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On Caputo fractional Bertrand curves in $\mathbb{E}^3$ and $\mathbb{E}_1^3$

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Abstract. In this article, we examine Bertrand curves in $\mathbb{E}^3$ and $\mathbb{E}_1^3$ by using the Caputo fractional derivative which we call α-Bertrand Curves. First, we consider α-Bertrand curves in $\mathbb{E}^3$ and we give a characterization of them. Then, we study α-Bertrand curves in $\mathbb{E}_1^3$ and we prove the necessary and sufficient condition for a α-Bertrand curves in $\mathbb{E}_1^3$ by considering time like, space like and null curves. We also give the related examples by using Python.

1. Introduction

Bertrand curves, named after the French mathematician Joseph Bertrand, are a fascinating class of curves that possess a remarkable property in Euclidean geometry. Bertrand curves have been extensively studied in differential geometry and have found applications in various fields, such as celestial mechanics and optics. Their attractive properties continue to captivate mathematicians and scientists, making them a compelling subject of investigation and interest. Many authors have studied Bertrand curves in the three-dimensional Euclidean space $\mathbb{E}^3$ and the three-dimensional Lorent-Minkowski space $\mathbb{E}_1^3$ such as [3, 10, 11, 22].

On the other hand, the fractional calculus is a generalization of calculus dealing with differentiation of non-integer order. It was first introduced by mathematicians [16, 21] in the late 17th century and early 18th century. Although it is not a new topic, the theory of fractional calculus has growing attention in recent years. Since it is an extention of the integer (real or complex) order classical integrals and derivatives, it serves as an effective and powerful tool to solve differential and integral equations. It has motivated the mathematicians, physicians and engineers; several different type of fractional derivatives (Riemann-Liouville, Caputo, Erd´elyi-Kober, Hadamard, Riesz, etc.). These have been introduced [5, 13, 20] and various kinds of real-world problems have been modeled using fractional derivatives in fields such as fluid mechanics, viscoelastic systems, signal and image processing, and stochastic systems, and so on [4, 24]. Furthermore, the fractional vector calculus, deformation tensors, fractional geometry of manifolds [12, 14] and fractional differential geometry of curves [2, 6–9, 15, 23] were also studied. Each definition of fractional derivative has distinct properties and is suitable for various applications. For example, the derivative of a constant is not zero for some kind of fractional derivatives except Caputo and conformable

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fractional derivatives. But in [1], the author claimed that the conformable derivative is not an operator of fractional order. For this reason, many mathematicians used the Caputo fractional derivative when they are examining the differential geometry of objects. For example, in [23] fractional differential geometry of curves (curvature of curve and Frenet-Serret formulas) was examined and a tangent vector of plane curve is defined by the Caputo fractional derivative. In [2] and [23], the authors introduced the use of the Caputo fractional derivative in the study of differential geometry of curves. In [19], the author examined differential geometry of curves in Lorentzian plane by using Caputo fractional derivative.

In this work, Bertrand curves in $\mathbb{E}^3$ and $\mathbb{E}^3_1$ are examined by using the Caputo fractional derivative. In order to investigate their differential geometric structures mathematical formulations, numerical simulations and illustrative examples are given.

2. Preliminaries

This section contains definition and properties of Caputo fractional derivative and Frenet frame with Euclidean and Minkowski metrics with respect to Caputo fractional derivative.

2.1. Caputo Fractional Derivative

The most well-known fractional derivatives are Riemann-Liouville and Caputo fractional derivatives. The reason of fractional derivative definition is not unique because each type of fractional derivative has some advantages and disadvantages compared to others. For instance, in Riemann-Liouville fractional derivative the derivative of a constant is not zero which contradicts with ordinary derivative. Even though Caputo fractional derivative seems like easy to deal with, it has disadvantages too. The most significant disadvantage of Caputo fractional derivative is the chain rule. The Caputo fractional derivative of composition of two functions gives very complicated expression, it involves infinite series which creates an obstacle when defining differential geometric objects. In order to solve this difficulty, in [23] Yajima used a simplification. In this work, the same argument of Yajima will be used.

Definition 2.1 (Riemann-Liouville fractional integral). *[23] Let f be an integrable function on nonnegative real numbers and $\alpha \in (0,1]$. The Riemann-Liouville fractional integral of f of order α is denoted by $\boldsymbol{I}^{\alpha}_{0+}$ and defined as*

$$\left(\boldsymbol{I}^{\alpha}_{0+}f\right)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}}d\tau \tag{1}$$

where $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$ is the gamma function.

Definition 2.2 (Riemann-Liouville fractional derivative). *[23] Let f be an integrable function on nonnegative real numbers and $\alpha \in (0,1]$. The Riemann-Liouville fractional derivative of f of order α is denoted by $\boldsymbol{D}^{\alpha}_{0+}$ and defined as*

$$\left(\boldsymbol{D}^{\alpha}_{0+}f\right)(t) = \frac{d}{dt}\left(\left(\boldsymbol{I}^{1-\alpha}_{0+}f\right)(t)\right) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha}}d\tau. \tag{2}$$

Definition 2.3 (Caputo fractional derivative). *[23] Let f be an integrable function on nonnegative real numbers and $\alpha \in (0,1]$. The Caputo fractional derivative of f of order α is denoted by ${}^{C}\boldsymbol{D}^{\alpha}_{0+}$ and defined as*

$$\left({}^{C}\boldsymbol{D}^{\alpha}_{0+}f\right)(t) = \left(\boldsymbol{I}^{1-\alpha}_{0+}\frac{df}{dt}\right)(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{1}{(t-\tau)^{\alpha}}\frac{df(\tau)}{d\tau}d\tau. \tag{3}$$

Corollary 2.4. *[23] The Caputo fractional derivative and Riemann Liouville fractional derivative of a function f of order $\alpha \in (0,1]$ has the following relationship.*

$$\left({}^{C}\boldsymbol{D}^{\alpha}_{0+}f\right)(t) = \left(\boldsymbol{D}^{\alpha}_{0+}f\right)(t) - \frac{t^{-\alpha}}{\Gamma(1-\alpha)}f(0). \tag{4}$$

From the definitions above, it is trivial that Caputo fractional derivative has adequate to define differential geometric objects such as curves, since the Caputo fractional derivative of a constant function is zero.

Theorem 2.5. *[23] Let f and t be two functions on appropriate intervals. Then, the Caputo fractional derivative of $f \circ t$ is given by*

$$\left({}^{C}\boldsymbol{D}^{\alpha}_{0+}(f \circ t)\right)(s) = \frac{(f(t(s)) - f(t(0))}{\Gamma(1-\alpha)} s^{1-\alpha} + \sum_{k=1}^{\infty} \binom{\alpha}{k} \frac{s^{k-\alpha}}{\Gamma(k-\alpha+1)} \frac{d^k f(t(s))}{ds^k}. \tag{5}$$

Remark 2.6. *For simplicity, we use the notation $\dfrac{d^{\alpha} f}{ds^{\alpha}}$ instead of the notation $\left({}^{C}\boldsymbol{D}^{\alpha}_{0+} f\right)(t)$ for Caputo fractional derivative.*

Since the expression for the chain rule is problematic in the Caputo fractional derivative, we use the Yajima's simplification [23]. We extract the first term of the series in the (5) and we refer the equation

$$\frac{d^{\alpha} f}{ds^{\alpha}} = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \frac{df}{dt} \frac{dt}{ds} \tag{6}$$

as the chain rule expression for the two functions in the Caputo fractional derivative throughout this paper.

Definition 2.7 (Caputo fractional derivative of a vector valued function). *Let $\boldsymbol{F} : \mathbb{R} \to \mathbb{E}^3$ be a vector valued function defined as*

$$\boldsymbol{F}(t) = (f_1(t), f_2(t), f_3(t))$$

where f_i, $i = 1, 2, 3$ are scalar functions. Then the Caputo fractional derivative of $\boldsymbol{F}$ of order α is given by

$$\frac{d^{\alpha}\boldsymbol{F}}{dt^{\alpha}} = \left(\frac{d^{\alpha} f_1}{dt^{\alpha}}, \frac{d^{\alpha} f_2}{dt^{\alpha}}, \frac{d^{\alpha} f_3}{dt^{\alpha}} \right).$$

Furthermore, if Caputo fractional derivative of $\boldsymbol{F}$ of order α exists, then $\boldsymbol{F}$ is called α-differentiable.

2.2. *Curves in $\mathbb{E}^3$ With Respect To Caputo Fractional Derivative*

In this section, we give the definitions and properties of curves in Euclidean 3-space with respect to Caputo fractional derivative which are given in [23].

Definition 2.8. *(α-arc length of a curve) [23] Suppose that the curve $\gamma : I \subset \mathbb{R} \to \mathbb{E}^3$ is parametrized by its arc length $\tilde{s}$. Let $\alpha \in (0, 1]$, then the α-arc length s of γ is defined as*

$$s = \left(\frac{\alpha^2}{\Gamma(2-\alpha)} \tilde{s} \right)^{\frac{1}{\alpha}}. \tag{7}$$

From the above definition, we can say that if γ is a curve parametrized by its α-arc length s and if $\tilde{s}$ is the arc length parameter of γ then,

$$\frac{d^{\alpha}\gamma}{ds^{\alpha}} = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \frac{df\gamma}{ds} \frac{ds}{d\tilde{s}}. \tag{8}$$

Remark 2.9. *[2] If a curve γ parametrized by α-arc length s, then by using (7),(8) we obtain*

$$\left\| \frac{d^{\alpha}\gamma}{ds^{\alpha}} \right\| = 1. \tag{9}$$

Since the Frenet frame of a curve γ is independent from the choice of parametrization of [2] , we can say that

$$\operatorname{span}\left\{\frac{d^\alpha\gamma}{ds^\alpha}, \left(\frac{d^\alpha\gamma}{ds^\alpha}\right)', \left(\frac{d^\alpha\gamma}{ds^\alpha}\right)''\right\} = \operatorname{span}\left\{\frac{d\gamma}{ds}, \left(\frac{d\gamma}{ds}\right)', \left(\frac{d\gamma}{ds}\right)''\right\} \forall\alpha \in (0,1], \tag{10}$$

where $'$ represents the derivative with respect to s. Therefore, the α-tangent vector of a curve γ is defined as $\mathbf{T}_\alpha = \dfrac{d^\alpha\gamma}{ds^\alpha}$. From (9), we have $\left(\dfrac{d^\alpha\gamma}{ds^\alpha}\right)' \perp \dfrac{d^\alpha\gamma}{ds^\alpha}$. Therefore the α-unit normal of a curve γ is defined as

$$\mathbf{N}_\alpha = \frac{\mathbf{T}'_\alpha}{\|\mathbf{T}'_\alpha\|}. \tag{11}$$

The function $\kappa_\alpha = \|\mathbf{T}'_\alpha\|$ is called the α-curvature of γ, and the vector $\mathbf{B}_\alpha = \mathbf{T}_\alpha \wedge \mathbf{N}_\alpha$ is called the α-binormal of γ. Moreover, the function $\tau_\alpha = \langle \mathbf{N}'_\alpha, \mathbf{B}_\alpha\rangle$ is called the α-torsion of γ.

Theorem 2.10. *[2] Let $\gamma : I \to \mathbb{E}^3$ be a curve parametrized by its α-arc length s. Let $\boldsymbol{T}_\alpha$, $\boldsymbol{N}_\alpha$, $\boldsymbol{B}_\alpha$, κ_α, τ_α be its α-tangent, α-normal, α-binormal, α-curvature and α-torsion of γ, respectively. Then, we have the following system of ordinary differential equations*

$$\begin{bmatrix} \boldsymbol{T}'_\alpha \\ \boldsymbol{N}'_\alpha \\ \boldsymbol{B}'_\alpha \end{bmatrix} = \begin{bmatrix} 0 & \kappa_\alpha & 0 \\ -\kappa_\alpha & 0 & \tau_\alpha \\ 0 & -\tau_\alpha & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}_\alpha \\ \boldsymbol{N}_\alpha \\ \boldsymbol{B}_\alpha \end{bmatrix}. \tag{12}$$

Remark 2.11. *[2] If we take $\alpha = 1$, then we obtain the Frenet frame with respect to ordinary derivative in Euclidean 3−space. In this case, we denote κ_1 and τ_1 as κ and τ, respectively.*

Theorem 2.12. *[2] Let $\gamma : I \to \mathbb{E}^3$ be a curve parametrized by an arbitrary parameter t. Let κ_α and τ_α be its α-curvature and α-torsion respectively. Define*

$$\phi(t) = \left(\frac{\Gamma(2-\alpha)}{\alpha}\right)^{\frac{1}{\alpha}} \left[\alpha \int_0^t \left\|\frac{d\gamma}{du}\right\| du\right]^{1-\frac{1}{\alpha}}, \tag{13}$$

then we have $\kappa_\alpha(t) = \phi(t)\kappa(t)$ and $\tau_\alpha(t) = \phi(t)\tau(t)$.

Corollary 2.13. *[2] Let $\gamma : I \to \mathbb{E}^3$ be a curve and let κ_α and τ_α be the nonzero α-curvature and nonzero α-torsion of γ , respectively. Then we have*

$$\frac{\kappa}{\kappa_\alpha} = \frac{\tau}{\tau_\alpha}, \ \forall\alpha \in (0,1]. \tag{14}$$

Corollary 2.14. *Let $\gamma : I \to \mathbb{E}^3$ be a curve with α-curvature κ_α and α-torsion τ_α. Then γ is a straight line if $\kappa_\alpha = 0$, a plane curve if $\tau_\alpha = 0$ and a generalized helix if $\dfrac{\kappa_\alpha}{\tau_\alpha} = const$.*

Theorem 2.15 (The Fundamental Theorem of Space Curves). *[2] Let $\kappa_\alpha > 0$ and τ_α be real-valued smooth functions on an open interval I which does not contain zero. Then, there exists a unit speed curve $\gamma : I \to \mathbb{E}^3$ parametrized by its α-arc length such that $\kappa_\alpha > 0$ and τ_α are its α-curvature and α-torsion, respectively. Further if $\beta \to \mathbb{E}^3$ is another curve admitting the same κ_α and τ_α, then $\beta(s) = M(\alpha(s))$, for a Euclidean motion M of $\mathbb{E}^3$.*

2.3. Curves in $\mathbb{E}^3_1$ With Respect To Caputo Fractional Derivative

Minkowski 3−space is one of the most commonly used non Euclidean spaces. It has numerous applications, especially in relativity. In this section we give definitions given by Lopez [18]. Furthermore, we generalize the concepts defined by [18],[11],[22] to the Caputo fractional derivative. First, we give the definition of Minkowski 3−space, then give definition of curves in Minkowski 3-space. We consider the curves into two parts, one is non-null curves and the other is null curves.

2.3.1. Basic Notations and Definitions in $\mathbb{E}_1^3$

Definition 2.16. *[18] The bilinear form $\langle\ ,\ \rangle : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is defined as*

$$\langle \boldsymbol{u}, \boldsymbol{v} \rangle = u_1 v_1 + u_2 v_2 - u_3 v_3 \tag{15}$$

*where $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$. Then, the bilinear form $\langle\ ,\ \rangle$ is called **Minkowski inner product**.*

Throughout this paper, we mention $\langle\ ,\ \rangle$ in order to state Minkowski inner product. Likewise, we denote Euclidean inner product by $\langle\ ,\ \rangle^*$. One of the well known orthonormal frames for $\mathbb{R}^3$ equipped by Euclidean inner product is $\{\mathbf{e}_1 = [1\ 0\ 0],\ \mathbf{e}_2 = [0\ 1\ 0],\ \mathbf{e}_3 = [0\ 0\ 1]\}$. This basis is also orthonormal basis for Minkowski $3-$space.

Definition 2.17. *[18] Minkowski norm $\|\ \| : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is defined as $\|\boldsymbol{u}\| = \sqrt{|\langle \boldsymbol{u}, \boldsymbol{u} \rangle|}$. As every norm defines a metric, one can define Minkowski metric as $d : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ such that $d(\boldsymbol{u}, \boldsymbol{v}) = \|\boldsymbol{u} - \boldsymbol{v}\|$.*

Definition 2.18. *[18] Minkowski $3-$space(Lorentz-Minkowski space) is a metric space defined as $\mathbb{E}_1^3 = (\mathbb{R}^3, d)$, where d is Minkowski metric generated by Minkowski inner product. Therefore Minkowski $3-$space is commonly denoted by $\mathbb{E}_1^3 = (\mathbb{R}^3, \langle\ ,\ \rangle)$.*

Definition 2.19. *[18] Let $\boldsymbol{u} = (u_1, u_2, u_3)$ and $\boldsymbol{v} = (v_1, v_2, v_3)$ be two vectors in $\mathbb{E}_1^3$. The wedge product of them is denoted by $\boldsymbol{u} \wedge \boldsymbol{v}$ and defined as*

$$\boldsymbol{u} \wedge \boldsymbol{v} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & -\boldsymbol{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{16}$$

where $\mathbf{e}_i$, $i = 1, 2, 3$, are standard basis vectors in $\mathbb{E}_1^3$.

Definition 2.20. *[18] Let $\boldsymbol{u}$ be a vector in $\mathbb{E}_1^3$. Then $\boldsymbol{u}$ is called,*

- ***spacelike** if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle > 0$ or $\boldsymbol{u} = \boldsymbol{0}$.*
- ***timelike** if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle < 0$.*
- ***lightlike** or **null** if $\langle \boldsymbol{u}, \boldsymbol{u} \rangle = 0$ with $\boldsymbol{u} \neq \boldsymbol{0}$.*

Definition 2.21. *[18] A smooth curve(or shortly, curve) is a differentiable map $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^3$, where I is an open interval.*

Definition 2.22. *A curve $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^3$ is said to be regular if $\alpha'(t) \neq \boldsymbol{0}$, $\forall t \in I$.*

Definition 2.23. *[18] A curve $\gamma(t)$ in $\mathbb{E}_1^3$ is said to be spacelike (respectively timelike, null) if $\gamma'(t)$ is a spacelike(respectively timelike, null) vector.*

Theorem 2.24. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a spacelike or timelike curve. Given $t_0 \in I, \exists \delta, \epsilon > 0$ and a diffeomorphism $\phi : (-\epsilon, \epsilon) \to (t_0 - \delta, t_0 + \delta)$ such that the curve $\beta : (-\epsilon, \epsilon) \to \mathbb{E}_1^3$ defined as $\beta = \alpha \circ \phi$ which satisfies $\|\beta'(s)\| = 1$, $\forall s \in (-\epsilon, \epsilon)$. Then α is parametrized by its arc length [18].*

Theorem 2.25. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}_1^3$ be a null curve whose trace is not a straight line (α'' is not null). Then there exists a new parametrization $\beta(s) = \alpha(\phi(s))$ such that $\|\beta''(s)\| = 1$, $\forall s$. We say α pseudo-parametrized by its arc length [18].*

2.3.2. Non-Null Curves in $\mathbb{E}_1^3$ With Respect To Caputo Fractional Derivative

In this section, we give the definition and properties of non-null curves in Minkowski 3−space with respect to Caputo fractional derivative. Note that each definition based on similar arguments, only the metric is different from the Euclidean case.

Definition 2.26. *(α-arc length of a curve) Suppose that the non-null curve $\gamma : I \subset \mathbb{R} \to \mathbb{E}_1^3$ is parametrized by its arc length $\tilde{s}$. Let $\alpha \in (0, 1]$, then the α-arc length s of γ is defined as*

$$s = \left(\frac{\alpha^2}{\Gamma(2-\alpha)} \tilde{s} \right)^{\frac{1}{\alpha}}. \tag{17}$$

From the above definition, we can say that if γ is a curve parametrized by its α-arc length s and if $\tilde{s}$ is the arc length parameter of γ then,

$$\frac{d^\alpha \gamma}{ds^\alpha} = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)} \frac{d\gamma}{ds} \frac{ds}{d\tilde{s}}. \tag{18}$$

Remark 2.27. *[2] If a non-null curve γ parametrized by α-arc length s, then by using (17),(18) we obtain*

$$\left\| \frac{d^\alpha \gamma}{ds^\alpha} \right\| = 1. \tag{19}$$

Since the Frenet frame of a non-null curve γ is independent of the choice of parametrization [2], we can write the following expression

$$\operatorname{span} \left\{ \frac{d^\alpha \gamma}{ds^\alpha}, \left(\frac{d^\alpha \gamma}{ds^\alpha} \right)', \left(\frac{d^\alpha \gamma}{ds^\alpha} \right)'' \right\} = \operatorname{span} \left\{ \frac{d\gamma}{ds}, \left(\frac{d\gamma}{ds} \right)', \left(\frac{d\gamma}{ds} \right)'' \right\} \forall \alpha \in (0, 1]. \tag{20}$$

Therefore, the α-tangent vector of a non-null curve γ is defined as $\mathbf{T}_\alpha = \frac{d^\alpha \gamma}{ds^\alpha}$. From (19), we have $\left(\frac{d^\alpha \gamma}{ds^\alpha} \right)' \perp \frac{d^\alpha \gamma}{ds^\alpha}$. Therefore the α-unit normal of a curve γ is defined as

$$\mathbf{N}_\alpha = \frac{\mathbf{T}'_\alpha}{\|\mathbf{T}'_\alpha\|}. \tag{21}$$

The function $\kappa_\alpha = \epsilon_2 \|\mathbf{T}'_\alpha\|$ is called the α-curvature of γ, and the vector $\mathbf{B}_\alpha = \mathbf{T}_\alpha \wedge \mathbf{N}_\alpha$ is called the α-binormal of γ where, $\epsilon_2 = \langle \mathbf{N}_\alpha, \mathbf{N}_\alpha \rangle$. Moreover, the function $\tau_\alpha = -\epsilon_2 \langle \mathbf{N}'_\alpha, \mathbf{B}_\alpha \rangle$ is called the α-torsion of γ.

Theorem 2.28. *Let $\gamma : I \to \mathbb{E}_1^3$ be a non-null curve parametrized by its α-arc length s. Let $\boldsymbol{T}_\alpha$, $\boldsymbol{N}_\alpha$, $\boldsymbol{B}_\alpha$, κ_α, τ_α be its α-tangent, α-normal, α-binormal, α-curvature and α-torsion of γ, respectively. Then, we have the following system of ordinary differential equations*

$$\begin{bmatrix} \boldsymbol{T}'_\alpha \\ \boldsymbol{N}'_\alpha \\ \boldsymbol{B}'_\alpha \end{bmatrix} = \begin{bmatrix} 0 & \epsilon_2 \kappa_\alpha & 0 \\ -\epsilon_1 \kappa_\alpha & 0 & \epsilon_3 \tau_\alpha \\ 0 & -\epsilon_2 \tau_\alpha & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}_\alpha \\ \boldsymbol{N}_\alpha \\ \boldsymbol{B}_\alpha \end{bmatrix} \tag{22}$$

where $\epsilon_1 = \langle \boldsymbol{T}_\alpha, \boldsymbol{T}_\alpha \rangle$, $\epsilon_2 = \langle \boldsymbol{N}_\alpha, \boldsymbol{N}_\alpha \rangle$, $\epsilon_3 = \langle \boldsymbol{B}_\alpha, \boldsymbol{B}_\alpha \rangle$.

Remark 2.29. *If we take $\alpha = 1$, then we obtain the Frenet frame in [11]. In this case, we denote κ_1 and τ_1 as κ and τ respectively.*

2.3.3. Null Curves in $\mathbb{E}_1^3$ With Respect To Caputo Fractional Derivative

In this section, we give the definition and properties of null curves in Minkowski 3−space with respect to Caputo fractional derivative.

Definition 2.30. *(Pseudo α-arc length of a curve) Suppose that the null curve $\gamma : I \subset \mathbb{R} \to \mathbb{E}_1^3$ is parametrized by its pseudo arc length $\tilde{s}$. Let $\alpha \in (0, 1]$, then the pseudo α-arc length s of γ is defined as*

$$s = \left(\frac{\alpha^2}{\Gamma(2-\alpha)}\tilde{s}\right)^{\frac{1}{\alpha}}. \tag{23}$$

From the above definition, we can say that if γ is a null curve parametrized by its pseudo α-arc length s and if $\tilde{s}$ is the pseudo arc length parameter of γ then,

$$\frac{d^\alpha\gamma}{ds^\alpha} = \frac{\alpha s^{1-\alpha}}{\Gamma(2-\alpha)}\frac{df\gamma}{ds}\frac{ds}{d\tilde{s}}. \tag{24}$$

Definition 2.31. *Let $\gamma : I \to \mathbb{E}_1^3$ be a null curve parametrized by its pseudo α-arc length s. Then the α-tangent vector $\boldsymbol{T}_\alpha$ is defined as*

$$\boldsymbol{T}_\alpha = \frac{d^\alpha\gamma}{ds^\alpha}. \tag{25}$$

Theorem 2.32. *Let $\gamma : I \to \mathbb{E}_1^3$ be a null curve parametrized by its pseudo α-arc length s. Then, we have the following system of ordinary differential equations*

$$\begin{bmatrix} \boldsymbol{T}'_\alpha \\ \boldsymbol{N}'_\alpha \\ \boldsymbol{B}'_\alpha \end{bmatrix} = \begin{bmatrix} 0 & \kappa_\alpha & 0 \\ \kappa_\alpha & 0 & -\tau_\alpha \\ 0 & -\tau_\alpha & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T}_\alpha \\ \boldsymbol{N}_\alpha \\ \boldsymbol{B}_\alpha \end{bmatrix}, \tag{26}$$

where $\epsilon_1 = \langle \boldsymbol{T}_\alpha, \boldsymbol{T}_\alpha \rangle$, $\epsilon_2 = \langle \boldsymbol{N}_\alpha, \boldsymbol{N}_\alpha \rangle$, $\epsilon_3 = \langle \boldsymbol{B}_\alpha, \boldsymbol{B}_\alpha \rangle$ defines the Frenet frame of γ. The vectors $\boldsymbol{T}_\alpha$, $\boldsymbol{N}_\alpha$, $\boldsymbol{B}_\alpha$, are called the α-tangent, α-normal, α-binormal of γ, respectively and the functions κ_α, τ_α are called the α-curvature and α-torsion of γ, respectively.

Remark 2.33. *If we take $\alpha = 1$ then we obtain the Frenet frame in [11]. In this case, we denote κ_1 and τ_1 as κ and τ respectively.*

The Frenet frame of a null curve γ is independent from the choice of parametrization, hence we can write the following expression

$$\text{span}\left\{\frac{d^\alpha\gamma}{ds^\alpha}, \left(\frac{d^\alpha\gamma}{ds^\alpha}\right)', \left(\frac{d^\alpha\gamma}{ds^\alpha}\right)''\right\} = \text{span}\left\{\frac{d\gamma}{ds}, \left(\frac{d\gamma}{ds}\right)', \left(\frac{d\gamma}{ds}\right)''\right\} \forall\alpha \in (0, 1]. \tag{27}$$

Hence, we can state the following theorem.

Theorem 2.34. *(Fundamental Theorem of Curves in $\mathbb{E}_1^3$) Let $\kappa_\alpha > 0$ and τ_α be smooth functions on real numbers defined on an open interval I which does not contain zero. Then, there is a curve $\gamma : I \to \mathbb{E}^3$ where $\kappa_\alpha > 0$ and τ_α are its α-curvature and α-torsion respectively. If there is another curve like this, then it is a translation or rotation of γ.*

Remark 2.35. *The fundamental theorem of curves in $\mathbb{E}_1^3$ is valid for both null and non-null curves.*

3. Fractional Bertrand Curves in $\mathbb{E}^3$

In this section, the definition of Bertrand curves in $\mathbb{E}^3$ is given with respect to Caputo fractional derivative and some important classifications are obtained by using the definitions and theorems in [10] whether a curve is Bertrand or not .

Definition 3.1. *A regular α-differentiable curve γ is said to be non degenerate if ${}^{C}D_t^{\alpha}\gamma \wedge ({}^{C}D_t^{\alpha})^2\gamma \neq \mathbf{0}$.*

Definition 3.2. *Let $\gamma : I \to \mathbb{E}^3$ be α-differentiable curve with $\kappa_\alpha \neq 0$. Then γ is called a fractional Bertrand curve or α-Bertrand curve if there exists $\bar{\gamma} : I \to \mathbb{E}^3$ such that*

$$\bar{\gamma} = \gamma + \lambda \mathbf{N}_\alpha \tag{28}$$

and $\bar{\mathbf{N}}_\alpha = \mathbf{N}_\alpha$, where $\bar{\mathbf{N}}_\alpha$ is the principal normal of $\bar{\gamma}$ and λ is a smooth function. γ and $\bar{\gamma}$ are called fractional Bertrand mates of order α or α-Bertrand mates.

Theorem 3.3. *If γ and $\bar{\gamma}$ are α-Bertrand mates in $\mathbb{E}^3$, i.e. $\bar{\gamma} = \gamma + \lambda \mathbf{N}_\alpha$, then λ is a constant function.*

Proof. As $\bar{\gamma}$ and γ are α-Bertrand mates, we have $\bar{\gamma} = \gamma + \lambda \mathbf{N}_\alpha$ and taking derivative with respect to α-arc length s of γ, we get

$$\frac{d\bar{\gamma}}{ds} = \frac{d\bar{\gamma}}{d\bar{s}}\frac{d\bar{s}}{ds} = \frac{d\gamma}{ds} + \frac{d\lambda}{ds}\mathbf{N}_\alpha + \lambda\frac{d\mathbf{N}_\alpha}{ds} \tag{29}$$

by using the fact that $\dfrac{d\gamma}{ds} = \dfrac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}}\dfrac{d^\alpha\gamma}{ds^\alpha}$ and we obtain

$$\frac{d\bar{s}}{ds}\frac{\Gamma(2-\alpha)}{\alpha\bar{s}^{1-\alpha}}\bar{\mathbf{T}}_\alpha = \left(\frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} - \lambda\kappa_\alpha\right)\mathbf{T}_\alpha + \frac{d\lambda}{ds}\mathbf{N}_\alpha + \lambda\tau_\alpha\mathbf{B}_\alpha. \tag{30}$$

Since, $\bar{\mathbf{N}}_\alpha = \mathbf{N}_\alpha$, we can take Euclidean inner product of both sides of (30) by $\mathbf{N}_\alpha$ and obtain $\dfrac{d\lambda}{ds} = 0$, which means λ is a constant. □

Theorem 3.4. *Let $\gamma : I \to \mathbb{E}^3$ be a α-differentiable non-degenerate curve with $\tau_\alpha \neq 0$ (torsion of γ w.r.t α-Frenet frame) and let A be a non-zero constant. Then, γ and $\bar{\gamma}$ are Bertrand mates with $\bar{\gamma} = \gamma + A\mathbf{N}_\alpha$ if and only if there exists a constant B such that $A\kappa_\alpha + B\tau_\alpha = \dfrac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}}$ and $B\kappa_\alpha - A\tau_\alpha \neq 0$.*

Proof. From the previous theorem, we have

$$\frac{d\bar{s}}{ds}\frac{\Gamma(2-\alpha)}{\alpha\bar{s}^{1-\alpha}}\bar{\mathbf{T}}_\alpha = \left(\frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} - A\kappa_\alpha\right)\mathbf{T}_\alpha + A\tau_\alpha\mathbf{B}_\alpha.$$

Then, due to the fact that $\mathbf{N}_\alpha = \bar{\mathbf{N}}_\alpha$, there exists a smooth angle function θ such that

$$\begin{bmatrix}\bar{\mathbf{B}}_\alpha\\ \bar{\mathbf{T}}_\alpha\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\mathbf{B}_\alpha\\ \mathbf{T}_\alpha\end{bmatrix}. \tag{31}$$

Hence, we obtain

$$\frac{\alpha\bar{s}^{1-\alpha}}{\Gamma(2-\alpha)}\left(\frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} - A\kappa_\alpha\right)\frac{ds}{d\bar{s}} = \cos\theta$$

and

$$\frac{\alpha\bar{s}^{1-\alpha}}{\Gamma(2-\alpha)}A\tau_\alpha\frac{ds}{d\bar{s}} = \sin\theta.$$

Thus,

$$-A\tau_\alpha \cos\theta + \left(\frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} - A\kappa_\alpha\right)\sin\theta = 0. \tag{32}$$

Taking the derivative of $\bar{\mathbf{T}}_\alpha = \cos\theta \mathbf{T}_\alpha + \sin\theta \mathbf{B}_\alpha$ with respect to s gives us

$$\frac{d\bar{s}}{ds}\bar{\kappa}_\alpha \bar{\mathbf{N}}_\alpha = -\theta' \sin\theta \mathbf{T}_\alpha + (\kappa_\alpha \cos\theta - \tau_\alpha \sin\theta)\mathbf{N}_\alpha + \theta' \cos\theta \mathbf{B}_\alpha.$$

Since $\mathbf{N}_\alpha = \bar{\mathbf{N}}_\alpha$, we have $\theta' = 0$ and therefore θ is a constant angle. From (31) we have $\sin\theta \neq 0$.

Hence, (32) implies

$$A\tau_\alpha \cot\theta + A\kappa_\alpha = \frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}}.$$

If we choose $B = A\cot\theta$, we get

$$A\kappa_\alpha + B\tau_\alpha = \frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}}.$$

Moreover,

$$\frac{d\bar{s}}{ds}\bar{\kappa}_\alpha = (\kappa_\alpha \cos\theta - \tau_\alpha \sin\theta) = \frac{\sin\theta}{A}(-A\tau_\alpha + B\kappa_\alpha) \neq 0.$$

Now, conversely assume that the conditions $A\kappa_\alpha + B\tau_\alpha = \dfrac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}}$ and $A\tau_\alpha - B\kappa_\alpha \neq 0$ hold and define $\bar{\gamma} = \gamma + A\mathbf{N}_\alpha$. Then we have

$$\frac{d\bar{s}}{ds}\frac{\Gamma(2-\alpha)}{\alpha \bar{s}^{1-\alpha}}\bar{\mathbf{T}}_\alpha = \left(\frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} - A\kappa_\alpha\right)\mathbf{T}_\alpha + A\tau_\alpha \mathbf{B}_\alpha = \tau_\alpha(B\mathbf{T}_\alpha + A\mathbf{B}_\alpha) \tag{33}$$

and taking derivative of (33) with respect to s gives us $\mathbf{N}_\alpha = \bar{\mathbf{N}}_\alpha$. □

4. Fractional Bertrand Curves in $\mathbb{E}_1^3$

In this section, we generalize Bertrand curves with respect to Caputo fractional derivative to the Minkowski 3-space by taking into account the definitions and classifications of [3, 11, 22].

4.1. *Spacelike Fractional Bertrand Curves in $\mathbb{E}_1^3$*

Spacelike curves can be categorized by type of their principal normals. In this section, we consider Caputo fractional spacelike curves having principal normals either spacelike or timelike. We give one definition for spacelike Bertrand curve but we give different categorizations for each case.

Definition 4.1. *A spacelike α-differentiable non-degenerate curve $\gamma : I \to \mathbb{E}_1^3$ with $\kappa_\alpha \neq 0$ is called a fractional Bertrand curve or α-Bertrand curve if there exists $\bar{\gamma} : I \to \mathbb{E}_1^3$ such that $\bar{\gamma} = \gamma + \lambda\mathbf{N}_\alpha$ and $\mathbf{N}_\alpha = \bar{\mathbf{N}}_\alpha$ where λ is a smooth function on I. γ and $\bar{\gamma}$ are called fractional Bertrand mates of order α or α-Bertrand mates.*

Theorem 4.2. *If spacelike curves γ and $\bar{\gamma}$ are α-Bertrand mates in $\mathbb{E}_1^3$ i.e. $\bar{\gamma} = \gamma + \lambda\mathbf{N}_\alpha$, then λ is a constant function.*

Proof. As $\bar{\gamma}$ and γ are α-Bertrand mates, we have $\bar{\gamma} = \gamma + \lambda\mathbf{N}_\alpha$ and taking derivative with respect to α-arc length s of γ, we get

$$\frac{d\bar{\gamma}}{ds} = \frac{d\bar{\gamma}}{d\bar{s}}\frac{d\bar{s}}{ds} = \frac{d\gamma}{ds} + \frac{d\lambda}{ds}\mathbf{N}_\alpha + \lambda\frac{d\mathbf{N}_\alpha}{ds}$$

by using the fact that $\dfrac{d\gamma}{ds} = \dfrac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}}\dfrac{d^{\alpha}\gamma}{ds^{\alpha}}$ and we obtain

$$\frac{d\bar{s}}{ds}\frac{\Gamma(2-\alpha)}{\alpha\bar{s}^{1-\alpha}}\bar{\mathbf{T}}_{\alpha} = \left(\frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} - \lambda\kappa_{\alpha}\right)\mathbf{T}_{\alpha} + \frac{d\lambda}{ds}\mathbf{N}_{\alpha} + \epsilon_3\lambda\tau_{\alpha}\mathbf{B}_{\alpha}. \tag{34}$$

Since $\bar{\mathbf{N}}_{\alpha} = \mathbf{N}_{\alpha}$, we can take Lorentzian inner product of both sides of (34) by $\mathbf{N}_{\alpha}$ and we get $\dfrac{d\lambda}{ds} = 0$ which means λ is a constant. □

Theorem 4.3. *Let $\gamma : I \to \mathbb{E}^3_1$ be a Caputo fractional non-degenerate spacelike curve parametrized by its α-arc length s with spacelike normal, nonzero curvature κ_{α} and torsion τ_{α}. Then γ is an α-Bertrand curve with α-Bertrand mate $\bar{\gamma}$ if and only if one of the following statements holds:*

(i) There exist constants λ, h satisfying

$$\frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} - A\kappa_{\alpha} = -h\lambda\tau_{\alpha},\ h^2 < 1,\ \tau_{\alpha} - h\kappa_{\alpha} \neq 0,\ h\tau_{\alpha} - \kappa_{\alpha} \neq 0$$

and in this case Bertrand mate $\bar{\gamma}$ is a timelike curve.

(ii) There exist constants λ, h satisfying

$$\frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} - A\kappa_{\alpha} = -h\lambda\tau_{\alpha},\ h^2 > 1,\ \tau_{\alpha} - h\kappa_{\alpha} \neq 0,\ h\tau_{\alpha} - \kappa_{\alpha} \neq 0$$

and in this case Bertrand mate $\bar{\gamma}$ is a spacelike curve with spacelike normal.

Proof. Let γ be a α-differentiable spacelike curve with spacelike normal and parametrized by its α-arc length s with non zero κ, τ.

(i) We prove the four conditions with three steps. Let $\bar{\gamma}$ be timelike and defined as

$$\bar{\gamma} = \gamma + \lambda\mathbf{N}_{\alpha}. \tag{35}$$

Taking derivative of (35) with respect to s gives

$$\frac{d\bar{s}}{ds}\frac{\Gamma(2-\alpha)}{\alpha\bar{s}^{1-\alpha}}\bar{\mathbf{T}}_{\alpha} = \left(\frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} - \lambda\kappa_{\alpha}\right)\mathbf{T}_{\alpha} + A\tau_{\alpha}\mathbf{B}_{\alpha}. \tag{36}$$

Step I.

If we define $\omega = \dfrac{d\bar{s}}{ds}\dfrac{\Gamma(2-\alpha)}{\alpha\bar{s}^{1-\alpha}}$ and $\rho = \dfrac{d\bar{s}}{ds}$, taking Minkowski inner product of equation (36) by itself gives

$$-\omega^2 = \left(\frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} - \lambda\kappa_{\alpha}\right)^2 - (\lambda\tau_{\alpha})^2.$$

Let $\delta = \dfrac{\frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} - \lambda\kappa_{\alpha}}{\omega}$ and $\mu = -\dfrac{\lambda\tau_{\alpha}}{\omega}$. Clearly it can be seen that $\delta^2 - \mu^2 = -1$, so we have

$$\bar{\mathbf{T}}_{\alpha} = \delta\mathbf{T}_{\alpha} + \mu\mathbf{B}_{\alpha}. \tag{37}$$

Taking the derivative of equation (37) with respect to s yields

$$\rho\bar{\kappa}_{\alpha}\bar{\mathbf{N}}_{\alpha} = \delta'\mathbf{T}_{\alpha} + (\delta\kappa_{\alpha} - \mu\tau_{\alpha})\mathbf{N}_{\alpha} + \mu'\mathbf{B}_{\alpha}. \tag{38}$$

Since $\mathbf{N}_\alpha = \bar{\mathbf{N}}_\alpha$, we conclude that $\delta' = \mu' = 0$ which implies δ and μ are constants.

Furthermore, we know that $\mu \neq 0$ by assumptions, hence we can define a constant $h = \dfrac{\delta}{\mu}$, which implies $\dfrac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} - \lambda\kappa_\alpha = -h\lambda\tau_\alpha$.

Step II.

On the other hand, we know $\omega\bar{\kappa}_\alpha\bar{\mathbf{N}}_\alpha = (\delta\kappa_\alpha - \mu\tau_\alpha)\mathbf{N}_\alpha$ and if we take Minkowski inner product of equation (38) by itself, we obtain

$$\rho^2\bar{\kappa}_\alpha^2 = \frac{(h\kappa_\alpha - \tau_\alpha)^2}{1-h^2},$$

hence we can say that $(h\kappa_\alpha - \tau_\alpha) \neq 0$ and $h^2 < 1$.

Step III.

If we define $\nu = \dfrac{\delta\kappa_\alpha - \mu\tau_\alpha}{\rho\bar{\kappa}_\alpha}$, then we can say

$$\bar{\mathbf{N}}_\alpha = \nu\mathbf{N}_\alpha. \tag{39}$$

Differentiation of (39) yields

$$\rho\bar{\tau}_\alpha\bar{\mathbf{B}}_\alpha = P\mathbf{T}_\alpha + Q\mathbf{B}_\alpha \tag{40}$$

where $P = \dfrac{\lambda\tau_\alpha(\tau_\alpha - h\kappa_\alpha)}{\omega\rho\bar{\kappa}_\alpha(1-h^2)}(h\tau_\alpha - \kappa_\alpha)$ and $Q = hP$, which proves $h\tau_\alpha - \kappa_\alpha \neq 0$.

For the proof of the sufficiency part, let us define a curve $\bar{\gamma}$ as $\bar{\gamma} = \gamma + \lambda\mathbf{N}_\alpha$ and $m_1 = sgn(\lambda\tau_\alpha)$, $m_2 = sgn(h\kappa_\alpha - \tau_\alpha)$, $m_3 = sgn(\kappa_\alpha - h\tau_\alpha)$. Then if we use the properties, we obtain

$$\bar{\mathbf{T}}_\alpha = -\frac{m_1}{\sqrt{1-h^2}}(h\mathbf{T}_\alpha + \mathbf{B}_\alpha),\ \bar{\mathbf{N}}_\alpha = -m_1m_2\mathbf{N}_\alpha,\ \bar{\mathbf{B}}_\alpha = \frac{m_1m_2m_3}{(\sqrt{1-h^2})}(\mathbf{T}_\alpha + h\mathbf{B}_\alpha).$$

From these equations, we conclude that $\bar{\gamma}$ is timelike.

(ii) Same argument can be applied to spacelike curves.

□

Theorem 4.4. *Let γ be α-differentiable spacelike curve with spacelike normal which is parametrized by its α-arc length with non-zero curvature κ_α and torsion τ_α. Assume that $\bar{\gamma}$ is $\alpha-$differentiable null curve with $\alpha-$curvature $\bar{\kappa}_\alpha = 1$. Then γ and $\bar{\gamma}$ are Bertrand curves if and only if there exists constants λ and $h = \pm 1$ satisfying $\dfrac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} - \lambda\kappa_\alpha = h\lambda\tau_\alpha$ and $h\kappa_\alpha + \tau_\alpha \neq 0$.*

Proof. An argument similar to proof of the Theorem 4.3 can be applied by using the definition of null curve. The reason of lack of the one condition comes from nullity. □

Theorem 4.5. *Let γ be $\alpha-$differentiable spacelike curve whose normal is timelike with nonzero κ_α and torsion τ_α. Then, γ is a Bertrand curve if and only if there exist constants λ and h where $h^2 < 1$ satisfying*

$$\frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} - \lambda\kappa_\alpha = h\lambda\tau_\alpha \tag{41}$$

and $\tau_\alpha - h\kappa_\alpha \neq 0$, $h\tau_\alpha + \kappa_\alpha \neq 0$.

Proof. It is easy to see this result by applying the method of the proof of Theorem 4.3. The differences of the less than/greater than sign and minus sign come from spacelike/timelike differences. □

Example 4.6. *Consider the spacelike curve with spacelike normal $\gamma : I \to \mathbb{E}_1^3$ defined by*

$$\gamma(s) = \left(\sqrt{2}s, \cosh(s), \sinh(s)\right).$$

By choosing $h = \frac{-1}{\sqrt{2}}$ and $\lambda = \frac{1}{2}$ the curve $\bar{\gamma}$ can be obtained as:

$$\bar{\gamma} = \gamma + \lambda \boldsymbol{N}_\alpha.$$

It can be easily verified that $\bar{\gamma}$ is a timelike. We want to sketch this curve for different values of α. To see variation along α properly, we choose the values for α as 0.1 and 0.9.

For $\alpha = 0.9$, we have

$$\bar{\gamma} = \Bigg(\frac{\sqrt{2}\left(s^{1.8} + 0.0447480047275141\right)}{s^{0.8}},$$
$$\frac{0.0447480047275141 \cdot \sinh(s)}{s^{0.8}} + 0.447480047275141 \cdot s^{0.2}\cosh(s) + \cosh(s),$$
$$\frac{0.0447480047275141 \cdot \cosh(s)}{s^{0.8}} + 0.447480047275141 \cdot s^{0.2}\sinh(s) + \sinh(s)\Bigg).$$

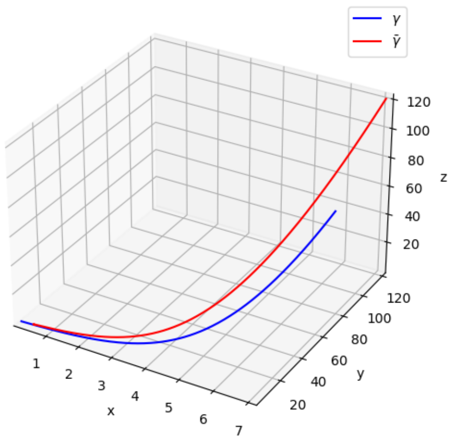

Figure 1: γ and its Bertrand mate $\bar{\gamma}$ for $\alpha = 0.9$

For $\alpha = 0.1$, we have $\bar{\gamma}$ as

$$\bar{\gamma} = \Bigg(\sqrt{2} \cdot \left(0.00486489896951851 \cdot s^{0.8} + s\right),$$
$$s^{0.8} \cdot \left(0.00540544329946501 \cdot s\cosh(s) + 0.00486489896951851\sinh(s)\right) + \cosh(s),$$
$$s^{0.8} \cdot \left(0.00540544329946501 \cdot s\sinh(s) + 0.00486489896951851\cosh(s)\right) + \sinh(s)\Bigg)$$

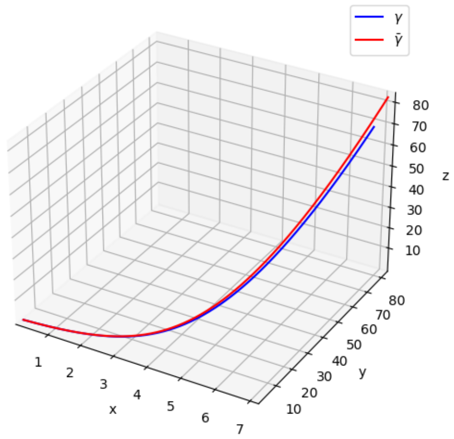

Figure 2: γ and its Bertrand mate $\bar{\gamma}$ for $\alpha = 0.1$

Example 4.7. *Consider the spacelike curve with spacelike normal $\gamma : I \to \mathbb{E}_1^3$ defined as*

$$\gamma(s) = \left(\sqrt{2}s, \cosh(s), \sinh(s)\right).$$

By choosing $\alpha = \frac{1}{2}$, $h = -\sqrt{2}$ and $\lambda = \frac{1}{3}$ the following curve can be obtained:

$$\bar{\gamma} = \gamma + \lambda \boldsymbol{N}_\alpha.$$

It can be easily verified that $\bar{\gamma}$ is spacelike curve with spacelike normal. We want to sketch this curve for different values of α. To see variation along α properly, we choose the values for α as 0.1 *and* 0.9.
For $\alpha = 0.9$, we have

$$\bar{\gamma} = \left(\frac{\sqrt{2}\left(s^{1.98} + 0.00330439284533693\right)}{s^{0.98}}, \right.$$
$$\frac{0.00330439284533693 \cdot \sinh(s)}{s^{0.98}} + 0.330439284533692 \cdot s^{0.02} \cosh(s) + \cosh(s),$$
$$\left. \frac{0.00330439284533693 \cdot \cosh(s)}{s^{0.98}} + 0.330439284533692 \cdot s^{0.02} \sinh(s) + \sinh(s) \right).$$

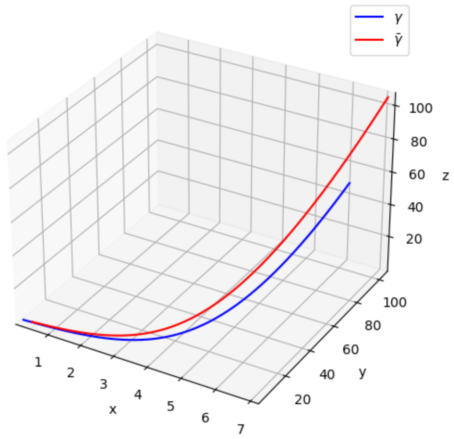

Figure 3: γ and its Bertrand mate $\bar{\gamma}$ for $\alpha = 0.9$

For $\alpha = 0.1$, we have $\bar{\gamma}$ as

$$\bar{\gamma} = \Big(\sqrt{2}\left(0.00324326597967901 s^{0.8} + s\right),$$
$$s^{0.8}\left(0.00360362886631001 \cdot s\cosh(s) + 0.00324326597967901 \cdot \sinh(s)\right) + \cosh(s),$$
$$s^{0.8}\left(0.00360362886631001 \cdot s\sinh(s) + 0.00324326597967901 \cdot \cosh(s)\right) + \sinh(s) \Big)$$

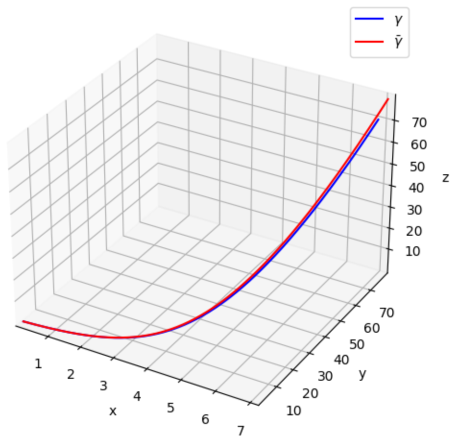

Figure 4: γ and its Bertrand mate $\bar{\gamma}$ for $\alpha = 0.1$

Example 4.8. *Consider the spacelike curve with spacelike normal $\gamma : I \to \mathbb{E}^3_1$ defined as*

$$\gamma(s) = \left(\sqrt{2}s, \cosh(s), \sinh(s)\right).$$

By choosing $\lambda = -1 - \sqrt{2}$, the following curve can be obtained:

$$\bar{\gamma} = \gamma + \lambda \mathbf{N}_{\alpha}.$$

We want to sketch this curve for different values of α. To see variation along α properly, we choose the values for α as 0.1 *and* 0.9*.*

For $\alpha = 0.9$, we have

$$\bar{\gamma} = \Bigg(\frac{\sqrt{2}\,(s^{1.8} - 0.03707046089}{s^{0.8}},$$
$$\frac{s^{0.8}\cosh(s) - 2.28390229446 \cdot (0.9460233055006 \cdot s\cosh(s) + 0.09460233055006 \cdot \sinh(s))}{s^{0.8}},$$
$$\frac{s^{0.8}\sinh(s) - 2.28390229446 \cdot (0.9460233055006 \cdot s\sinh(s) + 0.09460233055006 \cdot \cosh(s))}{s^{0.8}} \Bigg).$$

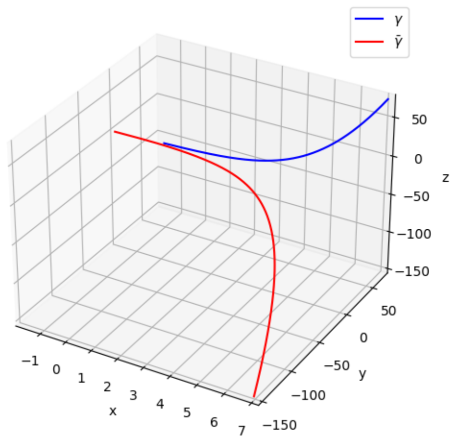

Figure 5: γ and its Bertrand mate $\bar{\gamma}$ for $\alpha = 0.9$

For $\alpha = 0.1$, we have

$$\bar{\gamma} = \Bigg(\sqrt{2}\left(-0.00972979793903703 \cdot s^{0.8} \cdot \left(1 + \sqrt{2}\right) + s\right),$$
$$- 0.103975413434764 \cdot s^{0.8} \cdot \left(1 + \sqrt{2}\right) \Big[0.103975413434764 \cdot s\cosh(s)$$
$$+ 0.0935778720912873 \cdot \sinh(s) \Big] + \cosh(s),$$
$$- 0.103975413434764 \cdot s^{0.8} \cdot \left(1 + \sqrt{2}\right) \Big[0.103975413434764 \cdot s\sinh(s)$$
$$+ 0.0935778720912873 \cdot \cosh(s) \Big] + \sinh(s) \Bigg)$$

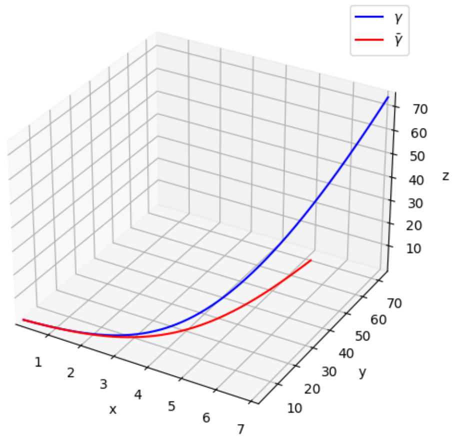

Figure 6: γ and its Bertrand mate $\bar{\gamma}$ for $\alpha = 0.1$

Example 4.9. *Consider the spacelike curve with timelike normal $\gamma : I \to \mathbb{E}_1^3$ defined as*

$$\gamma(s) = \left(\frac{s}{\sqrt{2}}, \frac{\sinh(s)}{\sqrt{2}}, \frac{\cosh(s)}{\sqrt{2}}\right).$$

By choosing $h = 1 + \sqrt{2}$, $\lambda = 1$, the following curve can be obtained:

$$\bar{\gamma} = \gamma + \lambda \boldsymbol{N}_\alpha.$$

It can be easily verified that $\bar{\gamma}$ is spacelike curve with timelike normal. We want to sketch this curve for different values of α. To see variation along α properly, we choose the values for α as 0.1 and 0.9.

For $\alpha = 0.9$, we have

$$\bar{\gamma} = \left(\frac{\sqrt{2}\left(s^{1.8} + 0.0894960094550281\right)}{2s^{0.8}},\right.$$
$$\frac{\sqrt{2}\left(s^{0.8}\sinh(s) + 0.894960094550282 \cdot s\sinh(s) + 0.0894960094550281 \cdot \cosh(s)\right)}{2s^{0.8}},$$
$$\left.\frac{\sqrt{2}\left(s^{0.8}\cosh(s) + 0.894960094550282 \cdot s\cosh(s) + 0.0894960094550281 \cdot \sinh(s)\right)}{2s^{0.8}}\right).$$

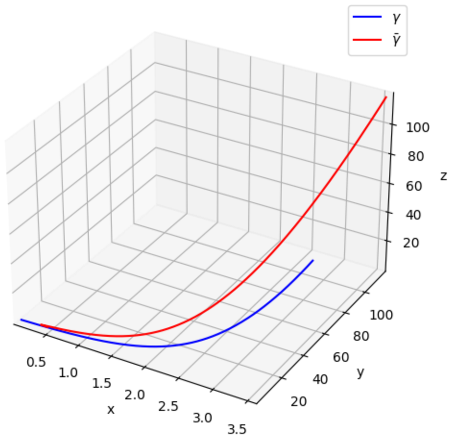

Figure 7: γ and its Bertrand mate $\bar{\gamma}$ for $\alpha = 0.9$

For $\alpha = 0.1$ *we have*

$$\bar{\gamma} = \left(\frac{\sqrt{2} \cdot (0.00972979793903703 \cdot s^{0.8} + s)}{2}, \right.$$
$$\frac{\sqrt{2}\left(s^{1.8} \cdot (0.01081088659893 \cdot \sinh(s) + 0.00972979793903703 \cdot \cosh(s)) + \sinh(s)\right)}{2},$$
$$\left. \frac{\sqrt{2}\left(s^{1.8} \cdot (0.01081088659893 \cdot \cosh(s) + 0.00972979793903703 \sinh(s)) + \cosh(s)\right)}{2} \right).$$

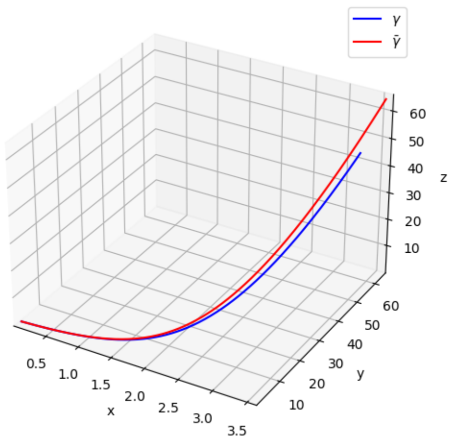

Figure 8: γ and its Bertrand mate $\bar{\gamma}$ for $\alpha = 0.1$

4.2. *Timelike Fractional Bertrand Curves in* $\mathbb{E}_1^3$

Unlike spacelike curves, timelike curves does not have sub-categories. In this part, we define $\alpha-$Bertrand curves for timelike curves and give classifications about Bertrand mates where one of them is timelike.

Definition 4.10. *An* $\alpha-$*differentiable timelike non-degenerate curve* $\gamma : I \to \mathbb{E}_1^3$ *with* $\kappa_\alpha \neq 0$ *is called fractional Bertrand curve or* $\alpha-$*Bertrand curve if there exists* $\bar{\gamma} : I \to \mathbb{E}_1^3$ *such that* $\bar{\gamma} = \gamma + \lambda \mathbf{N}_\alpha$ *and* $\mathbf{N}_\alpha = \bar{\mathbf{N}}_\alpha$ *where* λ *is a smooth function on I.*

Theorem 4.11. *If a timelike curve γ and $\bar{\gamma}$ are α-differentiable Bertrand curves in $\mathbb{E}^3_1$ with $\bar{\gamma} = \gamma + \lambda \mathbf{N}_\alpha$, then λ is a constant function.*

Proof. As $\bar{\gamma}$ and γ are Bertrand mates, we can write

$$\bar{\gamma} = \gamma + \lambda \mathbf{N}_\alpha.$$

Derivative with respect to $\alpha-$arc length s of γ is

$$\frac{\Gamma(2-\alpha)}{\alpha \bar{s}^{1-\alpha}} \bar{\mathbf{T}}_\alpha = \Big(\frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} + \lambda \kappa_\alpha\Big)\mathbf{T}_\alpha + \lambda' \mathbf{N}_\alpha + \lambda \tau_\alpha \mathbf{B}_\alpha.$$

Taking Minkowski inner product of both side by $\bar{\mathbf{N}}_\alpha$ gives the result. □

Theorem 4.12. *Let $\gamma : I \to \mathbb{E}^3_1$ be an $\alpha-$differentiable non-degenerate timelike curve parametrized by its $\alpha-$arc length s with spacelike normal, non-zero curvature κ_α and torsion τ_α. Then γ is Bertrand curve with Bertrand mate $\bar{\gamma}$ if and only if one of the following statements holds.*

1. *There exist constants λ, h satisfying*

$$h^2 < 1,\ \frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} + \lambda \kappa_\alpha = h \lambda \tau_\alpha,\ \tau_\alpha - h\kappa_\alpha \neq 0,\ h\tau_\alpha - \kappa_\alpha \neq 0,$$

and in this case Bertrand mate $\bar{\gamma}$ is a spacelike curve with spacelike normal.

2. *There exist constants λ, h satisfying*

$$h^2 > 1,\ \frac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} + \lambda \kappa_\alpha = h \lambda \tau_\alpha,\ \tau_\alpha - h\kappa_\alpha \neq 0,\ h\tau_\alpha - \kappa_\alpha \neq 0,$$

and in this case Bertrand mate $\bar{\gamma}$ is a timelike curve.

Proof. For both cases, the process we used for the proof of Theorem 4.3 works. □

Theorem 4.13. *Let γ be an $\alpha-$differentiable timelike curve parametrized by its $\alpha-$arc length s with non-zero curvature κ_α and torsion τ_α. Let $\bar{\gamma}$ be an $\alpha-$differentiable null curve with curvature $\bar{\kappa}_\alpha = 1$. Then, γ and $\bar{\gamma}$ are Bertrand mates if and only if there exists constants λ and $h^2 = 1$ satisfying $\dfrac{\Gamma(2-\alpha)}{\alpha s^{1-\alpha}} + \lambda\kappa_\alpha = h\lambda\tau_\alpha$ and $h\kappa_\alpha - \tau_\alpha \neq 0$.*

Proof. Again, using the definition of a null curve and applying the proof of the Theorem 4.3 gives us the result. Notice that one of the conditions is missing due to the nullity. □

Example 4.14. *Consider the spacelike curve γ with timelike normal $\gamma : I \to \mathbb{E}^3_1$ which is defined by*

$$\gamma(s) = \left(s,\ \sqrt{2}\cosh(s),\ \sqrt{2}\sinh(s)\right).$$

By choosing $h = \sqrt{2}$, $\lambda = -\frac{1}{2\sqrt{2}}$, the following curve can be obtained:

$$\bar{\gamma} = \gamma + \lambda \mathbf{N}_\alpha$$

It can be easily verified that $\bar{\gamma}$ is timelike.To see variation along α properly, we choose the values for α as 0.1 and 0.9. For $\alpha = 0.9$, we have

$$\bar{\gamma} = \left(\frac{s^{1.8} - 0.022374002363757\sqrt{2}}{s^{0.8}},\right.$$
$$- \frac{0.0447480047275141\sinh(s)}{s^{0.8}} - 0.447480047275141 s^{0.2}\cosh(s) + \sqrt{2}\cosh(s),$$
$$\left. - \frac{0.0447480047275141\cosh(s)}{s^{0.8}} - 0.447480047275141 s^{0.2}\sinh(s) + \sqrt{2}\sinh(s)\right).$$

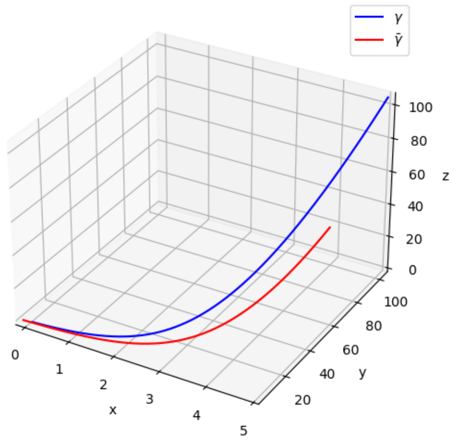

Figure 9: γ and its Bertrand mate $\bar{\gamma}$ for $\alpha = 0.9$

for $\alpha = 0.1$ *we have*

$$\bar{\gamma} = \Big(-0.00243244948475926 \cdot \sqrt{2} \cdot s^{0.8} + s,$$
$$-0.00486489896951851 \cdot s^{0.8} \sinh(s) - 0.00540544329946501 \cdot s^{1.8} \cosh(s) + \sqrt{2} \cosh(s),$$
$$-0.00486489896951851 \cdot s^{0.8} \cosh(s) - 0.00540544329946501 \cdot s^{1.8} \sinh(s) + \sqrt{2} \sinh(s) \Big)$$

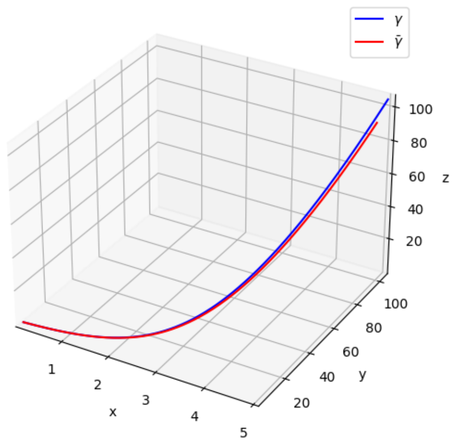

Figure 10: γ and its Bertrand mate $\bar{\gamma}$ for $\alpha = 0.1$

Example 4.15. *Consider the spacelike curve with timelike normal* $\gamma : I \to \mathbb{E}_1^3$ *which is defined by*

$$\gamma(s) = \left(s, \sqrt{2}\cosh(s), \sqrt{2}\sinh(s)\right).$$

By choosing $h = \frac{1}{\sqrt{2}}$, $\lambda = -\frac{\sqrt{2}}{3}$, *the following curve can be obtained:*

$$\bar{\gamma} = \gamma + \lambda \mathbf{N}_{\alpha}.$$

It can be easily verified that $\bar{\gamma}$ is spacelike. To see variation along α properly, we choose the values for α as 0.1 *and* 0.9. *For $\alpha = 0.9$ we have*

$$\bar{\gamma} = \Bigg(\frac{s^{1.8} - 0.029832003151676 \cdot \sqrt{2}}{s^{0.8}},$$
$$- \frac{0.0596640063033521 \cdot \sinh(s)}{s^{0.8}} - 0.596640063033521 \cdot s^{0.2} \cosh(s) + \sqrt{2} \cosh(s),$$
$$- \frac{0.0596640063033521 \cdot \cosh(s)}{s^{0.8}} - 0.596640063033521 \cdot s^{0.2} \sinh(s) + \sqrt{2} \sinh(s) \Bigg).$$

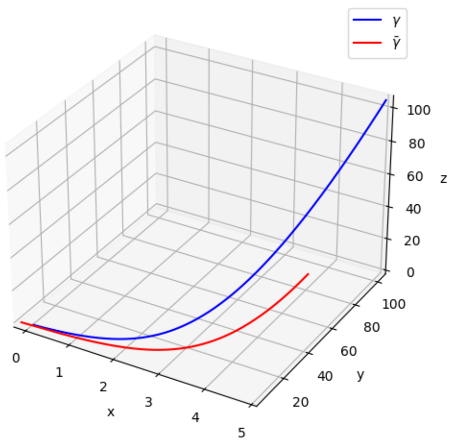

Figure 11: γ and its Bertrand mate $\bar{\gamma}$ for $\alpha = 0.9$

for $\alpha = 0.1$ we have

$$\bar{\gamma} = \Bigg(- 0.00324326597967901 \sqrt{2} s^{0.8} + s,$$
$$- 0.00648653195935802 s^{0.8} \sinh(s) - 0.00720725773262002 s^{1.8} \cosh(s) + \sqrt{2} \cosh(s),$$
$$- 0.00648653195935802 s^{0.8} \cosh(s) - 0.00720725773262002 s^{1.8} \sinh(s) + \sqrt{2} \sinh(s) \Bigg)$$

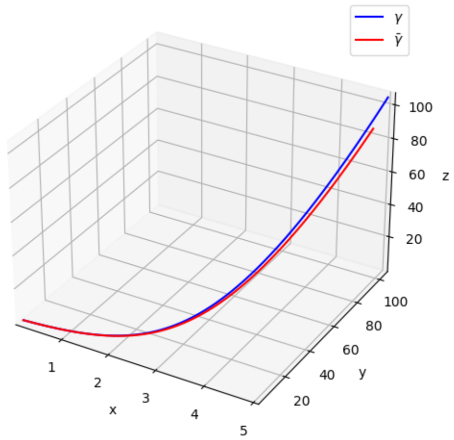

Figure 12: γ and its Bertrand mate $\bar{\gamma}$ for $\alpha = 0.1$

4.3. Null Fractional Bertrand Curves in $\mathbb{E}_1^3$

Although null curves are different from spacelike and timelike curves, the definition of null $\alpha-$Bertrand curves are almost same with spacelike and timelike cases, but they differ on classification. In this section, we examine null Bertrand curves by considering the Caputo fractional derivative in [11], [22], [3] and we obtain the same results for $\alpha = 1$.

Definition 4.16. *A null non-degenerate curve* $\gamma : I \to \mathbb{E}_1^3$ *with* $\kappa_\alpha \neq 0$ *is called fractional Bertrand curve or* α*-Bertrand curve if there exists* $\bar{\gamma} : I \to \mathbb{E}_1^3$ *such that* $\bar{\gamma} = \gamma + \lambda \mathbf{N}_\alpha$ *and* $\mathbf{N}_\alpha = \bar{\mathbf{N}}_\alpha$ *where* λ *is a smooth function on I. The curves* γ *and* $\bar{\gamma}$ *are called fractional Bertrand mates or* $\alpha-$*Bertrand mates.*

Theorem 4.17. *If null curves* γ *and* $\bar{\gamma}$ *are* α*-Bertrand mates in* $\mathbb{E}_1^3$*, i.e.* $\bar{\gamma} = \gamma + \lambda \mathbf{N}_\alpha$*, then* λ *is a constant function.*

Proof. The proof is similar to the other above cases. □

Theorem 4.18. *Let* γ *be non degenerate null curve.* γ *is said to be Bertrand curve if and only if* $\kappa_\alpha = 0$ *or* τ_α *is constant.*

Proof. The argument at the proof of the Theorem 4.3 also works here. □

Hence we have the following result.

Remark 4.19. *Let* γ *be a null Bertrand curve with nonzero curvature and let* $\bar{\gamma}$ *be its Bertrand mate. Then their curvature and torsion satisfy*

$$\kappa_\alpha \bar{\kappa}_\alpha = constant > 0, \ \tau_\alpha = \bar{\tau}_\alpha = constant \neq 0. \tag{42}$$

5. Conclusion

The Caputo fractional derivatives do not have extensive usage in differential geometry. However, its usage might still be a niche area of research because of the advantages like having more flexible approach to complex geometrical structures and leading to more accurate representations of facts in differential geometry. The application of Caputo fractional derivative to differential geometry of curves allows for a smoother and more accurate representation of curves and their geometric structures. Therefore, their practical applications require further exploration through specialized research. Since Caputo fractional differential operator is more flexible for analysis and share a set of properties that may be expected from a differential operator to be considered fractional, we use it to examine Bertrand curves.

6. Conflicts of interests

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

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