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The $\mathfrak{m}$-DMP inverse in Minkowski space and its applications

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Abstract. This paper first introduces a new generalized inverse in Minkowski space, called the $\mathfrak{m}$-DMP inverse, and discusses its algebraic and geometrical properties. The second objective is to characterize the $\mathfrak{m}$-DMP inverse equivalently by ranges, null spaces and matrix equations, and show its integral and limiting representations and several explicit expressions. Finally, the paper gives applications of the $\mathfrak{m}$-DMP inverse in solving a system of linear equations and a constrained optimization problem.

1. Introduction

Since Malik and Thome [21] defined the DMP inverse by using the Moore-Penrose inverse [27] and the Drazin inverse [5], there has been tremendous interest in developing the DMP inverse in recent years. Liu and Cai [17] proposed two iterative methods to compute the DMP inverse. The integral and determinantal representations for the DMP inverse were derived by [42] and [16], respectively. Ferreyra et al. [7] developed the maximal classes of matrices to determine the DMP inverse. Using the classical Cayley-Hamilton theorem, Wang et al. [35] gave an annihilating polynomial of the DMP inverse. Ma et al. [20] investigated characterizations, iterative methods, sign patterns and perturbation analysis for the DMP inverse as well as its applications in solving singular linear systems. Zuo et al. [44] presented further characterizations of the DMP inverse in terms of its range and null space. Furthermore, the notion of the DMP inverse was extended from square complex matrices to rectangular complex matrices [24], operators in Hilbert spaces [26], elements in rings [43], finite potent endomorphisms on arbitrary vector spaces [31], square matrices over the quaternion skew field [15], and tensors [32]. And, other extended forms of the DMP inverse were established by [10, 25].

In studying polarized light, Renardy [29] investigated the singular value decomposition in Minkowski space in order to quickly verify that a Mueller matrix maps the forward light cone into itself. Subsequently, the Minkowski inverse in Minkowski space was established by Meenakshi [22], who also gave a condition for a Mueller matrix to have a singular value decomposition in Minkowski space according to its Minkowski inverse. In the past two decades, a great deal of mathematical effort has been devoted to the study of the Minkowski inverse. More details of its properties, applications and generalizations can be found in [1, 8, 11–14, 18, 19, 23, 41]. Recently, Wang et al. defined the $\mathfrak{m}$-core inverse [36], $\mathfrak{m}$-core-EP inverse [37] and $\mathfrak{m}$-WG

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inverse [39] in Minkowski space, which can be regarded as extensions of the core inverse [2], core-EP inverse [28] and weak group inverse [34], respectively.

Inspired by the study of the DMP inverse and the generalized inverses in Minkowski space, the intention of this paper is to introduce a new generalized inverse in Minkowski space, called the $\mathfrak{m}$-DMP inverse, and discuss its properties, characterizations, representations and applications.

The primary contributions of the paper are summed up as follows:

(1) The definition of the $\mathfrak{m}$-DMP inverse in Minkowski space is given as the unique solution of a certain system of matrix equations. Based on its explicit expression, the canonical forms of the $\mathfrak{m}$-DMP inverse are also obtained in terms of the Hartwig-Spindelböck decomposition.

(2) The $\mathfrak{m}$-DMP inverse is represented as an outer inverse with prescribed range and null space, and some of its algebraic and geometrical properties are shown. On the converse, the $\mathfrak{m}$-DMP inverse is characterized by using its basic properties.

(3) Applying the full-rank factorization leads to an explicit formula of the $\mathfrak{m}$-DMP inverse. According to this result, we present an integral representation of the $\mathfrak{m}$-DMP inverse. And, a few limiting representations of the $\mathfrak{m}$-DMP inverse are proposed.

(4) We apply the $\mathfrak{m}$-DMP inverse to solve a system of linear equations in Minkowski space as well as a least norm problem. And, a condensed Cramer's rule for the unique solution of this system is stated.

The present paper is built up as follows. Some necessary notions, definitions and lemmas are recalled in Section 2. Section 3 is devoted to introducing the $\mathfrak{m}$-DMP inverse and its properties. Further characterizations and representations of the $\mathfrak{m}$-DMP inverse are shown in Section 4. Section 5 presents applications of the $\mathfrak{m}$-DMP inverse in solving a system of linear equations and an optimization problem. The conclusions are stated in Section 6.

2. Preliminaries

We use the following notations throughout this paper. Let $\mathbb{C}^n$, $\mathbb{C}^{m\times n}$ and $\mathbb{C}^{n\times n}_k$ be the sets of all complex n-dimensional vectors, complex $m \times n$ matrices, and complex $n \times n$ matrices with index k, respectively. The smallest nonnegative integer k satisfying $\mathrm{rank}(A^{k+1}) = \mathrm{rank}(A^k)$ is called the index of $A \in \mathbb{C}^{n\times n}$, denoted by $\mathrm{Ind}(A)$. The symbols A^*, $\mathrm{rank}(A)$, $\mathcal{R}(A)$, $\mathcal{N}(A)$, and $\|A\|_F$ stand for the conjugate transpose, rank, range, null space, and Frobenius norm of $A \in \mathbb{C}^{m\times n}$, respectively. We denote the identity matrix in $\mathbb{C}^{n\times n}$ by I_n, and the null matrix with appropriate orders by 0. The projector onto $\mathcal{S}$ along $\mathcal{T}$ is indicated by $P_{\mathcal{S},\mathcal{T}}$, where $\mathcal{S}, \mathcal{T} \subseteq \mathbb{C}^n$ are subspaces satisfying that their direct sum is $\mathbb{C}^n$, i.e., $\mathcal{S} \oplus \mathcal{T} = \mathbb{C}^n$.

Additionally, the Minkowski inner product [22, 29] of two elements x and y in $\mathbb{C}^n$ is defined by $(x, y) =< x, Gy >$, where $G = \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix}$ represents the Minkowski metric matrix with order n, and $< \cdot, \cdot >$ is the conventional Euclidean inner product. The complex linear space $\mathbb{C}^n$ with the Minkowski inner product is called the Minkowski space. Note that the Minkowski space is also an indefinite inner product space [11]. The Minkowski adjoint of $A \in \mathbb{C}^{m\times n}$ is $A^\sim = GA^*F$, where G and F are Minkowski metric matrices with orders n and m, respectively.

Next, we will review definitions of some generalized inverses.

Definition 2.1. *[27, 33] Let $A \in \mathbb{C}^{m\times n}$. Then the matrix $X \in \mathbb{C}^{n\times m}$ verifying*

$$AXA = A, \quad XAX = X, \quad (AX)^* = AX, \quad (XA)^* = XA,$$

is called the Moore-Penrose inverse of A, denoted by $A^\dagger$. In addition, if X satisfies $XAX = X$, then we call X an outer inverse of A. For subspaces $\mathcal{T} \subseteq \mathbb{C}^n$ and $\mathcal{S} \subseteq \mathbb{C}^m$, an outer inverse X of A with $\mathcal{R}(X) = \mathcal{T}$ and $\mathcal{N}(X) = \mathcal{S}$ is unique, and is denoted by $A^{(2)}_{\mathcal{T},\mathcal{S}}$.

Definition 2.2. *[5, 6] Let $A \in \mathbb{C}_k^{n\times n}$. Then the matrix $X \in \mathbb{C}^{n\times n}$ satisfying*

$$XAX = X, \quad AX = XA, \quad A^{k+1}X = A^k,$$

is called the Drazin inverse of A, denoted by A^D. In the case $\operatorname{Ind}(A) = 1$, the Drazin inverse of A reduces the group inverse of A, which is denoted by $A^{\#}$.

Definition 2.3. *[21] Let $A \in \mathbb{C}_k^{n\times n}$. Then we call the matrix $X \in \mathbb{C}^{n\times n}$ fulfilling*

$$XAX = X, \quad XA = A^D A, \quad A^k X = A^k A^{\dagger},$$

the DMP inverse of A, which is denoted by $A^{D,\dagger}$. And, $A^{D,\dagger} = A^D A A^{\dagger}$.

Definition 2.4. *[22] Let $A \in \mathbb{C}^{m\times n}$. If there exists a matrix $X \in \mathbb{C}^{n\times m}$ such that*

$$AXA = A, \quad XAX = X, \quad (AX)^{\sim} = AX, \quad (XA)^{\sim} = XA,$$

then X is called the Minkowski inverse of A, denoted by $A^{\mathfrak{m}}$.

Subsequently, we recall a few auxiliary lemmas which will be utilized later. First off, we mention the Hartwig-Spindelböck decomposition as an effective tool in studying generalized inverses.

Lemma 2.5. *(Hartwig-Spindelböck decomposition, [9]) Let $A \in \mathbb{C}^{n\times n}$ and $r = \operatorname{rank}(A)$. Then A can be represented in the from*

$$A = U\begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix}U^*, \tag{2.1}$$

where $U \in \mathbb{C}^{n\times n}$ is unitary, $\Sigma = \operatorname{diag}(\sigma_1, \sigma_2, ..., \sigma_r)$ is the diagonal matrix of singular values of A, $\sigma_i > 0$ $(i = 1, 2, ..., r)$, and $K \in \mathbb{C}^{r\times r}$ and $L \in \mathbb{C}^{r\times(n-r)}$ satisfy $KK^* + LL^* = I_r$.

Lemma 2.6. *[21, Formula (14)] Let $A \in \mathbb{C}^{n\times n}$ be given by (2.1). Then,*

$$A^D = U\begin{pmatrix} (\Sigma K)^D & \left((\Sigma K)^D\right)^2 \Sigma L \\ 0 & 0 \end{pmatrix}U^*. \tag{2.2}$$

Lemma 2.7. *[8, 22] Let $A \in \mathbb{C}^{n\times n}$ be given by (2.1), let the partition of the Minkowski metric matrix $G \in \mathbb{C}^{n\times n}$ be*

$$G = U\begin{pmatrix} G_1 & G_2 \\ G_2^* & G_4 \end{pmatrix}U^*, \tag{2.3}$$

where $G_1 \in \mathbb{C}^{r\times r}$, $G_2 \in \mathbb{C}^{r\times(n-r)}$ and $G_4 \in \mathbb{C}^{(n-r)\times(n-r)}$, and let

$$\Delta = \begin{pmatrix} K & L \end{pmatrix} U^* G U \begin{pmatrix} K^* \\ L^* \end{pmatrix}.$$

Then the following statements are equivalent:

(1) *$A^{\mathfrak{m}}$ exists;*

(2) $\operatorname{rank}(AA^{\sim}) = \operatorname{rank}(A^{\sim}A) = \operatorname{rank}(A)$;

(3) $\operatorname{rank}(A^{\sim}AA^{\sim}) = \operatorname{rank}(A)$;

(4) *G_1 and Δ are nonsingular.*

In this case,

$$A^{\mathfrak{m}} = GU\begin{pmatrix} K^*(G_1\Sigma\Delta)^{-1} & 0 \\ L^*(G_1\Sigma\Delta)^{-1} & 0 \end{pmatrix}U^*G. \tag{2.4}$$

Several significant properties of the Drazin inverse and the Minkowski inverse are referenced.

Lemma 2.8. *[11, Theorem 9] Let $A \in \mathbb{C}^{m\times n}$ be such that $A^{\mathfrak{m}}$ exists. Then,*

(1) $\mathcal{R}(A^{\mathfrak{m}}) = \mathcal{R}(A^{\sim})$ *and* $\mathcal{N}(A^{\mathfrak{m}}) = \mathcal{N}(A^{\sim})$;

(2) $AA^{\mathfrak{m}} = P_{\mathcal{R}(A),\mathcal{N}(A^{\sim})}$;

(3) $A^{\mathfrak{m}}A = P_{\mathcal{R}(A^{\sim}),\mathcal{N}(A)}$.

Lemma 2.9. *[33, Theorem 2.1.4] Let $A \in \mathbb{C}_k^{n\times n}$. Then,*

(1) $\mathcal{R}(A^D) = \mathcal{R}(A^k)$ *and* $\mathcal{N}(A^D) = \mathcal{N}(A^k)$;

(2) $AA^D = A^DA = P_{\mathcal{R}(A^k),\mathcal{N}(A^k)}$.

And, a few limiting expressions of an outer inverse with prescribed range and null space are reviewed.

Lemma 2.10. *[40, Theorem 2.1] Let $A \in \mathbb{C}^{m\times n}$, $X \in \mathbb{C}^{n\times p}$ and $Y \in \mathbb{C}^{p\times m}$. If $A^{(2)}_{\mathcal{R}(XY),\mathcal{N}(XY)}$ exists, then*

$$A^{(2)}_{\mathcal{R}(XY),\mathcal{N}(XY)} = \lim_{\lambda\to 0} X(\lambda I_p + YAX)^{-1}Y. \tag{2.5}$$

Lemma 2.11. *[38, Theorem 2.4] Let $A \in \mathbb{C}^{m\times n}$, and let $H \in \mathbb{C}^{n\times m}$ be such that $\mathcal{R}(H) = \mathcal{T}$ and $\mathcal{N}(H) = \mathcal{S}$, where $\mathcal{T}$ and $\mathcal{S}$ are subspaces of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. If $A^{(2)}_{\mathcal{T},\mathcal{S}}$ exists, then*

$$A^{(2)}_{\mathcal{T},\mathcal{S}} = \lim_{\lambda\to 0} H(\lambda I_m + AH)^{-1} \tag{2.6}$$

$$= \lim_{\lambda\to 0} (\lambda I_n + HA)^{-1}H. \tag{2.7}$$

3. The $\mathfrak{m}$-DMP inverse in Minkowski Space

The main purpose of this section is to introduce the $\mathfrak{m}$-DMP inverse in Minkowski space, and present some of its properties. We begin with considering the following system of matrix equations, whose unique solution is defined as the $\mathfrak{m}$-DMP inverse.

Theorem 3.1. *Let $A \in \mathbb{C}_k^{n\times n}$ with $\mathrm{rank}(A^{\sim}AA^{\sim}) = \mathrm{rank}(A)$. Then the system of matrix equations*

$$XAX = X, \quad XA = A^DA, \quad A^kX = A^kA^{\mathfrak{m}}, \tag{3.1}$$

has the unique solution

$$X = A^DAA^{\mathfrak{m}}.$$

Proof. Using the condition (3.1) and Lemma 2.9(2), we have that

$$X = XAX = A^DAX = (A^DA)^kX = (A^D)^kA^kA^{\mathfrak{m}} = A^DAA^{\mathfrak{m}},$$

which completes the proof. □

Definition 3.2. *Let $A \in \mathbb{C}_k^{n\times n}$ with $\mathrm{rank}(A^{\sim}AA^{\sim}) = \mathrm{rank}(A)$. The $\mathfrak{m}$-DMP inverse of A in Minkowski space, denoted by $A^{D,\mathfrak{m}}$, is defined as*

$$A^{D,\mathfrak{m}} = A^DAA^{\mathfrak{m}}. \tag{3.2}$$

Remark 3.3. *By comparing [36, Theorem 2.9] and Definition 3.2, it is obvious that the concept of the* $\mathfrak{m}$-DMP *inverse generalizes that of the* $\mathfrak{m}$*-core inverse, which is denoted by* $A^{\textcircled{\mathfrak{m}}}$ *. In other words, if* $A \in \mathbb{C}_1^{n\times n}$ *satisfies* $\operatorname{rank}(A^{\sim}AA^{\sim}) = \operatorname{rank}(A)$*, then* $A^{D,\mathfrak{m}} = A^{\textcircled{\mathfrak{m}}}$.

The following theorem gives the canonical representation of the $\mathfrak{m}$-DMP inverse in terms of the Hartwig-Spindelböck decomposition.

Theorem 3.4. *Let* $A \in \mathbb{C}_k^{n\times n}$ *be given by* (2.1) *with* $\operatorname{rank}(A^{\sim}AA^{\sim}) = \operatorname{rank}(A)$*, and let* G_1 *and* G_2 *be given by* (2.3). *Then* $A^{D,\mathfrak{m}}$ *has the decompositions in the forms*

$$A^{D,\mathfrak{m}} = U\begin{pmatrix} (\Sigma K)^D G_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^* G \tag{3.3}$$

$$= U\begin{pmatrix} (\Sigma K)^D & (\Sigma K)^D G_1^{-1} G_2 \\ 0 & 0 \end{pmatrix} U^*. \tag{3.4}$$

Proof. Inserting (2.1), (2.2) and (2.4) to (3.2), by direct calculation we infer that

$$\begin{aligned}
A^{D,\mathfrak{m}} =& U\begin{pmatrix} (\Sigma K)^D & \left((\Sigma K)^D\right)^2 \Sigma L \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix} U^* G U \begin{pmatrix} K^*(G_1\Sigma\Delta)^{-1} & 0 \\ L^*(G_1\Sigma\Delta)^{-1} & 0 \end{pmatrix} U^* G \\
=& U\begin{pmatrix} (\Sigma K)^D & \left((\Sigma K)^D\right)^2 \Sigma L \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Sigma \\ 0 \end{pmatrix}\begin{pmatrix} K & L \end{pmatrix} U^* G U \begin{pmatrix} K^* \\ L^* \end{pmatrix}\begin{pmatrix} \Delta^{-1}\Sigma^{-1}G_1^{-1} & 0 \end{pmatrix} U^* G \\
=& U\begin{pmatrix} (\Sigma K)^D & \left((\Sigma K)^D\right)^2 \Sigma L \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Sigma \\ 0 \end{pmatrix}\Delta\begin{pmatrix} \Delta^{-1}\Sigma^{-1}G_1^{-1} & 0 \end{pmatrix} U^* G \\
=& U\begin{pmatrix} (\Sigma K)^D G_1^{-1} & 0 \\ 0 & 0 \end{pmatrix} U^* G = U\begin{pmatrix} (\Sigma K)^D G_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} G_1 & G_2 \\ G_2^* & G_4 \end{pmatrix} U^* \\
=& U\begin{pmatrix} (\Sigma K)^D & (\Sigma K)^D G_1^{-1} G_2 \\ 0 & 0 \end{pmatrix} U^*,
\end{aligned}$$

which completes the proof. □

Remark 3.5. *Under the hypotheses of Theorem 3.4, we have an another outer inverse associated with A, that is,* $A^{\mathfrak{m},D} = A^{\mathfrak{m}}AA^D$*, which is called the dual* $\mathfrak{m}$-DMP *inverse of A in Minkowski space. Using the method analogous to the proof of* (3.3)*, we have*

$$A^{\mathfrak{m},D} = GU\begin{pmatrix} K^*\Delta^{-1}K(\Sigma K)^D & K^*\Delta^{-1}K((\Sigma K)^D)^2\Sigma L \\ L^*\Delta^{-1}K(\Sigma K)^D & L^*\Delta^{-1}K((\Sigma K)^D)^2\Sigma L \end{pmatrix} U^*.$$

It is expected that $A^{\mathfrak{m},D}$ *will have properties similar to that of* $A^{D,\mathfrak{m}}$.

Example 3.6. *Let*

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $\mathrm{rank}(A^{\sim}AA^{\sim}) = \mathrm{rank}(A) = 2$,

$$A^{\dagger} = \begin{pmatrix} 0.66667 & 0.33333 & -0.33333 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -0.33333 & 0.33333 & 0.66667 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A^{D} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^{D,\dagger} = \begin{pmatrix} 0.66667 & 0.33333 & -0.33333 & 0 & 0 \\ 0.66667 & 0.33333 & -0.33333 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A^{\mathfrak{m}} = \begin{pmatrix} 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^{D,\mathfrak{m}} = \begin{pmatrix} 2 & -1 & 1 & 0 & 0 \\ 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A^{\mathfrak{m},D} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Clearly, $A^{D,\mathfrak{m}}$ is a new generalized inverse of A, which is different from $A^{\dagger}$, A^{D}, $A^{D,\dagger}$ and $A^{\mathfrak{m}}$. It also shows that $A^{D,\mathfrak{m}}$ and $A^{\mathfrak{m},D}$ are different.

The following theorem gives some basic properties of the $\mathfrak{m}$-DMP inverse, which show that the $\mathfrak{m}$-DMP inverse is an outer inverse with prescribed range and null space.

Theorem 3.7. *Let $A \in \mathbb{C}_k^{n\times n}$ with $\mathrm{rank}(A^{\sim}AA^{\sim}) = \mathrm{rank}(A)$. Then,*

(1) $\mathrm{rank}(A^{D,\mathfrak{m}}) = \mathrm{rank}(A^{k})$;

(2) $\mathcal{R}(A^{D,\mathfrak{m}}) = \mathcal{R}(A^{k})$ *and* $\mathcal{N}(A^{D,\mathfrak{m}}) = \mathcal{N}(A^{k}A^{\mathfrak{m}})$;

(3) $A^{D,\mathfrak{m}} = A^{(2)}_{\mathcal{R}(A^{k}),\mathcal{N}(A^{k}A^{\mathfrak{m}})}$;

(4) $AA^{D,\mathfrak{m}} = P_{\mathcal{R}(A^{k}),\mathcal{N}(A^{k}A^{\mathfrak{m}})}$;

(5) $A^{D,\mathfrak{m}}A = A^{D}A = P_{\mathcal{R}(A^{k}),\mathcal{N}(A^{k})}$.

Proof. (1). It follows from Lemma 2.9 that

$$\mathrm{rank}(A^{D}) = \mathrm{rank}(A^{D}A) = \mathrm{rank}(A^{D}AA^{\mathfrak{m}}A) \le \mathrm{rank}(A^{D}AA^{\mathfrak{m}}) \le \mathrm{rank}(A^{D}),$$

which, together with (3.2), shows that $\mathrm{rank}(A^{D,\mathfrak{m}}) = \mathrm{rank}(A^{k})$.

(2). Using (3.2) and the item (1), we have $\mathcal{R}(A^{D,\mathfrak{m}}) = \mathcal{R}(A^{k})$ directly. Since

$$\mathrm{rank}(A^{k}) = \mathrm{rank}(A^{k}A^{\mathfrak{m}}A) \le \mathrm{rank}(A^{k}A^{\mathfrak{m}}) \le \mathrm{rank}(A^{k}),$$

again by the item (1) we get

$$\mathrm{rank}(A^{k}A^{\mathfrak{m}}) = \mathrm{rank}(A^{k}) = \mathrm{rank}(A^{D,\mathfrak{m}}). \tag{3.5}$$

Moreover,

$$\mathcal{N}(A^{k}A^{\mathfrak{m}}) \subseteq \mathcal{N}((A^{D}A)^{k}A^{\mathfrak{m}}) = \mathcal{N}(A^{D}AA^{\mathfrak{m}}) = \mathcal{N}(A^{D,\mathfrak{m}}),$$

implying $\mathcal{N}(A^{D,\mathfrak{m}}) = \mathcal{N}(A^{k}A^{\mathfrak{m}})$.

(3). It is obvious by (3.1) and the item (2).

(4). In terms of $\mathrm{Ind}(A) = k$ and the item (2), we infer that

$$\mathcal{R}(AA^{D,\mathfrak{m}}) = A\mathcal{R}(A^k) = \mathcal{R}(A^{k+1}) = \mathcal{R}(A^k).$$

Evidently, $\mathrm{rank}(AA^{D,\mathfrak{m}}) = \mathrm{rank}(A^k)$, which, together with the items (1) and (2), shows that

$$\mathcal{N}(AA^{D,\mathfrak{m}}) = \mathcal{N}(A^{D,\mathfrak{m}}) = \mathcal{N}(A^k A^{\mathfrak{m}}).$$

Then, since $A^{D,\mathfrak{m}}$ is an outer inverse of A, we have $AA^{D,\mathfrak{m}} = P_{\mathcal{R}(A^k),\mathcal{N}(A^kA^{\mathfrak{m}})}$.

(5). It is easily obtained by (3.2) and Lemma 2.9(2). □

It is a popular approach to characterize generalized inverses from the geometric point of view, for example, [2, Definiton 1], [21, Theorem 2.13], [24, Theorem 3.2], etc. So, the following theorem presents a geometric characterization of the $\mathfrak{m}$-DMP inverse.

Theorem 3.8. *Let $A \in \mathbb{C}_k^{n\times n}$ with $\mathrm{rank}(A^{\sim}AA^{\sim}) = \mathrm{rank}(A)$. Then $A^{D,\mathfrak{m}}$ is the unique matrix $X \in \mathbb{C}^{n\times n}$ such that*

$$AX = P_{\mathcal{R}(A^k),\mathcal{N}(A^kA^{\mathfrak{m}})}, \quad \mathcal{R}(X) \subseteq \mathcal{R}(A^k). \tag{3.6}$$

Proof. Obviously, from Theorem 3.7(2) and (4), we see that $A^{D,\mathfrak{m}}$ is a solution to (3.6). Then, we will prove the uniqueness of the solution of (3.6). Assume that both X_1 and X_2 are such that (3.6). Thus, since $AX_1 = AX_2 = P_{\mathcal{R}(A^k),\mathcal{N}(A^kA^{\mathfrak{m}})}$, we get $\mathcal{R}(X_1 - X_2) \subseteq \mathcal{N}(A) \subseteq \mathcal{N}(A^k)$. Moreover, it follows from $\mathcal{R}(X_1) \subseteq \mathcal{R}(A^k)$ and $\mathcal{R}(X_2) \subseteq \mathcal{R}(A^k)$ that $\mathcal{R}(X_1 - X_2) \subseteq \mathcal{R}(A^k)$. Hence, according to $\mathrm{Ind}(A) = k$, we directly obtain that

$$\mathcal{R}(X_1 - X_2) \subseteq \mathcal{R}(A^k) \cap \mathcal{N}(A^k) = \{0\},$$

i.e., $X_1 = X_2$. Therefore, $A^{D,\mathfrak{m}}$ is the unique solution to (3.6). □

Remark 3.9. *Let $A \in \mathbb{C}_1^{n\times n}$ with $\mathrm{rank}(A^{\sim}AA^{\sim}) = \mathrm{rank}(A)$. In terms of [36, Theorem 2.7(I) and Theorem 2.9] and Lemma 2.8(1), it can easily be obtained that $A^{\textcircled{\mathfrak{m}}} = A^{(1,2)}_{\mathcal{R}(A),\mathcal{N}(A^{\sim})}$. And, in view of Remark 3.3, it is a direct corollary of Theorem 3.8 that $A^{\textcircled{\mathfrak{m}}}$ is the unique matrix $X \in \mathbb{C}^{n\times n}$ such that $AX = P_{\mathcal{R}(A),\mathcal{N}(A^{\sim})}$ and $\mathcal{R}(X) \subseteq \mathcal{R}(A)$.*

The following theorem shows some new properties of the $\mathfrak{m}$-DMP inverse, which inherit from that of the DMP inverse [21, Proposition 2.14].

Theorem 3.10. *Let $A \in \mathbb{C}_k^{n\times n}$ be given by (2.1) with $\mathrm{rank}(A^{\sim}AA^{\sim}) = \mathrm{rank}(A)$. Then,*

(1) $A^{D,\mathfrak{m}} = A^D P_{\mathcal{R}(A),\mathcal{N}(A^{\sim})}$;

(2) $(A^{D,\mathfrak{m}})^l = \begin{cases} (A^DA^{\mathfrak{m}})^{\frac{l}{2}}, & \text{if } l \text{ is even}, \\ A(A^DA^{\mathfrak{m}})^{\frac{l+1}{2}}, & \text{if } l \text{ is odd}; \end{cases}$

(3) $A^{D,\mathfrak{m}} = (A^2A^{\mathfrak{m}})^D$;

(4) $((A^{D,\mathfrak{m}})^D)^D = A^{D,\mathfrak{m}}$;

(5) $AA^{D,\mathfrak{m}} = A^{D,\mathfrak{m}}A$ *if and only if* $A^{D,\mathfrak{m}} = A^D$ *if and only if* $\mathcal{N}(A^{\sim}) \subseteq \mathcal{N}(A^k)$;

(6) $A^{D,\mathfrak{m}} = 0$ *if and only if A is nilpotent;*

(7) $A^{D,\mathfrak{m}} = A$ *if and only if* $(\Sigma K)^D = \Sigma L$ *and* $L = KG_1^{-1}G_2$, *where G_1 and G_2 are given by* (2.3).

Proof. (1). It is clear by (3.2) and Lemma 2.8(2).

(2). From (3.2) we have that

$$(A^{D,\mathfrak{m}})^2 = A^D AA^{\mathfrak{m}} A^D AA^{\mathfrak{m}} = A^D AA^{\mathfrak{m}} AA^D A^{\mathfrak{m}} = A^D A^{\mathfrak{m}}.$$

Then, for an even number l, it follows that

$$(A^{D,\mathfrak{m}})^l = ((A^{D,\mathfrak{m}})^2)^{\frac{l}{2}} = (A^D A^{\mathfrak{m}})^{\frac{l}{2}}. \tag{3.7}$$

Moreover, if l is odd, then from (3.7) and (3.2), we get that

$$(A^{D,\mathfrak{m}})^l = A^{D,\mathfrak{m}}(A^{D,\mathfrak{m}})^{l-1} = A^{D,\mathfrak{m}}(A^D A^{\mathfrak{m}})^{\frac{l-1}{2}} = AA^D A^{\mathfrak{m}}(A^D A^{\mathfrak{m}})^{\frac{l-1}{2}} = A(A^D A^{\mathfrak{m}})^{\frac{l+1}{2}}.$$

(3). Using the Cline's Formula, i.e., $(XY)^D = X\left((YX)^D\right)^2 Y$ for $X \in \mathbb{C}^{m\times n}$ and $Y \in \mathbb{C}^{n\times m}$, from (3.2) we infer that

$$(A^2A^{\mathfrak{m}})^D = (A(AA^{\mathfrak{m}}))^D = A\left((AA^{\mathfrak{m}}A)^D\right)^2 AA^{\mathfrak{m}} = A(A^D)^2AA^{\mathfrak{m}} = A^DAA^{\mathfrak{m}} = A^{D,\mathfrak{m}}.$$

(4). Using again Cline's Formula, from (3.2), Lemma 2.8(2) and Lemma 2.9(1), we have

$$\begin{aligned}(A^{D,\mathfrak{m}})^D &= (A^D(AA^{\mathfrak{m}}))^D = A^D\left((AA^{\mathfrak{m}}A^D)^D\right)^2 AA^{\mathfrak{m}}\\ &= A^D\left((A^D)^D\right)^2 AA^{\mathfrak{m}} = (A^D)^DAA^{\mathfrak{m}} = (A^D)^{\#}AA^{\mathfrak{m}}.\end{aligned}$$

Then, again by Cline's Formula, we have

$$\begin{aligned}((A^{D,\mathfrak{m}})^D)^D &= ((A^D)^{\#}AA^{\mathfrak{m}})^D = (A^D)^{\#}\left((AA^{\mathfrak{m}}(A^D)^{\#})^D\right)^2 AA^{\mathfrak{m}}\\ &= (A^D)^{\#}\left(((A^D)^{\#})^{\#}\right)^2 AA^{\mathfrak{m}} = A^DAA^{\mathfrak{m}}.\end{aligned}$$

(5). According to (3.2), Lemma 2.8(2) and Lemma 2.9(2), we see that

$$\begin{aligned}AA^{D,\mathfrak{m}} = A^{D,\mathfrak{m}}A &\Leftrightarrow AA^D(AA^{\mathfrak{m}} - I_n) = 0 \Leftrightarrow \mathcal{N}(A^{\sim}) \subseteq \mathcal{N}(A^k)\\ &\Leftrightarrow A^D(AA^{\mathfrak{m}} - I_n) = 0 \Leftrightarrow A^{D,\mathfrak{m}} = A^D.\end{aligned}$$

(6). Using (3.3) and [21, Theorem 2.5], i.e., $A^{D,\dagger} = U\begin{pmatrix} (\Sigma K)^D & 0 \\ 0 & 0 \end{pmatrix}U^*$, we have that

$$A^{D,\mathfrak{m}} = 0 \Leftrightarrow (\Sigma K)^D = 0 \Leftrightarrow A^{D,\dagger} = 0,$$

which, together with [21, Proposition 2.14 (g)], i.e., $A^{D,\dagger} = 0 \Leftrightarrow A$ is nilpotent, shows the item (6) holds.

(7). It is obvious by (2.1) and (3.4). $\square$

Remark 3.11. *Under the hypotheses of Theorem 3.10, uisng* (2.1) *and* (2.4), *by direct calculation we have*

$$A^2A^{\mathfrak{m}} = U\begin{pmatrix} \Sigma KG_1^{-1} & 0 \\ 0 & 0 \end{pmatrix}U^*G.$$

Then, using (3.3) *in Theorem 3.4, we can verify that $A^{D,\mathfrak{m}}$ satisfies the definition of the Drazin inverse of $A^2A^{\mathfrak{m}}$, i.e., $A^{D,\mathfrak{m}} = (A^2A^{\mathfrak{m}})^D$. Thus, we succeed in avoiding the use of Cline's formula for the proof of Theorem 3.10*(3)*.*

4. Further characterizations and representations of the $\mathfrak{m}$-DMP inverse

We shall continue to characterize and represent the $\mathfrak{m}$-DMP inverse from different views in this section. We begin this section by characterizing the $\mathfrak{m}$-DMP inverse based on its essential properties obtained in Section 3.

Theorem 4.1. *Let* $A \in \mathbb{C}_k^{n\times n}$ *with* $\operatorname{rank}(A^{\sim}AA^{\sim}) = \operatorname{rank}(A)$, *and let* $X \in \mathbb{C}^{n\times n}$. *Then the following statements are equivalent:*

(1) $X = A^{D,\mathfrak{m}}$;

(2) $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$, $A^D X = A^D A^{\mathfrak{m}}$;

(3) $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$, $A^k X = A^k A^{\mathfrak{m}}$;

(4) $\mathcal{N}(A^k A^{\mathfrak{m}}) \subseteq \mathcal{N}(X)$, $XA = AA^D$;

(5) $\mathcal{N}(A^k A^{\mathfrak{m}}) \subseteq \mathcal{N}(X)$, $XA^{k+1} = A^k$.

Proof. (1) ⇒ (2). It is a direct corollary of Theorem 3.7(2) and (3.2).

(2) ⇒ (3). It is clear by premultiplying $A^D X = A^D A^{\mathfrak{m}}$ with A^{k+1}.

(3) ⇒ (4). Since $\operatorname{rank}(X) \leq \operatorname{rank}(A^k)$ by $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$, it follows from (3.5) and $A^k X = A^k A^{\mathfrak{m}}$ that $\operatorname{rank}(X) = \operatorname{rank}(A^k)$ and $\mathcal{N}(X) = \mathcal{N}(A^k A^{\mathfrak{m}})$. Then, from Lemma 2.9(2) and $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$, we infer that

$$\begin{aligned} A^k X = A^k A^{\mathfrak{m}} &\Rightarrow (A^D)^k A^k X A = (A^D)^k A^k A^{\mathfrak{m}} A \\ &\Rightarrow A^D A X A = A^D A A^{\mathfrak{m}} A \\ &\Rightarrow XA = A^D A. \end{aligned}$$

(4) ⇒(5). It is obvious by postmultiplying $XA = AA^D$ with A^k.

(5) ⇒ (1). Since $\mathcal{R}(A^k) \subseteq \mathcal{R}(X)$ and $\operatorname{rank}(A^k) \leq \operatorname{rank}(X)$ from $XA^{k+1} = A^k$, by (3.5) and $\mathcal{N}(A^k A^{\mathfrak{m}}) \subseteq \mathcal{N}(X)$ we have that $\mathcal{R}(X) = \mathcal{R}(A^k)$ and $\mathcal{N}(A^k A^{\mathfrak{m}}) = \mathcal{N}(X)$, which implies that there exists $Y \in \mathbb{C}^{n\times n}$ such that

$$YA^k A^{\mathfrak{m}} = X. \tag{4.1}$$

Then postmultiplying (4.1) with A gives that $YA^k = XA$, implying $\mathcal{N}(A^k) \subseteq \mathcal{N}(XA)$. Consequently, by Lemma 2.9(2), we get that

$$\begin{aligned} XA^{k+1} = A^k &\Rightarrow XA^{k+1}(A^D)^k X = A^k (A^D)^k X \\ &\Rightarrow XAA^D AX = A^D AX \\ &\Rightarrow XAX = X, \end{aligned}$$

implying $X = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^k A^{\mathfrak{m}})}$. Therefore, $X = A^{D,\mathfrak{m}}$ by Theorem 3.7(3). □

Theorem 4.2. *Let* $A \in \mathbb{C}_k^{n\times n}$ *with* $\operatorname{rank}(A^{\sim}AA^{\sim}) = \operatorname{rank}(A)$, *and let* $X \in \mathbb{C}^{n\times n}$. *Then the following statements are equivalent:*

(1) $X = A^{D,\mathfrak{m}}$;

(2) $AX^2 = X$, $AX = A^2 A^D A^{\mathfrak{m}}$;

(3) $AX^2 = X$, $AX = P_{\mathcal{R}(A^k),\mathcal{N}(A^D A^{\mathfrak{m}})}$;

(4) $AX^2 = X$, $A^k X = A^k A^{\mathfrak{m}}$.

Proof. (1) ⇒ (2). It is easily obtained by (3.2).

(2) ⇒ (3). According to (3.2) and Theorem 3.7(4), we have that

$$AX = A^2A^DA^{\mathfrak{m}} = AA^{D,\mathfrak{m}} = P_{\mathcal{R}(A^k),\mathcal{N}(A^DA^{\mathfrak{m}})}. \tag{4.2}$$

(3) ⇒ (4). Using (4.2), from $AX = P_{\mathcal{R}(A^k),\mathcal{N}(A^DA^{\mathfrak{m}})}$ we have that

$$A^kX = A^{k-1}P_{\mathcal{R}(A^k),\mathcal{N}(A^DA^{\mathfrak{m}})} = A^{k-1}A^2A^DA^{\mathfrak{m}} = A^kA^{\mathfrak{m}}.$$

(4) ⇒ (1). It follows from $AX^2 = X$ that

$$X = AXX = AAX^2X = A^2XX^2 = A^2AX^2X^2 = A^3XX^3 = ... = A^kX^{k+1},$$

which implies $\mathcal{R}(X) \subseteq \mathcal{R}(A^k)$. Then, by Lemma 2.9(2) and (3.2) we conclude that

$$\begin{aligned} A^kX = A^kA^{\mathfrak{m}} &\Rightarrow (A^D)^kA^kX = (A^D)^kA^kA^{\mathfrak{m}} \\ &\Rightarrow AA^DX = A^DAA^{\mathfrak{m}} \\ &\Rightarrow X = A^DAA^{\mathfrak{m}} = A^{D,\mathfrak{m}}. \end{aligned}$$

This completes the proof. □

Zuo et al. in [44, Theorem 3.8] gave an interesting result of the DMP inverse, that is, for $A \in \mathbb{C}_k^{n\times n}$,

$$A^{D,\dagger} = AA^\dagger(I_n - \overline{A}AA^\dagger)^D = (I_n - \overline{A}AA^\dagger)^DAA^\dagger, \tag{4.3}$$

where $\overline{A} = I_n - A$. The following theorem turns out analogous expressions of the $\mathfrak{m}$-DMP inverse.

Theorem 4.3. *Let $A \in \mathbb{C}_k^{n\times n}$ with $\mathrm{rank}(A^{\sim}AA^{\sim}) = \mathrm{rank}(A)$, and let $\overline{A} = I_n - A$. Then,*

$$\begin{aligned} A^{D,\mathfrak{m}} &= AA^{\mathfrak{m}}(I_n - \overline{A}AA^{\mathfrak{m}})^D \qquad (4.4) \\ &= (I_n - \overline{A}AA^{\mathfrak{m}})^DAA^{\mathfrak{m}}. \qquad (4.5) \end{aligned}$$

Proof. Using [5, Corollary 1], i.e., $(X + Y)^D = X^D + Y^D$, where $X, Y \in \mathbb{C}^{n\times n}$ satisfies $XY = YX = 0$, and a clear fact

$$(I_n - AA^{\mathfrak{m}})A^2A^{\mathfrak{m}} = A^2A^{\mathfrak{m}}(I_n - AA^{\mathfrak{m}}) = 0,$$

we can directly have that

$$(I_n - AA^{\mathfrak{m}} + A^2A^{\mathfrak{m}})^D = (I_n - AA^{\mathfrak{m}})^D + (A^2A^{\mathfrak{m}})^D = I_n - AA^{\mathfrak{m}} + (A^2A^{\mathfrak{m}})^D.$$

Hence, it follows from Theorem 3.10(3), Theorem 3.7(2) and Lemma 2.8(2) that

$$\begin{aligned} AA^{\mathfrak{m}}(I_n - \overline{A}AA^{\mathfrak{m}})^D &= AA^{\mathfrak{m}}(I_n - AA^{\mathfrak{m}} + A^2A^{\mathfrak{m}})^D \\ &= AA^{\mathfrak{m}}(I_n - AA^{\mathfrak{m}}) + AA^{\mathfrak{m}}(A^2A^{\mathfrak{m}})^D \\ &= AA^{\mathfrak{m}}A^{D,\mathfrak{m}} = A^{D,\mathfrak{m}}, \end{aligned}$$

and

$$\begin{aligned} (I_n - \overline{A}AA^{\mathfrak{m}})^DAA^{\mathfrak{m}} &= (I_n - AA^{\mathfrak{m}} + A^2A^{\mathfrak{m}})^DAA^{\mathfrak{m}} \\ &= (I_n - AA^{\mathfrak{m}})AA^{\mathfrak{m}} + (A^2A^{\mathfrak{m}})^DAA^{\mathfrak{m}} \\ &= A^{D,\mathfrak{m}}AA^{\mathfrak{m}} = A^{D,\mathfrak{m}}, \end{aligned}$$

which show that (4.4) and (4.5) are true. □

The full-rank factorization is also confirmed as a powerful tool to study the generalized inverses. And, using the full-rank factorization, Cline [4] and Zekraoui et al. [41] expressed the Drazin inverse and the Minkowski inverse, respectively. Based on their work, we present a new representation of the $\mathfrak{m}$-DMP inverse in the following theorem.

Theorem 4.4. *Let $A \in \mathbb{C}_k^{n\times n}$ with* $\operatorname{rank}(A^{\sim}AA^{\sim}) = \operatorname{rank}(A) > 0$ *and $A^k \neq 0$. Let*

$$A = B_1C_1, \quad C_1B_1 = B_2C_2, \quad C_2B_2 = B_3C_3, \quad ..., \quad C_{k-1}B_{k-1} = B_kC_k, \tag{4.6}$$

be such that B_1C_1 is a full-rank factorization of A, and $B_{i+1}C_{i+1}$ are full-rank factorizations of C_iB_i $(i = 1, 2, ..., k-1)$. Then

$$A^{D,\mathfrak{m}} = B_1 \cdots B_k(C_kB_k)^{-k}C_k \cdots C_2(B_1^{\sim}B_1)^{-1}B_1^{\sim}.$$

Proof. First, it follows from (4.6) that

$$C_k \cdots C_3C_2C_1B_1 = C_k \cdots C_3C_2B_2C_2 = \cdots = C_kB_kC_k \cdots C_2.$$

Then, applying [4, Formula (1.11)], i.e., $A^D = B_1 \cdots B_k(C_kB_k)^{-(k+1)}C_k \cdots C_1$, and [41, Theorem 8], i.e., $A^{\mathfrak{m}} = C_1^{\sim}(C_1C_1^{\sim})^{-1}(B_1^{\sim}B_1)^{-1}B_1^{\sim}$, to (3.2), we have that

$$\begin{aligned} A^{D,\mathfrak{m}} &= B_1 \cdots B_k(C_kB_k)^{-(k+1)}C_k \cdots C_1B_1C_1C_1^{\sim}(C_1C_1^{\sim})^{-1}(B_1^{\sim}B_1)^{-1}B_1^{\sim} \\ &= B_1 \cdots B_k(C_kB_k)^{-k}C_k \cdots C_2(B_1^{\sim}B_1)^{-1}B_1^{\sim}, \end{aligned}$$

which completes the proof. □

In terms of the full-rank decomposition, Zhou and Chen [42] derived integral representations of the DMP inverse, which do not require the restriction for the spectrum of a matrix. And, Kılıçman et al. [12] obtained an integral representation of the weighted Minkowski inverse. Motivated by their work, we show an integral representation of the $\mathfrak{m}$-DMP inverse as follows.

Theorem 4.5. *Let $A \in \mathbb{C}_k^{n\times n}$ with* $\operatorname{rank}(A^{\sim}AA^{\sim}) = \operatorname{rank}(A) > 0$ *and $A^k \neq 0$, and let the full-rank factorization of A be as in* (4.6). *Then,*

$$A^{D,\mathfrak{m}} = \int_0^{\infty} M\exp(-B_1^{\sim}B_1t)B_1^{\sim}dt, \tag{4.7}$$

where $M = B_1 \cdots B_k(C_kB_k)^{-k}C_k \cdots C_2$.

Proof. We first claim that $B^{\mathfrak{m}}$ exists. In fact,

$$\begin{aligned} \operatorname{rank}(B_1^{\sim}B_1) \leq \operatorname{rank}(B_1) = \operatorname{rank}(A) &= \operatorname{rank}(A^{\sim}AA^{\sim}) \\ &= \operatorname{rank}(C_1^{\sim}B_1^{\sim}B_1C_1C_1^{\sim}B_1^{\sim}) \leq \operatorname{rank}(B_1^{\sim}B_1). \end{aligned}$$

Then, $\operatorname{rank}(B_1) = \operatorname{rank}(B_1^{\sim}B_1) = \operatorname{rank}(B_1^{\sim}B_1B_1^{\sim})$ since $B_1^{\sim}$ is of full row rank. Thus, $B^{\mathfrak{m}}$ exists by Lemma 2.7(3). Furthermore, it follows from Lemma 2.8(1) and Theorem 3.7(2) that

$$\mathcal{N}(B_1^{\sim}) \subseteq \mathcal{N}(C_1^{\sim}B_1^{\sim}) = \mathcal{N}(A^{\sim}) = \mathcal{N}(A^{\mathfrak{m}}) \subseteq \mathcal{N}(A^kA^{\mathfrak{m}}) = \mathcal{N}(A^{D,\mathfrak{m}}),$$

which, together with Lemma 2.8(2), shows that

$$A^{D,\mathfrak{m}}B_1B_1^{\mathfrak{m}} = A^{D,\mathfrak{m}}. \tag{4.8}$$

Finally, applying Theorem 4.4 and [12, Corollary 8], i.e.,

$$X^{\mathfrak{m}} = \int_0^{\infty} \exp(-X^{\sim}Xt)X^{\sim}dt$$

for $X \in \mathbb{C}^{n\times n}$, to (4.8) gives (4.7) immediately. □

It is well known that the Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}$ can be expressed as a limit [3], i.e.,

$$A^{\dagger} = \lim_{\lambda\to 0}(\lambda I_n + A^*A)^{-1}A^*.$$

And, it has always been a hot topic to compute the generalized inverses by means of the limiting process. Ma et al. [20] and Kılıçman et al. [12] presented a few limiting expressions of the DMP inverse and weighted Minkowski inverse, respectively. The next theorem gives several limit representations for the $\mathfrak{m}$-DMP inverse.

Theorem 4.6. *Let $A \in \mathbb{C}_k^{n\times n}$ with* $\operatorname{rank}(A^{\sim}AA^{\sim}) = \operatorname{rank}(A)$. *Then,*

$$\begin{aligned} A^{D,\mathfrak{m}} &= \lim_{\lambda\to 0}A^k(\lambda I_n + A^{\mathfrak{m}}A^{k+1})^{-1}A^{\mathfrak{m}} && (4.9)\\ &= \lim_{\lambda\to 0}A^kA^{\mathfrak{m}}(\lambda I_n + A^{k+1}A^{\mathfrak{m}})^{-1} && (4.10)\\ &= \lim_{\lambda\to 0}(\lambda I_n + A^k)^{-1}A^kA^{\mathfrak{m}} && (4.11)\\ &= \lim_{\lambda\to 0}(\lambda I_n + A^k)^{-1}A^k(\lambda I_n + A^{\sim}A)^{-1}A^{\sim}. && (4.12)\end{aligned}$$

Proof. Since $\mathcal{R}(A^kA^{\mathfrak{m}}) = \mathcal{R}(A^k)$ from (3.5), using Theorem 3.7(3) we see that

$$A^{D,\mathfrak{m}} = A^{(2)}_{\mathcal{R}(A^k),\mathcal{N}(A^kA^{\mathfrak{m}})} = A^{(2)}_{\mathcal{R}(A^kA^{\mathfrak{m}}),\mathcal{N}(A^kA^{\mathfrak{m}})}. \tag{4.13}$$

Thus, applying (2.5), (2.6) and (2.7) to (4.13) yields (4.9), (4.10) and (4.11), respectively. Then, substituting [12, Corollary 11], i.e., $A^{\mathfrak{m}} = \lim_{\lambda\to 0}(\lambda I_n + A^{\sim}A)^{-1}A^{\sim}$, into (4.11) shows (4.12) immediately. This finishes the proof. □

Example 4.7. *Let us test the matrix A given in Example 3.6. Then,* $k := \operatorname{Ind}(A) = 2$,

$$\begin{aligned} B :&= A^k(\lambda I_n + A^{\mathfrak{m}}A^{k+1})^{-1}A^{\mathfrak{m}} = A^kA^{\mathfrak{m}}(\lambda I_n + A^{k+1}A^{\mathfrak{m}})^{-1}\\ &= (\lambda I_n + A^k)^{-1}A^kA^{\mathfrak{m}} = \begin{pmatrix} \frac{2}{\lambda+1} & \frac{-1}{\lambda+1} & \frac{1}{\lambda+1} & 0 & 0\\ \frac{2}{\lambda+1} & \frac{-1}{\lambda+1} & \frac{1}{\lambda+1} & 0 & 0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\end{pmatrix},\end{aligned}$$

$$C := (\lambda I_n + A^k)^{-1}A^k(\lambda I_n + A^{\sim}A)^{-1}A^{\sim} = \begin{pmatrix} \frac{\lambda+2}{(\lambda+1)^3} & \frac{-\lambda-1}{(\lambda+1)^3} & \frac{1}{(\lambda+1)^3} & 0 & 0\\ \frac{\lambda+2}{(\lambda+1)^3} & \frac{-\lambda-1}{(\lambda+1)^3} & \frac{1}{(\lambda+1)^3} & 0 & 0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\end{pmatrix}.$$

It is easy to check that $\lim_{\lambda\to 0}B = \lim_{\lambda\to 0}C = A^{D,\mathfrak{m}}$, *where $A^{D,\mathfrak{m}}$ has been shown in Example 3.6 and so is omitted.*

5. Applications of the $\mathfrak{m}$-DMP inverse in solving some equations

Our motivation in this section arises mainly from the work that Ma et al. [20] solved singular linear systems by using DMP inverse, and gave a condensed Cramer's rule for computing the DMP-inverse solution. We start with a consideration on a system of linear equations.

Theorem 5.1. *Let $A \in \mathbb{C}_k^{n\times n}$ with* $\operatorname{rank}(A^{\sim}AA^{\sim}) = \operatorname{rank}(A)$ *and $b \in \mathbb{C}^n$, and let a system of linear equations be*

$$A^kx = A^kA^{\mathfrak{m}}b. \tag{5.1}$$

Then the general solution of the system (5.1) *is*

$$x = A^{D,\mathfrak{m}}b + (I_n - A^{D,\mathfrak{m}}A)v, \tag{5.2}$$

where arbitrary $v \in \mathbb{C}^n$. Moreover,

$$x = A^{D,\mathfrak{m}}b$$

is the unique solution to the system (5.1) *on* $\mathcal{R}(A^k)$.

Proof. It is clear by (3.2) that $A^{D,\mathfrak{m}}b$ is a solution to (5.1). Hence, using Theorem 3.7(5), we have that the set of all solutions of (5.1) is

$$\left\{ A^{D,\mathfrak{m}}b + \alpha \;\middle|\; \alpha \in \mathcal{N}(A^k) \right\} = \left\{ A^{D,\mathfrak{m}}b + \alpha \;\middle|\; \alpha \in \mathcal{R}(I_n - A^{D,\mathfrak{m}}A) \right\},$$

which shows that the general solution of (5.1) is (5.2). Moreover, since $\mathcal{R}(A^k) \oplus \mathcal{N}(A^k) = \mathbb{C}^n$ by $\mathrm{Ind}(A) = k$, using Theorem 3.7(2) we see that $A^{D,\mathfrak{m}}b \in \mathcal{R}(A^k)$ is the unique solution to (5.1) on $\mathcal{R}(A^k)$. □

Wang et al. [36] considered an interesting least-squares problem in Frobenius norm, that is,

$$\|(AA^\dagger)^\sim Ax - b\|_F = \min \text{ subject to } x \in \mathcal{R}(A),$$

where $A \in \mathbb{C}_1^{n\times n}$ with $\mathrm{rank}(A^\sim A) = \mathrm{rank}(A) < n$, and $b \in \mathbb{C}^n$. In the following theorem, we discuss an analogous optimization problem on the $\mathfrak{m}$-DMP inverse.

Theorem 5.2. *Let $A \in \mathbb{C}_k^{n\times n}$ be given by* (2.1) *with $r := \mathrm{rank}(A^\sim AA^\sim) = \mathrm{rank}(A) < n$. Let $b = GU\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{C}^n$, where $b_1 \in \mathbb{C}^r$ satisfies $G_1^{-1}b_1 \in \mathcal{R}\left((\Sigma K)^D\right)$ and G_1 is given by* (2.3)*, and $b_2 \in \mathbb{C}^{n-r}$. Then*

$$\min_{x\in\mathcal{R}(A^k)} \|(AA^\dagger)^\sim Ax - b\|_F = \|b_2\|_F\,. \tag{5.3}$$

Moreover,

$$x = A^{D,\mathfrak{m}}b$$

is the unique solution of (5.3).

Proof. For every $x \in \mathcal{R}(A^k)$, it follows from Lemma 2.9(1) that there exits $y \in \mathbb{C}^n$ such that $x = A^D y$. Put $y = U\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, where $y_1 \in \mathbb{C}^r$ and $y_2 \in \mathbb{C}^{n-r}$. Using [2, Formula 2.2], i.e., $AA^\dagger = U\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}U^*$, from (2.1), (2.3) and (2.2) we infer that

$$\begin{aligned}
\left\|(AA^\dagger)^\sim Ax - b\right\|_F &= \left\|(AA^\dagger)^\sim AA^D y - b\right\|_F \\
&= \Bigg\|GU\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} G_1 & G_2 \\ G_2^* & G_4 \end{pmatrix}\begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix} \\
&\qquad \begin{pmatrix} (\Sigma K)^D & \left((\Sigma K)^D\right)^2 \Sigma L \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} - GU\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}\Bigg\|_F \\
&= \left\|\begin{pmatrix} G_1(\Sigma K)^D\Sigma K y_1 + G_1(\Sigma K)^D\Sigma L y_2 - b_1 \\ -b_2 \end{pmatrix}\right\|_F \\
&= \left(\left\|G_1(\Sigma K)^D\Sigma K y_1 + G_1(\Sigma K)^D\Sigma L y_2 - b_1\right\|_F^2 + \|b_2\|_F^2\right)^{\frac{1}{2}} \geq \|b_2\|_F\,.
\end{aligned}$$

Since $\Sigma K(\Sigma K)^D G_1^{-1}b_1 = G_1^{-1}b_1$ by the condition $G_1^{-1}b_1 \in \mathcal{R}\left((\Sigma K)^D\right)$, we have that if

$$y_1 = (\Sigma K)^D\Sigma L y_2 - G_1^{-1}b_1 \text{ and } y_2 \in \mathbb{C}^{n-r}, \tag{5.4}$$

then

$$G_1(\Sigma K)^D \Sigma K y_1 + G_1(\Sigma K)^D \Sigma L y_2 - b_1 = 0,$$

which implies that $\left\|(AA^\dagger)^{\sim} Ax - b\right\|_F$ assumes the minimum value,

$$\min_{x\in\mathcal{R}(A^k)} \|(AA^\dagger)^{\sim} Ax - b\|_F = \|b_2\|_F .$$

Therefore, by (2.2), (5.4) and (3.3), it follows that

$$\begin{aligned} x = A^D y &= U\begin{pmatrix} (\Sigma K)^D & \left((\Sigma K)^D\right)^2 \Sigma L \\ 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \\ &= U\begin{pmatrix} (\Sigma K)^D(-(\Sigma K)^D \Sigma L y_2 - G_1^{-1} b_1) + \left((\Sigma K)^D\right)^2 \Sigma L y_2 \\ 0 \end{pmatrix} \\ &= U\begin{pmatrix} (\Sigma K)^D G_1^{-1} b_1 \\ 0 \end{pmatrix} = A^{D,\mathrm{m}} b, \end{aligned}$$

which shows that $x = A^{D,\mathrm{m}} b$ is the unique solution of (5.3). □

Remark 5.3. *Let $A \in \mathbb{C}_k^{n\times n}$ be given by* (2.1) *with $r := \operatorname{rank}(A) < n$, and let $b = U\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{C}^n$, where $b_1 \in \mathbb{C}^r$ is such that $b_1 \in \mathcal{R}\left((\Sigma K)^D\right)$, and $b_2 \in \mathbb{C}^{n-r}$. In terms of the same argument in Theorem 5.2, we have that*

$$\min_{x\in\mathcal{R}(A^k)} \|Ax - b\|_F = \|b_2\|_F . \tag{5.5}$$

Futhermore, $x = A^{D,\dagger} b$ is the unique solution of (5.5).

We end up this section with presenting a condensed Cramer's rule to directly calculate the unique solution of (5.1) or (5.3). Let the determinant of $A \in \mathbb{C}^{n\times n}$ be $\det(A)$, and by $A(i \to b)$ we denote a matrix obtained by replacing the ith column of $A \in \mathbb{C}^{n\times n}$ with $b \in \mathbb{C}^n$.

Theorem 5.4. *Let $A \in \mathbb{C}_k^{n\times n}$ with $\operatorname{rank}(A^{\sim}AA^{\sim}) = \operatorname{rank}(A)$, $b \in \mathbb{C}^n$, and $t = \operatorname{rank}(A^k)$. Assume $V \in \mathbb{C}^{n\times(n-t)}$ and $W \in \mathbb{C}^{(n-t)\times n}$ are such that $\mathcal{R}(V) = \mathcal{N}(A^k)$ and $\mathcal{N}(W) = \mathcal{R}(A^k)$. Denote $E = V(WV)^{-1}W$. Then the components of $x = A^{D,\mathrm{m}} b$ are given by*

$$x_i = \frac{\det((A^k + E)(i \to A^k A^{\mathrm{m}} b))}{\det(A^k + E)},\ i = 1, 2, .., n. \tag{5.6}$$

Proof. Since [20, Theorem 3.1] has proved that E exists, $\mathcal{N}(E) = \mathcal{R}(A^k)$, and $(A^k + E)^{-1} = (A^k)^D + E^\#$, from Lemma 2.9(2) and (3.2) we see that the system of linear equations

$$(A^k + E)x = A^k A^{\mathrm{m}} b \tag{5.7}$$

has the unique solution

$$x = (A^k + E)^{-1} A^k A^{\mathrm{m}} b = (A^k)^D A^k A^{\mathrm{m}} b + E^\# A^k A^{\mathrm{m}} b = A^D A A^{\mathrm{m}} b = A^{D,\mathrm{m}} b.$$

Finally, applying Cramer's rule [30] to the nonsingular linear system (5.7) gives (5.6) immediately. □

Example 5.5. *Consider the matrix A given in Example 3.6, and let*

$$b = \begin{pmatrix} 0.15735 \\ 0.15735 \\ 0.1415 \\ -0.1 \\ -0.2 \end{pmatrix}, V = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, W = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Then, $k := \mathrm{Ind}(A) = 2$*, and the Hartwig-Spindelböck decomposition of* A *is* $A = U\begin{pmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{pmatrix}U^*$*, where*

$$U = \begin{pmatrix} -0.40825 & 0.70711 & 0 & 0 & 0.57735 \\ -0.8165 & 0 & 0 & 0 & -0.57735 \\ -0.40825 & -0.70711 & 0 & 0 & 0.57735 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \Sigma = \begin{pmatrix} 1.7321 & 0 \\ 0 & 1 \end{pmatrix},$$

$$K = \begin{pmatrix} 0.28868 & -0.5 \\ -0.28868 & 0.5 \end{pmatrix}, L = \begin{pmatrix} 0 & -0.70711 & -0.40825 \\ 0 & -0.70711 & 0.40825 \end{pmatrix}.$$

Furthermore,

$$G_1 = \begin{pmatrix} -0.66667 & -0.57735 \\ -0.57735 & 0 \end{pmatrix}, b_1 = \begin{pmatrix} 0.12201 \\ 0.21132 \end{pmatrix}, b_2 = \begin{pmatrix} 0.2 \\ 0.1 \\ 0.1 \end{pmatrix},$$

$$(\Sigma K)^D = \begin{pmatrix} 0.5 & -0.86603 \\ -0.28868 & 0.5 \end{pmatrix}, A^k = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, E = V(WV)^{-1}W = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then it can easily be checked that $\mathrm{rank}(A^k) = 1$, $G_1^{-1}b_1 \in \mathcal{R}\left((\Sigma K)^D\right)$, $\mathcal{R}(V) = \mathcal{N}(A^k)$ *and* $\mathcal{N}(W) = \mathcal{R}(A^k)$*. Using Theorems 5.1 and 5.2, we have that the unique solution of the system* (5.1) *on* $\mathcal{R}(A^k)$ *or the system* (5.3) *is*

$$x = A^{D,\mathfrak{m}}b = \begin{pmatrix} 0.29885 & 0.29885 & 0 & 0 & 0 \end{pmatrix}^*, \tag{5.8}$$

and

$$\min_{x \in \mathcal{R}(A^k)} \|(AA^\dagger)^\sim Ax - b\|_F = \|b_2\|_F = 0.24495.$$

And, it is easy to check that the solution x *calculated by* (5.6) *in Theorem 5.4 is equal to* x *given in* (5.8).

6. Conclusions

This paper defines the $\mathfrak{m}$-DMP inverse in Minkowski space, and shows some of its properties, characterizations, representations, and applications in solving a system of linear equations and a constrained least norm problem.

Not only because the $\mathfrak{m}$-DMP inverse, as a new generalized inverse, is an extension of the DMP inverse in Minkowski space, but also because of the wide research background of the DMP inverse, we are convinced that the $\mathfrak{m}$-DMP inverse still has more potential results and applications to explore. Several future directions for the research of the $\mathfrak{m}$-DMP inverse can be described as follows:

(1) The perturbation analysis and iterative methods for the $\mathfrak{m}$-DMP inverse will be two topics worth studying.

(2) Generalizing generalized inverses by weighting is always an important part in studying generalized inverses. And, Meng [24] defined W-weighted DMP inverse of a rectangular matrix, which is a generalization of the DMP inverse of a square matrix. It is equally interesting to discuss the $\mathfrak{m}$-DMP inverse for rectangular matrices.

(3) Inspired by the work of [25], it is natural to ask what interesting characterizations and applications for the two new matrix classes $A^DAA^\sim$ and $A^\sim AA^D$ can be obtained.

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