial inner product over $x\otimes y$ in (18), then we get

$$\begin{aligned}
&\left\langle\mathcal{P}_{f,\otimes}(A,B)(x\otimes y),x\otimes y\right\rangle\\
&\geq f(u)\langle(1\otimes B)(x\otimes y),x\otimes y\rangle\\
&+\varphi(u)\langle(A\otimes 1-u1\otimes B)(x\otimes y),x\otimes y\rangle\\
&=f(u)\langle(1\otimes B)(x\otimes y),x\otimes y\rangle\\
&+\varphi(u)\left[\langle(A\otimes 1)(x\otimes y),x\otimes y\rangle-u\langle 1\otimes B(x\otimes y),x\otimes y\rangle\right].
\end{aligned} \tag{26}$$

Observe that for $x, y \in H$ with $\|x\| = \|y\| = 1$, we have

$$\begin{aligned}\langle (1 \otimes B)(x \otimes y), x \otimes y \rangle &= \langle (1x \otimes By), x \otimes y \rangle \\ &= \langle 1x, x \rangle \langle By, y \rangle = \|x\|^2 \langle By, y \rangle = \langle By, y \rangle\end{aligned}$$

and

$$\begin{aligned}\langle (A \otimes 1)(x \otimes y), x \otimes y \rangle &= \langle Ax \otimes 1y, x \otimes y \rangle \\ &= \langle Ax, x \rangle \langle 1y, y \rangle = \langle Ax, x \rangle \|y\|^2 = \langle Ax, x \rangle\end{aligned}$$

and by (26) we deduce (23).

If we take $u = \frac{\langle Ax,x \rangle}{\langle By,y \rangle}$ in (23), then we get

$$\begin{aligned}&\left\langle \mathcal{P}_{f,\otimes}(A, B)(x \otimes y), x \otimes y \right\rangle \\ &\geq \left[f\left(\frac{\langle Ax, x \rangle}{\langle By, y \rangle} \right) - \varphi\left(\frac{\langle Ax, x \rangle}{\langle By, y \rangle} \right) \frac{\langle Ax, x \rangle}{\langle By, y \rangle} \right] \langle By, y \rangle \\ &+ \varphi\left(\frac{\langle Ax, x \rangle}{\langle By, y \rangle} \right) \langle Ax, x \rangle \\ &= f\left(\frac{\langle Ax, x \rangle}{\langle By, y \rangle} \right) \langle By, y \rangle,\end{aligned}$$

which gives (25). □

Corollary 3.3. *Assume that f is convex on $(0, \infty)$, $0 < m \leq A, B \leq M$ for some constants m, M and $\varphi \in \partial f$, then*

$$\begin{aligned}\mathcal{P}_{f,\otimes}(A, B) \geq& \left[f\left(\frac{m+M}{2} \right) - \varphi\left(\frac{m+M}{2} \right) \frac{m+M}{2} \right] (1 \otimes B) \\ &+ \varphi\left(\frac{m+M}{2} \right) (A \otimes 1)\end{aligned} \tag{27}$$

and, if f is differentiable,

$$\begin{aligned}\mathcal{P}_{f,\otimes}(A, B) \geq& \left[f\left(\frac{m+M}{2} \right) - f'\left(\frac{m+M}{2} \right) \frac{m+M}{2} \right] (1 \otimes B) \\ &+ f'\left(\frac{m+M}{2} \right) (A \otimes 1).\end{aligned} \tag{28}$$

Also

$$\begin{aligned}\mathcal{P}_{f,\otimes}(A, B) \geq& \left(\frac{f(M) - f(m)}{M - m} \right) (A \otimes 1) \\ &+ \left(\frac{2}{M - m} \int_m^M f(u)\, du - \frac{Mf(M) - mf(m)}{M - m} \right) (1 \otimes B).\end{aligned} \tag{29}$$

Proof. If we take the integral mean in (18), then we get

$$\begin{aligned}\mathcal{P}_{f,\otimes}(A, B) \geq& \left(\frac{1}{M - m} \int_m^M f(u)\, du \right) (1 \otimes B) \\ &+ \left(\frac{1}{M - m} \int_m^M \varphi(u)\, du \right) (A \otimes 1) \\ &- \left(\frac{1}{M - m} \int_m^M \varphi(u)\, u du \right) (1 \otimes B).\end{aligned} \tag{30}$$

Observe that, since $\varphi \in \partial\Phi$, hence

$$\frac{1}{M-m}\int_m^M \varphi(u)\,du = \frac{f(M)-f(m)}{M-m}$$

and

$$\begin{aligned}\frac{1}{M-m}\int_m^M u\varphi(u)\,du &= \frac{1}{M-m}\left[uf(u)\Big|_m^M - \int_m^M f(u)\,du\right]\\ &= \frac{Mf(M)-mf(m)}{M-m} - \frac{1}{M-m}\int_m^M f(u)\,du.\end{aligned}$$

Therefore

$$\begin{aligned}&\left(\frac{1}{M-m}\int_m^M f(u)\,du\right)(1\otimes B) + \left(\frac{1}{M-m}\int_m^M \varphi(u)\,du\right)(A\otimes 1)\\ &-\left(\frac{1}{M-m}\int_m^M \varphi(u)\,udu\right)(1\otimes B).\\ &=\left(\frac{f(M)-f(m)}{M-m}\right)(A\otimes 1)\\ &+\left(\frac{2}{M-m}\int_m^M f(u)\,du - \frac{Mf(M)-mf(m)}{M-m}\right)(1\otimes B)\end{aligned}$$

and by (30) we obtain (29). □

Theorem 3.4. *Assume that f is continuously differentiable convex on $(0,\infty)$, $A, B > 0$ and $u \in (0,\infty)$, then*

$$\mathcal{P}_{f,\otimes}(A,B) \le f(u)(1\otimes B) + \mathcal{P}^{\dagger}_{f',\otimes}(A,B) - u\mathcal{P}_{f',\otimes}(A,B), \tag{31}$$

where for a continuous function g on $(0,\infty)$,

$$\begin{aligned}\mathcal{P}^{\dagger}_{g,\otimes}(A,B) &:= \int_{[0,\infty)}\int_{[0,\infty)} tg\left(\frac{t}{s}\right)dE(t)\otimes dF(s) \\ &= (A\otimes 1)\,g\left(A\otimes B^{-1}\right)\\ &= (A\otimes 1)\,g\left((A\otimes 1)(1\otimes B)^{-1}\right).\end{aligned} \tag{32}$$

Proof. By the gradient inequality we have

$$f(x) \le f(u) + (x-u)f'(x) \tag{33}$$

for all $x, u \in (0,\infty)$.

If we take $x = \frac{t}{s}$ in (33) and multiply with s, then we get

$$sf\left(\frac{t}{s}\right) \le sf(u) + tf'\left(\frac{t}{s}\right) - usf'\left(\frac{t}{s}\right) \tag{34}$$

for all $t, s \in (0,\infty)$.

We consider the spectral resolutions of A and B given by

$$A = \int_{[0,\infty)} tdE(t) \text{ and } B = \int_{[0,\infty)} sdF(s).$$

If we take in (34) the integral $\int_{[0,\infty)}\int_{[0,\infty)}$ over $dE(t)\otimes dF(s)$, then we get

$$\begin{aligned}&\int_{[0,\infty)}\int_{[0,\infty)} sf\left(\frac{t}{s}\right)dE(t)\otimes dF(s) \\ &\le f(u)\int_{[0,\infty)}\int_{[0,\infty)} s dE(t)\otimes dF(s) \\ &+\int_{[0,\infty)}\int_{[0,\infty)} tf'\left(\frac{t}{s}\right)dE(t)\otimes dF(s) \\ &-u\int_{[0,\infty)}\int_{[0,\infty)} sf'\left(\frac{t}{s}\right)dE(t)\otimes dF(s),\end{aligned} \tag{35}$$

which gives the desired inequality (31). □

Corollary 3.5. *With the assumptions of Theorem 3.4 and for $x, y \in H$ with $\|x\| = \|y\| = 1$, we have*

$$\begin{aligned}&\left\langle \mathcal{P}_{f,\otimes}(A,B)(x\otimes y), x\otimes y\right\rangle \\ &\le f(u)\langle By,y\rangle + \left\langle \mathcal{P}^{\dagger}_{f',\otimes}(A,B)(x\otimes y), x\otimes y\right\rangle \\ &-u\left\langle \mathcal{P}_{f',\otimes}(A,B)(x\otimes y), x\otimes y\right\rangle,\end{aligned} \tag{36}$$

for all $u > 0$.

In particular, if we take $u = \frac{\langle Ax,x\rangle}{\langle By,y\rangle}$ in (23) then we get the Jensen's type inequality of interest

$$\begin{aligned}0 &\le \frac{\left\langle \mathcal{P}_{f,\otimes}(A,B)(x\otimes y), x\otimes y\right\rangle}{\langle By,y\rangle} - f\left(\frac{\langle Ax,x\rangle}{\langle By,y\rangle}\right) \\ &\le \frac{\left\langle \mathcal{P}^{\dagger}_{f',\otimes}(A,B)(x\otimes y), x\otimes y\right\rangle}{\langle By,y\rangle} \\ &- \frac{\langle Ax,x\rangle}{\langle By,y\rangle^2}\left\langle \mathcal{P}_{f',\otimes}(A,B)(x\otimes y), x\otimes y\right\rangle.\end{aligned} \tag{37}$$

Corollary 3.6. *With the assumptions of Theorem 3.4 and if $0 < m \le A, B \le M$ for some constants m, M, then*

$$\begin{aligned}\mathcal{P}_{f,\otimes}(A,B) &\le f\left(\frac{m+M}{2}\right)(1\otimes B) + \mathcal{P}^{\dagger}_{f',\otimes}(A,B) \\ &- \frac{m+M}{2}\mathcal{P}_{f',\otimes}(A,B)\end{aligned} \tag{38}$$

and

$$\begin{aligned}\mathcal{P}_{f,\otimes}(A,B) &\le \left(\frac{1}{M-m}\int_m^M f(u)\,du\right)(1\otimes B) \\ &+ \mathcal{P}^{\dagger}_{f',\otimes}(A,B) - \frac{m+M}{2}\mathcal{P}_{f',\otimes}(A,B).\end{aligned} \tag{39}$$

We also have:

Theorem 3.7. *Assume that f is convex on $(0,\infty)$, $A, B > 0$ with spectra $\mathrm{Sp}(A) \subset I$, $\mathrm{Sp}(B) \subset J$ and such that $0 < \gamma \le \frac{t}{s} \le \Gamma$ for $t \in I$ and $s \in J$, then*

$$\begin{aligned}\mathcal{P}_{f,\otimes}(A,B) &\le [f(u) - u\varphi(u)](1\otimes B) + \varphi(u)(A\otimes 1) \\ &+ [f'_-(\Gamma) - f'_+(\gamma)]\,|A\otimes 1 - u(1\otimes B)|\end{aligned} \tag{40}$$

for $u \in [\gamma,\Gamma]$ and $\varphi \in \partial f$.

Proof. Observe that, by the gradient inequality we have

$$\begin{aligned} f(x) &\le f(u) + (x-u)\varphi(x) \\ &= f(u) + (x-u)\varphi(u) + (x-u)[\varphi(x) - \varphi(u)] \end{aligned} \tag{41}$$

for $x, u > 0$ and $\varphi \in \partial f$.

Since φ is monotonic nondrecreasing, then

$$\begin{aligned} 0 &\le (f'(x) - f'(u))(x-u) = \left|(f'(x) - f'(u))(x-u)\right| \\ &= \left|f'(x) - f'(u)\right| |x-u| \le [f'_-(\Gamma) - f'_+(\gamma)] |x-u|, \end{aligned}$$

for $x, u \in [\gamma, \Gamma]$ and by (41)

$$f(x) \le f(u) + (x-u)\varphi(u) + [f'_-(\Gamma) - f'_+(\gamma)] |x-u| \tag{42}$$

for $x, u \in [\gamma, \Gamma]$.

If we take in (42) $x = \frac{t}{s}$ and multiply with s, then we get

$$sf\left(\frac{t}{s}\right) \le sf(u) + (t-us)\varphi(u) + [f'_-(\Gamma) - f'_+(\gamma)] |t-us| \tag{43}$$

for $t, s > 0$ with $\frac{t}{s}, u \in [\gamma, \Gamma]$.

We consider the spectral resolutions of A and B given by

$$A = \int_I t dE(t) \text{ and } B = \int_J s dF(s).$$

If we take in (34) the integral $\int_I \int_J$ over $dE(t) \otimes dF(s)$, then we get

$$\begin{aligned} &\int_I \int_J sf\left(\frac{t}{s}\right) dE(t) \otimes dF(s) \\ &\le f(u) \int_I \int_J s dE(t) \otimes dF(s) + \varphi(u) \int_I \int_J (t-us) dE(t) \otimes dF(s) \\ &+ [f'_-(\Gamma) - f'_+(\gamma)] \int_I \int_J |t-us| dE(t) \otimes dF(s), \end{aligned}$$

which, as above, gives the desired result (40). □

Corollary 3.8. *With the assumptions of Theorem 3.7 and for $x, y \in H$ with $\|x\| = \|y\| = 1$, we have*

$$\begin{aligned} &\left\langle \mathcal{P}_{f,\otimes}(A,B)(x \otimes y), x \otimes y \right\rangle \\ &\le [f(u) - u\varphi(u)] \langle By, y\rangle + \langle Ax, x\rangle \varphi(u) \\ &+ [f'_-(\Gamma) - f'_+(\gamma)] \langle |A \otimes 1 - u(1 \otimes B)| (x \otimes y), x \otimes y\rangle \end{aligned} \tag{44}$$

for all $u \in [\gamma, \Gamma]$.

In particular, if we take $u = \frac{\langle Ax,x\rangle}{\langle By,y\rangle} \in [\gamma, \Gamma]$ in (44) then we get the reverse of Jensen's inequality

$$\begin{aligned} 0 &\le \frac{\left\langle \mathcal{P}_{f,\otimes}(A,B)(x \otimes y), x \otimes y \right\rangle}{\langle By, y\rangle} - f\left(\frac{\langle Ax, x\rangle}{\langle By, y\rangle}\right) \\ &\le [f'_-(\Gamma) - f'_+(\gamma)] \\ &\times \left\langle \frac{1}{\langle By, y\rangle} \left| A \otimes 1 - \frac{\langle Ax, x\rangle}{\langle By, y\rangle}(1 \otimes B)\right| (x \otimes y), x \otimes y \right\rangle. \end{aligned} \tag{45}$$

4. Some Examples

Consider the function $\Pi_r(u) = u^r - 1$, $u \geq 0$, $r \geq 1$, then by (19) we get

$$\mathcal{P}_{\Pi_r,\otimes}(A,B) \geq r u^{r-1}(A \otimes 1) - [(r-1)u^r + 1](1 \otimes B), \tag{46}$$

for $A, B > 0$ and $u > 0$.

If there exist the constants m_1, M_1, m_2 and M_2 with

$$0 < m_1 \leq A \leq M_1, m_2 \leq B \leq M_2, \tag{47}$$

then we can take in Theorem 3.7 $\gamma = \frac{m_1}{M_2}$ and $\Gamma = \frac{M_1}{m_2}$ and from (40) we derive

$$\begin{aligned}\mathcal{P}_{\Pi_r,\otimes}(A,B) &\leq A \otimes 1 - [(r-1)u^r + 1](1 \otimes B) \\ &\quad + r\left(\left(\frac{M_1}{m_2}\right)^{r-1} - \left(\frac{m_1}{M_2}\right)^{r-1}\right)|A \otimes 1 - u(1 \otimes B)|.\end{aligned} \tag{48}$$

For $x, y \in H$ with $\|x\| = \|y\| = 1$, we have by (25) that

$$\frac{\langle \mathcal{P}_{\Pi_r,\otimes}(A,B)(x \otimes y), x \otimes y \rangle}{\langle By, y \rangle} \geq \left(\frac{\langle Ax, x \rangle}{\langle By, y \rangle}\right)^r - 1 \tag{49}$$

for $A, B > 0$.

If the condition (47) is satisfied, then by (45) we get

$$\begin{aligned}0 &\leq \frac{\left\langle \mathcal{P}_{f,\otimes}(A,B)(x \otimes y), x \otimes y \right\rangle}{\langle By, y \rangle} - \left(\frac{\langle Ax, x \rangle}{\langle By, y \rangle}\right)^r + 1 \\ &\leq r\left(\left(\frac{M_1}{m_2}\right)^{r-1} - \left(\frac{m_1}{M_2}\right)^{r-1}\right) \\ &\times \left\langle \frac{1}{\langle By, y \rangle}\left|A \otimes 1 - \frac{\langle Ax, x \rangle}{\langle By, y \rangle}(1 \otimes B)\right|(x \otimes y), x \otimes y \right\rangle\end{aligned} \tag{50}$$

for $x, y \in H$ with $\|x\| = \|y\| = 1$.

If we take the convex function $f = (\cdot)\ln(\cdot)$, then we get by (19) that

$$\mathcal{P}_{(\cdot)\ln(\cdot),\otimes}(A,B) \geq (\ln u + 1)(A \otimes 1) - u(1 \otimes B), \tag{51}$$

for $A, B > 0$ and $u > 0$.

By (25) we obtain

$$\frac{\left\langle \mathcal{P}_{(\cdot)\ln(\cdot),\otimes}(A,B)(x \otimes y), x \otimes y \right\rangle}{\langle Ax, x \rangle} \geq \ln\left(\frac{\langle Ax, x \rangle}{\langle By, y \rangle}\right) \tag{52}$$

for $x, y \in H$ with $\|x\| = \|y\| = 1$.

If the condition (47) is satisfied, then by (40) we obtain

$$\begin{aligned}\mathcal{P}_{(\cdot)\ln(\cdot),\otimes}(A,B) &\leq (\ln u + 1)(A \otimes 1) - u(1 \otimes B) \\ &\quad + \ln\left(\frac{M_1 M_2}{m_2 m_1}\right)|A \otimes 1 - u(1 \otimes B)|\end{aligned} \tag{53}$$

for $u \in \left[\frac{m_1}{M_2}, \frac{M_1}{m_2}\right]$.

From (45) we also derive

$$0 \leq \frac{\left\langle \mathcal{P}_{(\cdot)\ln(\cdot),\otimes}(A,B)(x\otimes y), x\otimes y\right\rangle}{\langle Ax,x\rangle} - \ln\left(\frac{\langle Ax,x\rangle}{\langle By,y\rangle}\right) \tag{54}$$
$$\leq \ln\left(\frac{M_1 M_2}{m_2 m_1}\right)$$
$$\times \left\langle \frac{1}{\langle Ax,x\rangle}\left|A\otimes 1 - \frac{\langle Ax,x\rangle}{\langle By,y\rangle}(1\otimes B)\right|(x\otimes y), x\otimes y\right\rangle$$

for x, $y \in H$ with $\|x\| = \|y\| = 1$.

By choosing other convex functions, one can derive several similar inequalities. The details are omitted.

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