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Some algebraic structures about Nijenhuis operators

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Abstract. In this paper, we aim to study the basic properties of Nijenhuis operators on coalgebras and present three algebraic structures: Nijenhuis admissible coassociative Yang-Baxter equations (cYBes for short), Nijenhuis 2-cycles and admissible Nijenhuis systems. We also prove that new solutions of cYBes can be obtained by Nijenhuis admissible cYBes. The equivalence between Nijenhuis admissible cYBes and Nijenhuis 2-cycles is given. A new way to construct Nijenhuis operators on coalgebra is provided.

1. Introduction and preliminaries

Nijenhuis operators on an associative algebra firstly were introduced [1] in the study of quantum bi-Hamiltonian systems. In order to establish the bialgebra theory of Nijenhuis algebras, Nijenhuis coalgebra was presented in [8]. A **Nijenhuis coalgebra** is a pair (C, N) including a coalgebra (C, Δ) and a **Nijenhuis operator** N **on** C, i.e. a linear map $N : C \longrightarrow C$ satisfies

$$N(c_1) \otimes N(c_2) + N^2(c)_1 \otimes N^2(c)_2 = N(N(c)_1) \otimes N(c)_2 + N(c)_1 \otimes N(N(c)_2), \quad \forall\, c \in C. \tag{1}$$

In [2], the authors studied the free Nijenhuis algebras and then obtained the universal enveloping Nijenhuis algebra of an NS algebra. (Bi)hom-Nijienhuis operators and T^*-extensions of (Bi)hom-Lie superalgebras were considered in [3, 6]. Dendriform-Nijenhuis bialgebras and related associative Yang-Baxter equations were introduced in [10]. The authors [4] introduced the notions of compatible $\mathcal{O}$–operators and other algebraic structures related to Nijenhuis operators. In order to obtain the coquasitriangular infinitesimal bialgebras the coassociative Yang-Baxter equations were given in [9] studied in [7]. The authors in [5] introduced a generalization of Nijenhuis operators that lead to BiHom-NS-algebras along BiHom-associative algebras. In this paper, we try to investigate the properties of Nijenhuis operators on coalgebras.

In Section 2, we mainly give some properties of Nijenhuis coalgebras and Nijenhuis bicomodules. Let (C, N) be a Nijenhuis coalgebra. $(C, \tilde{N} := -id_C - N)$ is also a Nijenhuis coalgebra. We also provide an equivalent characterization of Nijenhuis coalgebras (see Theorem 2.3).

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Let C be a coalgebra and $N : C \longrightarrow C$ a linear map. For all $k \in \mathbb{N}$ (set of natural numbers), define the linear maps $\Delta_{N^k} : C \longrightarrow C \otimes C$ by

$$\Delta_{N^k}(c) = N^k(c_1) \otimes c_2 + c_1 \otimes N^k(c_2) - N^k(c)_1 \otimes N^k(c)_2. \tag{2}$$

When (C, N) is a Nijenhuis coalgebra, then, in Theorem 2.4, we obtain that (C, Δ_{N^k}) is a coalgebra, N^j is also a Nijenhuis operator on the coalgebra (C, Δ_{N^k}), any linear combinations of N^k and N^j are Nijenhuis operators and N^j is a coalgebra homomorphism from (C, Δ_{N^k}) to $(C, \Delta_{N^{k+j}})$. Nijenhuis coalgebras and Nijenhuis bicomodules (Definition 2.8) can be characterized by the infinitesimal deformation.

In Section 3, some algebraic structures about Nijenhuis operators on coalgebras are investigated. Firstly, we prove that if S is admissible to (C, N) associated to (M, ρ_ℓ, ρ_r), then $(M, \rho_{\tilde{\ell}}, \rho_{\tilde{r}})$ is a bicomodule over (C, Δ_N) (see Proposition 3.4). We note here this admissible condition is motivated by the dual representation (see Proposition 3.1 and Definition 3.2). Secondly, $\mathcal{O}$-operator on a bicomodule (M, ρ_ℓ, ρ_r) over C is introduced in Definition 3.5 which can induce a coalgebra structure on the bicomodule M (see Proposition 3.6). Thirdly, the notion of Nijenhuis admissible coassociative Yang-Baxter equation (cYBe for short) in (C, N) is presented based on a cYBe satisfying two additional conditions (see Definition 3.9) and furthermore, new solutions of cYBe can be constructed from this new structures (see Theorem 3.11). Fourthly, we give the concept of Nijenhuis 2-cycle on (C, N) which is controlled by a new 2-cycle r_N defined by a Nijenhuis operator (see Definition 3.15). Some equivalent characterization between Nijenhuis admissible cYBe and Nijenhuis 2-cycle is obtained in Theorem 3.16 and a new Nijenhuis operator is constructed in Theorem 3.18. Lastly, a special algebraic structure named by admissible Nijenhuis system (Definition 3.20) is derived and one equivalent description is also provided in Theorem 3.21.

In Section 4, we propose the notion of Nijenhuis Hopf algebra which will be studied later.

Notations: Throughout this paper, we fix a field K. All vector spaces, tensor products, and linear homomorphisms are over K. All the vector spaces and algebras are finite dimensional unless otherwise specified. We use Sweedler's notation [11] for terminologies on coalgebras. For a coalgebra C, we write comultiplication $\Delta(c) = c_1 \otimes c_2$, for any $c \in C$, in which we often omit the summation symbols for convenience.

2. Nijenhuis comodules on Nijenhuis coalgebras

In this section, we will investigate the properties of Nijenhuis coalgebras and Nijenhuis comodules.

2.1. Nijenhuis coalgebras

In this subsection, we investigate some properties of Nijenhuis coalgebras.

Remark 2.1. *Let (C, N) be a Nijenhuis coalgebra. Then*

(a) *For all $c \in C$, define a linear map $\Delta_N : C \longrightarrow C \otimes C$ by*

$$\Delta_N(c) = N(c_1) \otimes c_2 + c_1 \otimes N(c_2) - N(c)_1 \otimes N(c)_2, \tag{3}$$

then (C, Δ_N) is a coalgebra.

(b) *By Eqs.(1) and (3), we have*

$$(N \otimes N) \circ \Delta = \Delta_N \circ N. \tag{4}$$

Lemma 2.2. *Let (C, N) be a Nijenhuis coalgebra. Define*

$$\tilde{N} := -id_C - N. \tag{5}$$

Then $(C, \tilde{N})$ is also a Nijenhuis coalgebra.

Proof. For all $c \in C$, we prove Eq.(1) for $\tilde{N}$ as follows.

$$\begin{aligned}
\tilde{N}(c_1) \otimes \tilde{N}(c_2) &= c_1 \otimes c_2 + c_1 \otimes N(c_2) + N(c_1) \otimes c_2 + N(c_1) \otimes N(c_2) \\
&\overset{(1)}{=} c_1 \otimes c_2 + c_1 \otimes N(c_2) + N(c_1) \otimes c_2 + N(N(c)_1) \otimes N(c)_2 \\
&\quad + N(c)_1 \otimes N(N(c)_2) - N^2(c)_1 \otimes N^2(c)_2 \\
&= \tilde{N}(\tilde{N}(c)_1) \otimes \tilde{N}(c)_2 + \tilde{N}(c)_1 \otimes \tilde{N}(\tilde{N}(c)_2) - \tilde{N}^2(c)_1 \otimes \tilde{N}^2(c)_2,
\end{aligned}$$

finishing the proof. □

Next we provide an equivalent characterization of Nijenhuis coalgebras.

Theorem 2.3. *Let C be a coalgebra, $N : C \longrightarrow C$ a linear map. Then (C, N) is a Nijenhuis coalgebra if and only if there exists a linear map $G : C \longrightarrow C \otimes C$ such that*

$$N(c_1) \otimes N(c_2) = GN(c), \quad \tilde{N}(c_1) \otimes \tilde{N}(c_2) + \tilde{N}(c)_1 \otimes \tilde{N}(c)_2 = -G\tilde{N}(c), \tag{6}$$

where the linear map $\tilde{N}$ is defined by Eq.(5).

Proof. ($\Longrightarrow$) Let N be a Nijenhuis operator on a coalgebra C. Set $G = \Delta_N$ given in Eq.(3), then we obtain Eq.(6) by Eq.(4) and the proof of Lemma 2.2.

($\Longleftarrow$) Assume that there exists a linear map $G : C \longrightarrow C \otimes C$ such that Eq.(6) holds. Then we have

$$\begin{aligned}
-G(c) &= GN(c) - G(c) - GN(c) \\
&= GN(c) + G\tilde{N}(c) \\
&= N(c_1) \otimes N(c_2) - \tilde{N}(c_1) \otimes \tilde{N}(c_2) - \tilde{N}(c)_1 \otimes \tilde{N}(c)_2 \\
&= N(c_1) \otimes N(c_2) - (c_1 \otimes c_2 + c_1 \otimes N(c_2) + N(c_1) \otimes c_2 \\
&\quad + N(c_1) \otimes N(c_2)) + c_1 \otimes c_2 + N(c)_1 \otimes N(c)_2 \\
&= -c_1 \otimes N(c_2) - N(c_1) \otimes c_2 + N(c)_1 \otimes N(c)_2.
\end{aligned}$$

So by Eq.(6), we know that N is a Nijenhuis operator. □

The following results are good properties of Nijenhuis coalgebras.

Theorem 2.4. *Let (C, N) be a Nijenhuis coalgebra.*

(a) *(C, Δ_{N^k}) is a coalgebra.*
(b) *The coalgebras $(C, (\Delta_{N^j})_{N^k})$ and $(C, \Delta_{N^{j+k}})$ coincide.*
(c) *N^j is also a Nijenhuis operator on the coalgebra (C, Δ_{N^k}).*
(d) *Any linear combinations of N^k and N^j are Nijenhuis operators on (C, Δ).*
(e) *Any linear combinations of Δ_{N^k} and Δ_{N^j} make C into a coalgebra.*
(f) *N^j is a coalgebra homomorphism from (C, Δ_{N^k}) to $(C, \Delta_{N^{k+j}})$.*

Proof. (a) By Eq.(1) and iterative method, for all $c \in C$, we have

$$\Delta_{N^k} N(c) = (N \otimes N)\Delta_{N^{k-1}}(c). \tag{7}$$

Then

$$\begin{aligned}
\Delta_{N^k} N^j(c) &= \Delta_{N^k} N N^{j-1}(c) \overset{(7)}{=} (N \otimes N)\Delta_{N^{k-1}} N^{j-1}(c) = \cdots \\
&\overset{(7)}{=} (N^\ell \otimes N^\ell)\Delta_{N^{k-\ell}} N^{j-\ell}(c) = \cdots \overset{(7)}{=} \begin{cases} (N^k \otimes N^k)\Delta N^{j-k}(c), & j \geq k \\ (N^j \otimes N^j)\Delta_{N^{k-j}}(c), & j \leq k \end{cases}.
\end{aligned} \tag{8}$$

Next we will prove the coassociativity for Δ_{N^k}.

$(id \otimes \Delta_{N^k})\Delta_{N^k}(c) - (\Delta_{N^k} \otimes id)\Delta_{N^k}(c)$

$$\begin{aligned}
&\overset{(2)}{=} N^k(c_1)\otimes N^k(c_2)\otimes c_3 + N^k(c_1)\otimes c_2\otimes N^k(c_3) - N^k(c_1)\otimes N^k(c_2)_1\otimes N^k(c_2)_2\\
&\quad + c_1\otimes N^k(N^k(c_2)_1)\otimes N^k(c_2)_2 + c_1\otimes N^k(c_2)_1\otimes N^k(N^k(c_2)_2) - c_1\otimes N^{2k}(c_2)_1\\
&\quad \otimes N^{2k}(c_2)_2 - N^k(c)_1\otimes N^k(N^k(c)_2)\otimes N^k(c)_3 - N^k(c)_1\otimes N^k(c)_2\otimes N^k(N^k(c)_3)\\
&\quad + N^k(c)_1\otimes N^k(N^k(c)_2)_1\otimes N^k(N^k(c)_2)_2 - (N^k(N^k(c_1)_1)\otimes N^k(c_1)_2\otimes c_2\\
&\quad + N^k(c_1)_1\otimes N^k(N^k(c_1)_2)\otimes c_2 - N^{2k}(c_1)_1\otimes N^{2k}(c_1)_2\otimes c_2 + N^k(c_1)\otimes c_2\otimes N^k(c_3)\\
&\quad + c_1\otimes N^k(c_2)\otimes N^k(c_3) - N^k(c_1)_1\otimes N^k(c_1)_2\otimes N^k(c_2) - N^k(N^k(c)_1)\otimes N^k(c)_2\otimes N^k(c)_3\\
&\quad - N^k(c)_1\otimes N^k(N^k(c)_2)\otimes N^k(c)_3 + N^k(N^k(c)_1)_1\otimes N^k(N^k(c)_1)_2\otimes N^k(c)_2)\\
&\overset{(2)(8)}{=} -N^k(c_1)\otimes N^k(c_2)_1\otimes N^k(c_2)_2 - N^k(c)_1\otimes N^k(c)_2\otimes N^k(N^k(c)_3) + N^k(c)_1\otimes N^k(N^k(c)_2)_1\\
&\quad \otimes N^k(N^k(c)_2)_2 - (-N^k(c_1)_1\otimes N^k(c_1)_2\otimes N^k(c_2) - N^k(N^k(c)_1)\otimes N^k(c)_2\otimes N^k(c)_3\\
&\quad + N^k(N^k(c)_1)_1\otimes N^k(N^k(c)_1)_2\otimes N^k(c)_2)\\
&\overset{(2)(8)}{=} -N^k(N^k(c)_1)\otimes N^k(c)_2\otimes N^k(c)_3 - N^k(c)_1\otimes N^k(N^k(c)_2)_1\otimes N^k(N^k(c)_2)_2 + N^{2k}(c)_1\otimes\\
&\quad N^{2k}(c)_2\otimes N^{2k}(c)_3 - N^k(c)_1\otimes N^k(c)_2\otimes N^k(N^k(c)_3) + N^k(c)_1\otimes N^k(N^k(c)_2)_1\otimes N^k(N^k(c)_2)_2\\
&\quad -(-N^k(N^k(c)_1)_1\otimes N^k(N^k(c)_1)_2\otimes N^k(c)_2 - N^k(c)_1\otimes N^k(c)_2\otimes N^k(N^k(c)_3) + N^{2k}(c)_1\otimes\\
&\quad N^{2k}(c)_2\otimes N^{2k}(c)_3 - N^k(N^k(c)_1)\otimes N^k(c)_2\otimes N^k(c)_3 + N^k(N^k(c)_1)_1\otimes N^k(N^k(c)_1)_2\otimes N^k(c)_2)\\
&= 0.
\end{aligned}$$

Thus, (C, Δ_{N^k}) is a coalgebra.

(b) We only prove the case of $j \geq k$. For all $c \in C$, one gets

$$\begin{aligned}
(\Delta_{N^j})_{N^k}(c) &\overset{(2)}{=} (N^k\otimes id)\Delta_{N^j}(c) + (id\otimes N^k)\Delta_{N^j}(c) - \Delta_{N^j}N^k(c)\\
&\overset{(2)(7)}{=} N^{k+j}(c_1)\otimes c_2 + N^k(c_1)\otimes N^j(c_2) - N^k(N^j(c)_1)\otimes N^j(c)_2 + N^j(c_1)\otimes N^k(c_2)\\
&\quad + c_1\otimes N^{k+j}(c_2) - N^j(c)_1\otimes N^k(N^j(c)_2) - (N^k\otimes N^k)\Delta_{N^{j-k}}(c)\\
&\overset{(2)}{=} N^{k+j}(c_1)\otimes c_2 + N^k(c_1)\otimes N^j(c_2) - N^k(N^j(c)_1)\otimes N^j(c)_2 + N^j(c_1)\otimes N^k(c_2)\\
&\quad + c_1\otimes N^{k+j}(c_2) - N^j(c)_1\otimes N^k(N^j(c)_2) - N^j(c_1)\otimes N^k(c_2) - N^k(c_1)\otimes N^j(c_2)\\
&\quad + N^k(N^{j-k}(c)_1)\otimes N^k(N^{j-k}(c)_2)\\
&= N^{k+j}(c_1)\otimes c_2 + c_1\otimes N^{k+j}(c_2) - N^{k+j}(c)_1\otimes N^{k+j}(c)_2 + N^{k+j}(c)_1\otimes N^{k+j}(c)_2\\
&\quad - N^k(N^j(c)_1)\otimes N^j(c)_2 - N^j(c)_1\otimes N^k(N^j(c)_2) + N^k(N^{j-k}(c)_1)\otimes N^k(N^{j-k}(c)_2)\\
&\overset{(2)}{=} N^{k+j}(c_1)\otimes c_2 + c_1\otimes N^{k+j}(c_2) - N^{k+j}(c)_1\otimes N^{k+j}(c)_2 - \Delta_{N^k}N^j(c)\\
&\quad + N^k(N^{j-k}(c)_1)\otimes N^k(N^{j-k}(c)_2)\\
&\overset{(2)(8)}{=} \Delta_{N^{j+k}}(c).
\end{aligned}$$

(c) By Item (b), we obtain

$$\Delta_{N^{k+j}}(c) = (\Delta_{N^k})_{N^j}(c) \overset{(2)}{=} (N^j\otimes id)\Delta_{N^k}(c) + (id\otimes N^j)\Delta_{N^k}(c) - \Delta_{N^k}N^j(c).$$

By Eq.(8), one has $(N^j\otimes N^j)\Delta_{N^k}(c) = (N^j\otimes id)\Delta_{N^k}N^j(c) + (id\otimes N^j)\Delta_{N^k}N^j(c) - \Delta_{N^k}N^{2j}(c)$, i.e., N^j is a Nijenhuis operator on the coalgebra (C, Δ_{N^k}).

(d) Assume that N^k and N^j are Nijenhuis operators on (C, Δ). Set $N' = \lambda N^k + \gamma N^j$, where λ and γ are two parameters. By Eqs.(2) and (8), we can obtain that N' is a Nijenhuis operator on (C, Δ) via direct computation.

(e) For all $c \in C$, one has

$$\begin{aligned}
N^k(c_1)\otimes N^j(c_2) + N^j(c_1)\otimes N^k(c_2) &\overset{(2)(8)}{=} \Delta_{N^k}N^j(c) + \Delta_{N^j}N^k(c)\\
&\overset{(2)}{=} N^k(N^j(c)_1)\otimes N^j(c)_2 + N^j(c)_1\otimes N^k(N^j(c)_2) - N^{k+j}(c)_1\otimes N^{k+j}(c)_2
\end{aligned}$$

$$+N^j(N^k(c)_1)\otimes N^k(c)_2+N^k(c)_1\otimes N^j(N^k(c)_2)-N^{k+j}(c)_1\otimes N^{k+j}(c)_2.$$

Then by the above equation we can check that $(id\otimes\Delta_{N'})\Delta_{N'}=(\Delta_{N'}\otimes id)\Delta_{N'}$ holds, where $N'=\lambda N^k+\gamma N^j$ with two parameters λ and γ.

(f) can be proved by Eq.(8). □

2.2. *Nijenhuis bicomodules*

Next we recall the notion of Nijenhuis bicomodule over a coalgebra in [8].

Let (C,Δ) be a coalgebra and M a vector space, $\rho_\ell: M\longrightarrow C\otimes M$ (write $\rho_\ell(m)=m_{(-1)}\otimes m_{(0)}$) and $\rho_r: M\longrightarrow M\otimes C$ (write $\rho_r(m)=m_{[0]}\otimes m_{[1]}$) be two linear maps. The triple (M,ρ_ℓ,ρ_r) is a **bicomodule over** (C,Δ) if, for all $m\in M$,

$$\text{Left comodules: } m_{(-1)}\otimes m_{(0)(-1)}\otimes m_{(0)(0)}=m_{(-1)1}\otimes m_{(-1)2}\otimes m_{(0)}, \tag{9}$$

$$\text{Right comodules: } m_{[0][0]}\otimes m_{[0][1]}\otimes m_{[1]}=m_{[0]}\otimes m_{[1]1}\otimes m_{[1]2}, \tag{10}$$

$$m_{(-1)}\otimes m_{(0)[0]}\otimes m_{(0)[1]}=m_{[0](-1)}\otimes m_{[0](0)}\otimes m_{[1]}. \tag{11}$$

Let (M,ρ_ℓ,ρ_r) be a bicomodule over (C,Δ) and (M',ρ_ℓ',ρ_r') a bicomodule over (C',Δ'). Then the pair (f,g), where $f: C\longrightarrow C'$ and $g: M\longrightarrow M'$ are two linear maps, is a **homomorphism from** (M,ρ_ℓ,ρ_r) **to** (M',ρ_ℓ',ρ_r') if $\Delta'\circ f=(f\otimes f)\circ\Delta$, $\rho_\ell'\circ g=(f\otimes g)\circ\rho_\ell$ and $\rho_r'\circ g=(g\otimes f)\circ\rho_r$.

Lemma 2.5. *Denote the usual pairing between the dual space M^* and M by $\langle\,,\,\rangle: M^*\times M\to K$. Let C be a coalgebra and (M,ρ_ℓ,ρ_r) be a bicomodule over C. Assume that $\{e_1,e_2,\cdots,e_n\}$ is a basis for M and $\{e^1,e^2,\cdots,e^n\}$ is the dual basis. Define two linear maps ${\rho_r}^*$ and ${\rho_\ell}^*$ on M^* by*

$${\rho_r}^*(m^*)=e_{i[1]}\otimes e^i\langle m^*,e_{i[0]}\rangle,\ \ \textit{and}\ \ {\rho_\ell}^*(m^*)=e^i\otimes e_{i(-1)}\langle m^*,e_{i(0)}\rangle. \tag{12}$$

Then the triple $(M^*,{\rho_r}^*,{\rho_\ell}^*)$ is a bicomodule over C, called **dual bicomodule** *of (M,ρ_ℓ,ρ_r).*

Proof. We only prove that Eq.(9) holds for $(M^*,{\rho_r}^*)$ as follows and others can be verified similarly.

$$\begin{aligned}(id\otimes{\rho_r}^*){\rho_r}^*(m^*) &= e_{i[1]}\otimes e_{j[1]}\otimes e^j\langle e^i,e_{j[0]}\rangle\langle m^*,e_{i[0]}\rangle\\ &= e_{j[0][1]}\otimes e_{j[1]}\otimes e^j\langle m^*,e_{j[0][0]}\rangle\\ &\overset{(10)}{=} e_{j[1]1}\otimes e_{j[1]2}\otimes e^j\langle m^*,e_{j[0]}\rangle=(\Delta\otimes id){\rho_r}^*(m^*),\end{aligned}$$

as desired. □

Nijenhuis coalgebras and Nijenhuis bicomodules can be characterized by the following way.

Let C,M be two vector spaces, $\rho_\ell,\sigma_\ell: M\longrightarrow C\otimes M$, $\rho_r,\sigma_r: M\longrightarrow M\otimes C$ and $\Delta,\tau: C\longrightarrow C\otimes C$ be six linear maps, λ,γ be two parameters. Set $\Delta_{\lambda,\gamma}=\lambda\Delta+\gamma\tau$. Then $(C,\Delta_{\lambda,\gamma})$ is a coalgebra if and only if the following conditions hold:

$$(id\otimes\Delta)\Delta=(\Delta\otimes id)\Delta, \tag{13}$$

$$(id\otimes\tau)\Delta+(id\otimes\Delta)\tau=(\tau\otimes id)\Delta+(\Delta\otimes id)\tau, \tag{14}$$

$$(id\otimes\tau)\tau=(\tau\otimes id)\tau, \tag{15}$$

that is to say, (C,Δ), (C,τ) are coalgebras and Eq.(14) holds.

Let $(C,\Delta_{\lambda,\gamma})$ be a coalgebra. Set $\rho_\ell^{\lambda,\gamma}=\lambda\rho_\ell+\gamma\sigma_\ell$ and $\rho_r^{\lambda,\gamma}=\lambda\rho_r+\gamma\sigma_r$. Then $(M,\rho_\ell^{\lambda,\gamma},\rho_r^{\lambda,\gamma})$ is a bicomodule over $(C,\Delta_{\lambda,\gamma})$ if and only if the following conditions hold:

$$\begin{aligned}(id\otimes\rho_\ell)\rho_\ell&=(\Delta\otimes id)\rho_\ell,\\ (id\otimes\sigma_\ell)\rho_\ell+(id\otimes\rho_\ell)\sigma_\ell&=(\tau\otimes id)\rho_\ell+(\Delta\otimes id)\sigma_\ell,\\ (id\otimes\sigma_\ell)\sigma_\ell&=(\tau\otimes id)\sigma_\ell,\\ (\rho_r\otimes id)\rho_r&=(id\otimes\Delta)\rho_r,\end{aligned}$$

$$(id\otimes\tau)\rho_r+(id\otimes\Delta)\sigma_r=(\sigma_r\otimes id)\rho_r+(\rho_r\otimes id)\sigma_r,$$
$$(id\otimes\tau)\sigma_r=(\sigma_r\otimes id)\sigma_r,$$
$$(id\otimes\rho_r)\rho_\ell=(\rho_\ell\otimes id)\rho_r,$$
$$(id\otimes\sigma_r)\rho_\ell+(id\otimes\rho_r)\sigma_\ell=(\sigma_\ell\otimes id)\rho_r+(\rho_\ell\otimes id)\sigma_r,$$
$$(id\otimes\sigma_r)\sigma_\ell=(\sigma_\ell\otimes id)\sigma_r,$$

which means that (M,ρ_ℓ,ρ_r) is a bicomodule over (C,Δ), (M,σ_ℓ,σ_r) is a bicomodule over (C,τ) and $(M,\rho_\ell^{1,1},\rho_r^{1,1})$ is a bicomodule over $(C,\Delta_{1,1})$.

Theorem 2.6. *[8] Let (M,ρ_ℓ,ρ_r) be a bicomodule over (C,Δ) and $(M,\rho_\ell^{\lambda,\gamma},\rho_r^{\lambda,\gamma})$ a bicomodule over $(C,\Delta_{\lambda,\gamma})$. Then $(\lambda id_C+\gamma N,\lambda id_M+\gamma S)$, where $N\in End(C)$ and $S\in End(M)$, is a homomorphism from (M,ρ_ℓ,ρ_r) over (C,Δ) to $(M,\rho_\ell^{\lambda,\gamma},\rho_r^{\lambda,\gamma})$ over $(C,\Delta_{\lambda,\gamma})$ if and only if*

$$\tau N=(N\otimes N)\Delta, \tag{16}$$
$$\tau+\Delta N=(id_C\otimes N)\Delta+(N\otimes id_C)\Delta, \tag{17}$$
$$\sigma_\ell S=(N\otimes S)\rho_\ell, \tag{18}$$
$$\sigma_\ell+\rho_\ell S=(id_C\otimes S)\rho_\ell+(N\otimes id_M)\rho_\ell, \tag{19}$$
$$\sigma_r S=(S\otimes N)\rho_r, \tag{20}$$
$$\sigma_r+\rho_r S=(S\otimes id_C)\rho_r+(id_M\otimes N)\rho_r, \tag{21}$$

Remark 2.7. *Eqs.(16) and (17) imply that (C,Δ,N) is a Nijenhuis coalgebra.*

Based on Eqs.(18)-(21), one gets that

Definition 2.8. *Let (C,Δ,N) be a Nijenhuis coalgebra and M a vector space. Let $\rho_\ell:M\longrightarrow C\otimes M$, $\rho_r:M\longrightarrow M\otimes C$ and $S:M\longrightarrow M$ be three linear maps and (M,ρ_ℓ,ρ_r) a bicomodule over (C,Δ). The 4-tuple (M,ρ_ℓ,ρ_r,S) is a* **Nijenhuis bicomodule over** *(C,N) if*

$$N(m_{(-1)})\otimes S(m_{(0)})+S^2(m)_{(-1)}\otimes S^2(m)_{(0)}=S(m)_{(-1)}\otimes S(S(m)_{(0)})+N(S(m)_{(-1)})\otimes S(m)_{(0)}, \tag{22}$$
$$S(m_{[0]})\otimes N(m_{[1]})+S^2(m)_{[0]}\otimes S^2(m)_{[1]}=S(S(m)_{[0]})\otimes S(m)_{[1]}+S(m)_{[0]}\otimes N(S(m)_{[1]}). \tag{23}$$

Remark 2.9. *(1) Let (C,Δ,N) be a Nijenhuis coalgebra. As we know, $(C,\Delta_L:=\Delta)$ is a left (C,Δ)-comodule, at the same time $(C,\Delta_R:=\Delta)$ is a right (C,Δ)-comodule and (C,Δ_L,Δ_R) is a (C,Δ)-bicomodule. Then (C,Δ_L,Δ_R,N) is a Nijenhuis bicomodule over (C,N).*

(2) Let (C,Δ,N) be a Nijenhuis coalgebra, M a vector space, $\rho_\ell:M\longrightarrow C\otimes M$, $\rho_r:M\longrightarrow M\otimes C$, $S:M\longrightarrow M$ three linear maps. Define a coproduct

$$\Delta_{C\oplus M}(c+m):=c_1\otimes c_2+m_{(-1)}\otimes m_{(0)}+m_{[0]}\otimes m_{[1]} \tag{24}$$

on $C\oplus M$ and a linear operator $N_{C\oplus M}$ on $C\oplus M$ by $N_{C\oplus M}(c+m):=N(c)+S(m)$. Then $(C\oplus M,N_{C\oplus M})$ together with the coproduct Eq.(24) is a Nijenhuis coalgebra if and only if (M,ρ_ℓ,ρ_r) is a bicomodule over (C,Δ) and (M,ρ_ℓ,ρ_r,S) is a Nijenhuis bicomodule over (C,N).

The resulting Nijenhuis coalgebra is denoted by $(A\times_{\ell,r}M,N+S)$ and is called the **semi-direct coproduct** *of (C,N) by its bicomodule (M,ρ_ℓ,ρ_r,S).*

3. Structures related to Nijenhuis operators on coalgebras

3.1. Dual Nijenhuis bicomodules

Proposition 3.1. *Let (C,N) be a Nijenhuis coalgebra, (M,ρ_ℓ,ρ_r) be a bicomodule over C and $S:M\longrightarrow M$ a linear map. Then the 4-tuple $(M^*,{\rho_r}^*,{\rho_\ell}^*,S^*)$ is a Nijenhuis bicomodule over (C,N) if and only if the following two equations hold:*

$$N(m_{(-1)})\otimes S(m_{(0)})+S(m)_{(-1)}\otimes S(S(m)_{(0)})=N(S(m)_{(-1)})\otimes S(m)_{(0)}+m_{(-1)}\otimes S^2(m_{(0)}), \tag{25}$$
$$S(m_{[0]})\otimes N(m_{[1]})+S(S(m)_{[0]})\otimes S(m)_{[1]}=S(m)_{[0]}\otimes N(S(m)_{[1]})+S^2(m_{[0]})\otimes m_{[1]}. \tag{26}$$

Proof. By Definition 2.8, $(M^*, \rho_r{}^*, \rho_\ell{}^*, S^*)$ is a Nijenhuis bicomodule over (C, N) if and only if, for all $m^* \in M^*$,

$$(N \otimes S^*)\rho_r{}^*(m^*) + \rho_r{}^*(S^{2*}(m^*)) - (id \otimes S^*)\rho_r{}^*(S^*(m^*)) - (N \otimes id)\rho_r{}^*(S^*(m^*)) = 0, \tag{27}$$

and

$$(S^* \otimes N)\rho_\ell{}^*(m^*) + \rho_\ell{}^*(S^{2*}(m^*)) - (S^* \otimes id)\rho_\ell{}^*(S^*(m^*)) - (id \otimes N)\rho_\ell{}^*(S^*(m^*)) = 0. \tag{28}$$

hold.

For all $c^* \in C^*$ and $m \in M$, one calculates

$$\begin{aligned}
&\langle (N \otimes S^*)\rho_r{}^*(m^*) + \rho_r{}^*(S^{2*}(m^*)) - (id \otimes S^*)\rho_r{}^*(S^*(m^*)) - (N \otimes id)\rho_r{}^*(S^*(m^*)), c^* \otimes m\rangle \\
&\overset{(12)}{=} \langle N(e_{i[1]}), c^*\rangle\langle S^*(e^i), m\rangle\langle m^*, e_{i[0]}\rangle + \langle e_{i[1]}, c^*\rangle\langle e^i, m\rangle\langle S^{2*}(m^*), e_{i[0]}\rangle \\
&\quad -\langle e_{i[1]}, c^*\rangle\langle S^*(e^i), m\rangle\langle S^*(m^*), e_{i[0]}\rangle - \langle N(e_{i[1]}), c^*\rangle\langle e^i, m\rangle\langle S^*(m^*), e_{i[0]}\rangle \\
&= \langle N(S(m)_{[1]}), c^*\rangle\langle m^*, S(m)_{[0]}\rangle + \langle m_{[1]}, c^*\rangle\langle m^*, S^2(m_{[0]})\rangle - \langle S(m)_{[1]}, c^*\rangle\langle m^*, S(S(m)_{[0]})\rangle \\
&\quad -\langle N(m_{[1]}), c^*\rangle\langle m^*, S(m_{[0]})\rangle \\
&= \langle m^* \otimes c^*, S(m)_{[0]} \otimes N(S(m)_{[1]}) + S^2(m_{[0]}) \otimes m_{[1]} - S(S(m)_{[0]}) \otimes S(m)_{[1]} - S(m_{[0]}) \otimes N(m_{[1]})\rangle,
\end{aligned}$$

then Eq.(27) is equivalent to Eq.(26). Likewise, one can check that Eq.(28) holds if and only if Eq.(25) holds. □

Definition 3.2. *Let (C, N) be a Nijenhuis coalgebra, (M, ρ_ℓ, ρ_r) be a bicomodule over C and $S : M \longrightarrow M$ a linear map. Then S* **is admissible to** *(C, N)* **associated to** *(M, ρ_ℓ, ρ_r) if Eqs.(25) and (26) hold. We call that $\mathbb{S} : C \longrightarrow C$ is admissible to (C, N) associated to (C, Δ_L, Δ_R), for all $c \in C$, satisfying*

$$N(c_1) \otimes \mathbb{S}(c_2) + \mathbb{S}(c)_1 \otimes \mathbb{S}(\mathbb{S}(c)_2) = N(\mathbb{S}(c)_1) \otimes \mathbb{S}(c)_2 + c_1 \otimes \mathbb{S}^2(c_2), \tag{29}$$

$$\mathbb{S}(c_1) \otimes N(c_2) + \mathbb{S}(\mathbb{S}(c)_1) \otimes \mathbb{S}(c)_2 = \mathbb{S}(c)_1 \otimes N(\mathbb{S}(c)_2) + \mathbb{S}^2(c_1) \otimes c_2. \tag{30}$$

Example 3.3. *Let (C, Δ, N) be a Nijenhuis coalgebra. Then N^* is admissible to (C, N) associated to $(C^*, \Delta_R{}^*, \Delta_L{}^*)$.*

Proof. It is a direct conclusion by $(C, \Delta_L, \Delta_R, N)$ is a Nijenhuis bicomodule over (C, Δ, N). □

Proposition 3.4. *Let (C, N) be a Nijenhuis coalgebra, (M, ρ_ℓ, ρ_r) be a bicomodule over C and $S : M \longrightarrow M$ a linear map. For all $m \in M$, define two linear maps $\rho_{\tilde{\ell}} : M \longrightarrow C \otimes M$ and $\rho_{\tilde{r}} : M \longrightarrow M \otimes C$ by*

$$\rho_{\tilde{\ell}}(m) = N(m_{(-1)}) \otimes m_{(0)} - m_{(-1)} \otimes S(m_{(0)}) + S(m)_{(-1)} \otimes S(m)_{(0)}, \tag{31}$$

$$\rho_{\tilde{r}}(m) = m_{[0]} \otimes N(m_{[1]}) - S(m_{[0]}) \otimes m_{[1]} + S(m)_{[0]} \otimes S(m)_{[1]}. \tag{32}$$

If S is admissible to (C, N) associated to (M, ρ_ℓ, ρ_r), then $(M, \rho_{\tilde{\ell}}, \rho_{\tilde{r}})$ is a bicomodule over (C, Δ_N).

Proof. For all $m \in M$, we have

$$\begin{aligned}
&(id \otimes \rho_{\tilde{\ell}})\rho_{\tilde{\ell}}(m) - (\Delta_N \otimes id)\rho_{\tilde{\ell}}(m) \\
&\overset{(3)(31)}{=} N(m_{(-1)}) \otimes N(m_{(0)(-1)}) \otimes m_{(0)(0)} - N(m_{(-1)}) \otimes m_{(0)(-1)} \otimes S(m_{(0)(0)}) + N(m_{(-1)}) \otimes S(m_{(0)})_{(-1)} \\
&\quad \otimes S(m_{(0)})_{(0)} - m_{(-1)} \otimes N(S(m_{(0)})_{(-1)}) \otimes S(m_{(0)})_{(0)} + m_{(-1)} \otimes S(m_{(0)})_{(-1)} \otimes S(S(m_{(0)})_{(0)}) \\
&\quad -m_{(-1)} \otimes S^2(m_{(0)})_{(-1)} \otimes S^2(m_{(0)})_{(0)} + S(m)_{(-1)} \otimes N(S(m)_{(0)(-1)}) \otimes S(m)_{(0)(0)} \\
&\quad -S(m)_{(-1)} \otimes S(m)_{(0)(-1)} \otimes S(S(m)_{(0)(0)}) + S(m)_{(-1)} \otimes S(S(m)_{(0)})_{(-1)} \otimes S(S(m)_{(0)})_{(0)} \\
&\quad -(N(N(m_{(-1)})_1) \otimes N(m_{(-1)})_2 \otimes m_0 + N(m_{(-1)})_1 \otimes N(N(m_{(-1)})_2) \otimes m_0 - N^2(m_{(-1)})_1 \\
&\quad \otimes N^2(m_{(-1)})_2 \otimes m_0 - N(m_{(-1)1}) \otimes m_{(-1)2} \otimes S(m_{(0)}) - m_{(-1)1} \otimes N(m_{(-1)2}) \otimes S(m_{(0)}) + N(m_{(-1)})_1 \\
&\quad \otimes N(m_{(-1)})_2 \otimes S(m_{(0)}) + N(S(m)_{(-1)1}) \otimes S(m)_{(-1)2} \otimes S(m)_{(0)} + S(m)_{(-1)1} \otimes N(S(m)_{(-1)2}) \\
&\quad \otimes S(m)_{(0)} - N(S(m)_{(-1)})_1 \otimes N(S(m)_{(-1)})_2 \otimes S(m)_{(0)}) \\
&\overset{(1)(25)}{=} N(m_{(-1)}) \otimes N(m_{(0)(-1)}) \otimes m_{(0)(0)} - N(m_{(-1)}) \otimes m_{(0)(-1)} \otimes S(m_{(0)(0)}) + N(S(m)_{(-1)}) \otimes S(m)_{(0)(-1)}
\end{aligned}$$

$$
\begin{aligned}
&\otimes S(m)_{(0)(0)} + m_{(-1)} \otimes S^2(m_{(0)})_{(-1)} \otimes S^2(m_{(0)})_{(0)} - S(m)_{(-1)} \otimes S(m_{(0)})_{(-1)} \otimes S(S(m_{(0)})_{(0)})\\
&-m_{(-1)} \otimes N(S(m_{(0)})_{(-1)}) \otimes S(m_{(0)})_{(0)} + m_{(-1)} \otimes S(m_{(0)})_{(-1)} \otimes S(S(m_{(0)})_{(0)}) - m_{(-1)}\\
&\otimes S^2(m_{(0)})_{(-1)} \otimes S^2(m_{(0)})_{(0)} + S(m)_{(-1)} \otimes N(S(m)_{(0)(-1)}) \otimes S(m)_{(0)(0)} - S(m)_{(-1)} \otimes S(m)_{(0)(-1)}\\
&\otimes S(S(m)_{(0)(0)}) + S(m)_{(-1)} \otimes S(S(m)_{(0)})_{(-1)} \otimes S(S(m)_{(0)})_{(0)} - (N(m_{(-1)1}) \otimes N(m_{(-1)2}) \otimes m_{(0)}\\
&-N(m_{(-1)1}) \otimes m_{(-1)2} \otimes S(m_{(0)}) - m_{(-1)} \otimes N(S(m_{(0)})_{(-1)}) \otimes S(m_{(0)})_{(0)} - m_{(-1)} \otimes m_{(0)(-1)}\\
&\otimes S^2(m_{(0)(0)}) + m_{(-1)} \otimes S(m_{(0)})_{(-1)} \otimes S(S(m_{(0)})_{(0)}) + N(S(m)_{(-1)})_1 \otimes N(S(m)_{(-1)})_2 \otimes S(m)_{(0)}\\
&+m_{(-1)1} \otimes m_{(-1)2} \otimes S^2(m_{(0)}) - S(m)_{(-1)1} \otimes S(m)_{(-1)2} \otimes S(S(m)_{(0)}) + N(S(m)_{(-1)1}) \otimes S(m)_{(-1)2}\\
&\otimes S(m)_{(0)} + S(m)_{(-1)1} \otimes N(S(m)_{(-1)2}) \otimes S(m)_{(0)} - N(S(m)_{(-1)})_1 \otimes N(S(m)_{(-1)})_2 \otimes S(m)_{(0)})\\
\overset{(9)}{=}\ & 0,
\end{aligned}
$$

then Eq.(9) holds for $\rho_{\tilde{\ell}}$ and Δ_N. Similarly, by Eqs.(1), (3), (10), (26) and (32), one can obtain $(\rho_{\tilde{r}} \otimes id)\rho_{\tilde{r}}(m) - (id \otimes \Delta_N)\rho_{\tilde{r}}(m) = 0$, by Eqs.(11), (25), (26), (31) and (32), we get $(id \otimes \rho_{\tilde{r}})\rho_{\tilde{\ell}}(m) - (\rho_{\tilde{\ell}} \otimes id)\rho_{\tilde{r}}(m) = 0$. Thus $(M, \rho_{\tilde{\ell}}, \rho_{\tilde{r}})$ is a bicomodule over (C, Δ_N). □

3.2. $\mathcal{O}$-operators on bicomodules

Definition 3.5. *Let C be a coalgebra, (M, ρ_ℓ, ρ_r) a bicomodule over C. A linear map $T : C \longrightarrow M$ is an* **$\mathcal{O}$-operator on a bicomodule** (M, ρ_ℓ, ρ_r) **over** *C if, for all $c \in C$,*

$$T(c_1) \otimes T(c_2) = T(T(c)_{(-1)}) \otimes T(c)_{(0)} + T(c)_{[0]} \otimes T(T(c)_{[1]}). \tag{33}$$

Proposition 3.6. *Let $T : C \longrightarrow M$ be an $\mathcal{O}$-operator on a bicomodule (M, ρ_ℓ, ρ_r) over a coalgebra C. Define a linear map $\Delta_T : M \longrightarrow M \otimes M$ by*

$$\Delta_T(m) = T(m_{(-1)}) \otimes m_{(0)} + m_{[0]} \otimes T(m_{[1]}). \tag{34}$$

Then (C, Δ_T) is a coalgebra.

Proof. We check the coassociativity as follows. For all $m \in M$, we have

$$
\begin{aligned}
&(id \otimes \Delta_T)\Delta_T(m) - (\Delta_T \otimes id)\Delta_T(m)\\
&\overset{(34)}{=} T(m_{(-1)}) \otimes T(m_{(0)(-1)}) \otimes m_{(0)(0)} + T(m_{(-1)}) \otimes m_{(0)[0]} \otimes T(m_{(0)[1]}) + m_{[0]} \otimes T(T(m_{[1]})_{(-1)})\\
&\quad \otimes T(m_{[1]})_{(0)} + m_{[0]} \otimes T(m_{[1]})_{[0]} \otimes T(T(m_{[1]})_{[1]}) - T(T(m_{(-1)})_{(-1)}) \otimes T(m_{(-1)})_{(0)} \otimes m_{(0)}\\
&\quad -T(T(m_{(-1)})_{[0]}) \otimes T(T(m_{(-1)})_{[1]}) \otimes m_{(0)} - T(m_{[0](-1)}) \otimes m_{[0](0)} \otimes T(m_{[1]}) - m_{[0][0]}\\
&\quad \otimes T(m_{[0][1]}) \otimes T(m_{[1]})\\
&\overset{(33)(9)(10)(11)}{=} 0,
\end{aligned}
$$

finishing the proof. □

3.3. Nijenhuis admissible cYBes

Let C be a coalgerbra and $\mathfrak{B} : C \otimes C \longrightarrow K$ be a bilinear form. Define the linear map $\mathfrak{B}^\# : C \longrightarrow C^*$ by

$$\langle \mathfrak{B}^\#(a), b\rangle = \mathfrak{B}(a, b), \forall\, a, b \in C. \tag{35}$$

$\mathfrak{B}$ is **nondegenerate** if the linear map $\mathfrak{B}^\#$ is an isomorphism. $\mathfrak{B}$ is **skew-symmetric** if $\mathfrak{B}(a, b) = -\mathfrak{B}(b, a)$.

The notion of coassociative Yang-Baxter equation was introduced in [9].

Definition 3.7. *Let C be a coalgerbra, $\mathfrak{B} : C \otimes C \longrightarrow K$ be a bilinear form. We call*

$$\mathfrak{B}(a_1, b)\mathfrak{B}(a_2, c) + \mathfrak{B}(a, c_1)\mathfrak{B}(b, c_2) - \mathfrak{B}(b_1, c)\mathfrak{B}(a, b_2) = 0 \tag{36}$$

the **coassiciative Yang-Baxter equation (cYBe for short) in** *C.*

Lemma 3.8. *Let C be a coalgebra and $\mathfrak{B}$ be skew-symmetric. Then $\mathfrak{B}$ is a solution of cYBe in C if and only if $\mathfrak{B}^{\#}$ is an O-operator on the bicomodule $(C^*, {\Delta_R}^*, {\Delta_L}^*)$ over C.*

Proof. By Eq.(12), we have ${\Delta_R}^*(c^*) = c^{*(-1)} \otimes c^{*(0)} = e_{i2} \otimes e^i\langle c^*, e_{i1}\rangle$ and ${\Delta_L}^*(c^*) = c^{*[0]} \otimes c^{*[1]} = e^i\langle c^*, e_{i2}\rangle \otimes e_{i1}$.
($\Longrightarrow$) For all $c \in C$, one has

$$\begin{aligned}
&\langle \mathfrak{B}^{\#}(c_1) \otimes \mathfrak{B}^{\#}(c_2) - \mathfrak{B}^{\#}(\mathfrak{B}^{\#}(c)^{(-1)}) \otimes \mathfrak{B}^{\#}(c)^{(0)} - \mathfrak{B}^{\#}(c)^{[0]} \otimes \mathfrak{B}^{\#}(\mathfrak{B}^{\#}(c)^{[1]}), a \otimes b\rangle \\
&\quad= \langle \mathfrak{B}^{\#}(c_1), a\rangle\langle \mathfrak{B}^{\#}(c_2), b\rangle - \langle \mathfrak{B}^{\#}(e_{i2}), a\rangle\langle e^i, b\rangle\langle \mathfrak{B}^{\#}(c), e_{i1}\rangle - \langle e^i, a\rangle\langle \mathfrak{B}^{\#}(c), e_{i2}\rangle\langle \mathfrak{B}^{\#}(e_{i1}), b\rangle \\
&\quad\overset{(35)}{=} \mathfrak{B}(c_1, a)\mathfrak{B}(c_2, b) - \mathfrak{B}(e_{i2}, a)\langle e^i, b\rangle\mathfrak{B}(c, e_{i1}) - \langle e^i, a\rangle\mathfrak{B}(c, e_{i2})\mathfrak{B}(e_{i1}, b) \\
&\quad= \mathfrak{B}(a, c_1)\mathfrak{B}(b, c_2) - \mathfrak{B}(a, b_2)\mathfrak{B}(b_1, c) + \mathfrak{B}(a_2, c)\mathfrak{B}(a_1, b) \\
&\quad\overset{(36)}{=} 0,
\end{aligned}$$

which implies that $\mathfrak{B}^{\#}$ is an O-operator on the bicomodule $(C^*, {\Delta_R}^*, {\Delta_L}^*)$ over C.
($\Longleftarrow$) can be proved similarly. □

Definition 3.9. *Let (C, N) be a Nijenhuis coalgebra. Eq.(36) together with the following two conditions:*

$$\mathfrak{B}(a, N(b)) = \mathfrak{B}(N(a), b), \tag{37}$$

$$\mathfrak{B}(N(a)_1, b)N(a)_2 + N(b)_1\mathfrak{B}(N(b)_2, a) = N(b_1)\mathfrak{B}(b_2, a) + \mathfrak{B}(a_1, b)N(a_2), \tag{38}$$

where $a, b \in C$, is called **Nijenhuis admissible cYBe in** (C, N).

Remark 3.10.

(1) *Eq.*(37) $\Leftrightarrow \mathfrak{B}^{\#}N = N^*\mathfrak{B}^{\#} \Leftrightarrow N\mathfrak{B}^{\#-1} = \mathfrak{B}^{\#-1}N^*$(*when $\mathfrak{B}$ is nondegenerate*);

(2) *Eq.*(38) $\Leftrightarrow (N^* \otimes \mathfrak{B}^{\#}){\Delta_L}^* + (\mathfrak{B}^{\#} \otimes N^*){\Delta_R}^* = (\mathfrak{B}^{\#} \otimes id){\Delta_R}^*N^* + (id \otimes \mathfrak{B}^{\#}){\Delta_L}^*N^*$.

Theorem 3.11. *Let (C, N) be a Nijenhuis coalgebra and $\mathfrak{B}$ be a skew-symmetric solution of Nijenhuis admissible cYBe in (C, N). Then*

(a) *$\mathfrak{B}_N$ is the skew-symmetric solution of cYBe in C, where $\mathfrak{B}_N$ is given by $\mathfrak{B}_N(a, b) = \langle \mathfrak{B}^{\#}_N(a), b\rangle = \mathfrak{B}(N(a), b)$.*
(b) *Any linear combinations $\mathfrak{B}' = \lambda\mathfrak{B} + \gamma\mathfrak{B}_N$, λ and γ two parameters, of $\mathfrak{B}$ and $\mathfrak{B}_N$ are the skew-symmetric solutions of cYBe in C.*

Proof. (a) For all $a, b, c \in C$, one calculates

$$\begin{aligned}
&\mathfrak{B}_N(a_1, b)\mathfrak{B}_N(a_2, c) + \mathfrak{B}_N(a, c_1)\mathfrak{B}_N(b, c_2) - \mathfrak{B}_N(b_1, c)\mathfrak{B}_N(a, b_2) \\
&\quad= \mathfrak{B}(N(a), c_1)\mathfrak{B}(N(b), c_2) + \mathfrak{B}(N(a_1), b)\mathfrak{B}(N(a_2), c) - \mathfrak{B}(N(b_1), c)\mathfrak{B}(N(a), b_2) \\
&\quad\overset{(37)}{=} \mathfrak{B}(a, N(c_1))\mathfrak{B}(b, N(c_2)) + \mathfrak{B}(N(a_1), b)\mathfrak{B}(N(a_2), c) - \mathfrak{B}(N(b_1), c)\mathfrak{B}(N(a), b_2) \\
&\quad\overset{(1)}{=} \mathfrak{B}(a, N(N(c)_1))\mathfrak{B}(b, N(c)_2) + \mathfrak{B}(a, N(c)_1)\mathfrak{B}(b, N(N(c)_2)) - \mathfrak{B}(a, N^2(c)_1)\mathfrak{B}(b, N^2(c)_2) \\
&\qquad+\mathfrak{B}(N(a_1), b)\mathfrak{B}(N(a_2), c) - \mathfrak{B}(N(b_1), c)\mathfrak{B}(N(a), b_2) \\
&\quad\overset{(37)(38)}{=} \mathfrak{B}(N(a), N(c)_1)\mathfrak{B}(b, N(c)_2) + \mathfrak{B}(a, N(c)_1)\mathfrak{B}(N(b), N(c)_2) - \mathfrak{B}(a, N^2(c)_1)\mathfrak{B}(b, N^2(c)_2) \\
&\qquad+\mathfrak{B}(N(a_1), b)\mathfrak{B}(N(a_2), c) - \mathfrak{B}(N(b_1), c)\mathfrak{B}(N(a), b_2) + \mathfrak{B}(N(a)_1, b)\mathfrak{B}(N(a)_2, N(c)) \\
&\qquad+\mathfrak{B}(N(b)_1, N(c))\mathfrak{B}(N(b)_2, a) - \mathfrak{B}(N(b_1), N(c))\mathfrak{B}(b_2, a) - \mathfrak{B}(a_1, b)\mathfrak{B}(N(a_2), N(c)) \\
&\quad\overset{(36)(37)}{=} \mathfrak{B}(b_1, N(c))\mathfrak{B}(N(a), b_2) - \mathfrak{B}(N(a)_1, b)\mathfrak{B}(N(a)_2, N(c)) + \mathfrak{B}(N(b)_1, N(c))\mathfrak{B}(a, N(b)_2) \\
&\qquad-\mathfrak{B}(a_1, N(b))\mathfrak{B}(a_2, N(c)) - \mathfrak{B}(b_1, N^2(c))\mathfrak{B}(a, b_2) + \mathfrak{B}(a_1, b)\mathfrak{B}(a_2, N^2(c)) \\
&\qquad+\mathfrak{B}(a_1, N(b))\mathfrak{B}(a_2, N(c)) - \mathfrak{B}(b_1, N(c))\mathfrak{B}(N(a), b_2) + \mathfrak{B}(N(a)_1, b)\mathfrak{B}(N(a)_2, N(c)) \\
&\qquad-\mathfrak{B}(N(b)_1, N(c))\mathfrak{B}(a, N(b)_2) + \mathfrak{B}(b_1, N^2(c))\mathfrak{B}(a, b_2) - \mathfrak{B}(a_1, b)\mathfrak{B}(a_2, N^2(c)) \\
&\quad= 0
\end{aligned}$$

and

$$\mathfrak{B}_N(b,a)=\mathfrak{B}(N(b),a)=-\mathfrak{B}(a,N(b))\overset{(37)}{=}-\mathfrak{B}(N(a),b)=-\mathfrak{B}_N(a,b).$$

Thus $\mathfrak{B}_N$ is the skew-symmetric solution of cYBe.

(b) For all $a,b,c\in C$, we only check that Eq.(36) holds for $\mathfrak{B}'$ as follows and the skew-symmetry is obvious.

$$\begin{aligned}
&\mathfrak{B}'(a_1,b)\mathfrak{B}'(a_2,c)+\mathfrak{B}'(a,c_1)\mathfrak{B}'(b,c_2)-\mathfrak{B}'(b_1,c)\mathfrak{B}'(a,b_2)\\
&= \lambda^2\mathfrak{B}(a_1,b)\mathfrak{B}(a_2,c)+\lambda\gamma\mathfrak{B}(a_1,b)\mathfrak{B}_N(a_2,c)+\lambda\gamma\mathfrak{B}_N(a_1,b)\mathfrak{B}(a_2,c)\\
&\quad+\gamma^2\mathfrak{B}_N(a_1,b)\mathfrak{B}_N(a_2,c)+\lambda^2\mathfrak{B}(a,c_1)\mathfrak{B}(a,b_1)+\lambda\gamma\mathfrak{B}(a,c_1)\mathfrak{B}_N(b,c_2)\\
&\quad+\lambda\gamma\mathfrak{B}_N(a,c_1)\mathfrak{B}(b,c_2)+\gamma^2\mathfrak{B}_N(a,c_1)\mathfrak{B}_N(b,c_2)-\lambda^2\mathfrak{B}(b_1,c)\mathfrak{B}(a,b_2)\\
&\quad-\lambda\gamma\mathfrak{B}(b_1,c)\mathfrak{B}_N(a,b_2)-\lambda\gamma\mathfrak{B}_N(b_1,c)\mathfrak{B}(a,b_2)-\gamma^2\mathfrak{B}_N(b_1,c)\mathfrak{B}_N(a,b_2)\\
&\overset{(36)}{=} \lambda\gamma[\mathfrak{B}(a_1,b)\mathfrak{B}_N(a_2,c)+\mathfrak{B}_N(a_1,b)\mathfrak{B}(a_2,c)+\mathfrak{B}(a,c_1)\mathfrak{B}_N(b,c_2)\\
&\quad+\mathfrak{B}_N(a,c_1)\mathfrak{B}(b,c_2)-\mathfrak{B}(b_1,c)\mathfrak{B}_N(a,b_2)-\mathfrak{B}_N(b_1,c)\mathfrak{B}(a,b_2)\\
&\overset{(38)}{=} \lambda\gamma[\mathfrak{B}(N(a)_1,b)\mathfrak{B}(N(a)_2,c)+\mathfrak{B}(N(b)_1,c)\mathfrak{B}(N(b)_2,a)+\mathfrak{B}(N(a_1),b)\mathfrak{B}(a_2,c)\\
&\quad-\mathfrak{B}(b_1,c)\mathfrak{B}(N(a),b_2)+\mathfrak{B}(a,c_1)\mathfrak{B}(N(b),c_2)+\mathfrak{B}(N(a),c_1)\mathfrak{B}(b,c_2)]\\
&\overset{(37)}{=} \lambda\gamma[\mathfrak{B}(N(a)_1,b)\mathfrak{B}(N(a)_2,c)-\mathfrak{B}(N(b)_1,c)\mathfrak{B}(a,N(b)_2)+\mathfrak{B}(a_1,N(b))\mathfrak{B}(a_2,c)\\
&\quad-\mathfrak{B}(b_1,c)\mathfrak{B}(N(a),b_2)+\mathfrak{B}(a,c_1)\mathfrak{B}(N(b),c_2)+\mathfrak{B}(N(a),c_1)\mathfrak{B}(b,c_2)]\\
&\overset{(36)}{=} 0,
\end{aligned}$$

finishing the proof. □

3.4. Nijenhuis 2-cycles

Definition 3.12. *Let C be a coalgebra and $r=r^1\otimes r^2\in C\otimes C$. We call r a* **2-cycle** *on a coalgebra C if*

$$r^1{}_1\otimes r^1{}_2\otimes r^2+r^2\otimes r^1{}_1\otimes r^1{}_2+r^1{}_2\otimes r^2\otimes r^1{}_1=0. \tag{39}$$

r is called **anti-symmetric** *if $r^1\otimes r^2=-r^2\otimes r^1$.*

Define the linear map $r^\#:C^*\longrightarrow C$ by

$$\langle r^\#(a^*),b^*\rangle=\langle a^*,r^1\rangle\langle b^*,r^2\rangle,\forall\, a^*,b^*\in C^*.$$

r is **nondegenerate** *if the linear map $r^\#$ is an isomorphism.*

Lemma 3.13. *Let C be a coalgebra and $r\in C\otimes C$ be anti-symmetric and nondegenerate. Then r is a 2-cycle if and only if $r^{\#-1}:C\longrightarrow C^*$ is an $\mathcal{O}$-operator on the bicomodule $(C^*,\Delta_R{}^*,\Delta_L{}^*)$ over C.*

Proof. ($\Longrightarrow$) $r^{\#-1}$ is an $\mathcal{O}$-operator on the bicomodule $(C^*,\Delta_R{}^*,\Delta_L{}^*)$ if and only if

$$r^{\#-1}(c_1)\otimes r^{\#-1}(c_2)=r^{\#-1}(r^{\#-1}(c)^{(-1)})\otimes r^{\#-1}(c)^{(0)}+r^{\#-1}(c)^{[0]}\otimes r^{\#-1}(r^{\#-1}(c)^{[1]}).$$

Let $r^{\#-1}(c)=c^*$, then we have

$$r^{\#-1}(c_1)\otimes r^{\#-1}(c_2)-r^{\#-1}(c^{*(-1)})\otimes c^{*(0)}-c^{*[0]}\otimes r^{\#-1}(c^{*[1]})=0. \tag{40}$$

Applying $r^\#\otimes r^\#$ to Eq.(40), one has

$$c_1\otimes c_2-c^{*(-1)}\otimes r^\#(c^{*(0)})-r^\#(c^{*[0]})\otimes c^{*[1]}=0. \tag{41}$$

Next we verify that Eq.(41) holds as follows. For all $a^*,b^*\in C^*$,

$$\begin{aligned}
&\langle b^*\otimes a^*,c_1\otimes c_2-c^{*(-1)}\otimes r^\#(c^{*(0)})-r^\#(c^{*[0]})\otimes c^{*[1]}\rangle\\
&= \langle b^*\otimes a^*,c_1\otimes c_2\rangle-\langle b^*\otimes a^*,c^{*(-1)}\otimes r^\#(c^{*(0)})\rangle-\langle b^*\otimes a^*,r^\#(c^{*[0]})\otimes c^{*[1]}\rangle
\end{aligned}$$

$$\begin{aligned}
&\overset{(12)}{=} \langle b^*\otimes a^*, \Delta(r^\#(c^*))\rangle - \langle b^*\otimes a^*, e_{i2}\otimes r^\#(e^i)\rangle\langle c^*, e_{i1}\rangle - \langle b^*\otimes a^*, r^\#(e^i)\otimes e_{i1}\rangle\langle c^*, e_{i2}\rangle\\
&= \langle c^*, r^1\rangle\langle b^*, r^2{}_1\rangle\langle a^*, r^2{}_2\rangle - \langle b^*, e_{i2}\rangle\langle c^*, e_{i1}\rangle\langle e^i, r^1\rangle\langle a^*, r^2\rangle - \langle c^*, e_{i2}\rangle\langle e^i, r^1\rangle\langle b^*, r^2\rangle\langle a^*, e_{i1}\rangle\\
&= -\langle c^*, r^2\rangle\langle b^*, r^1{}_1\rangle\langle a^*, r^1{}_2\rangle - \langle b^*, r^1{}_2\rangle\langle c^*, r^1{}_1\rangle\langle a^*, r^2\rangle - \langle c^*, r^1{}_2\rangle\langle b^*, r^2\rangle\langle a^*, r^1{}_1\rangle\\
&\overset{(39)}{=} 0,
\end{aligned}$$

finishing the proof.

($\Longleftarrow$) It can be proved similarly. □

By Lemma 3.8 and Lemma 3.13, we obtain

Corollary 3.14. *Let C be a coalgebra and $\mathfrak{B}\in(C\otimes C)^*$ be a skew-symmetric bilinear form. Then $\mathfrak{B}$ is a nondegenerate solution of cYBe if and only if r is 2-cycle, where*

$$r^1\otimes r^2 = e_i\otimes \mathfrak{B}^{\#-1}(e^i). \tag{42}$$

Definition 3.15. *Let (C,N) a Nijenhuis coalgebra and $r\in C\otimes C$ be a 2-cycle. Then r is a* **Nijenhuis 2-cycle on** *(C,N) if*

$$N(r^1)\otimes r^2 = r^1\otimes N(r^2) \tag{43}$$

and $r_N\in C\otimes C$ defined by $r_N = N(r^1)\otimes r^2$ is also a 2-cycle, i.e.

$$r^1{}_1\otimes r^1{}_2\otimes N(r^2) + N(r^2)\otimes r^1{}_1\otimes r^1{}_2 + r^1{}_2\otimes N(r^2)\otimes r^1{}_1 = 0. \tag{44}$$

Theorem 3.16. *Let (C,N) be a Nijenhuis coalgebra and the bilinear form $\mathfrak{B}\in(C\otimes C)^*$ be nondegenerate. Then $\mathfrak{B}$ is a skew-symmetric solution of Nijenhuis admissible cYBe in (C,N) if and only if r is an anti-symmetric Nijenhuis 2-cycle on (C,N), where r is given by Eq.(42).*

Proof. It is obvious that $\mathfrak{B}$ is skew-symmetric if and only if r is anti-symmetric. Next we only need to prove that Eqs.(37) and (38) hold for $\mathfrak{B}$ and N if and only if Eqs.(43) and (44) hold for r and N. For all $\alpha,\beta\in C^*$, we have

$$\begin{aligned}
&\langle \alpha\otimes\beta, N(r^1)\otimes r^2\rangle = \langle\alpha, N(r^1)\rangle\langle\beta, r^2\rangle = \langle N^*(\alpha), r^1\rangle\langle\beta, r^2\rangle \overset{(42)}{=} \langle N^*(\alpha), e_i\rangle\langle\beta, \mathfrak{B}^{\#-1}(e^i)\rangle\\
&\quad = \langle N^*(\alpha), e_i\rangle\langle \mathfrak{B}^{\#-1*}(\beta), e^i\rangle = \langle N^*(\alpha), \mathfrak{B}^{\#-1*}(\beta)\rangle = \langle \mathfrak{B}^{\#-1}N^*(\alpha), \beta\rangle;\\
&\langle \alpha\otimes\beta, r^1\otimes N(r^2)\rangle = \langle\alpha, r^1\rangle\langle\beta, N(r^2)\rangle = \langle\alpha, r^1\rangle\langle N^*(\beta), r^2\rangle \overset{(42)}{=} \langle\alpha, e_i\rangle\langle N^*(\beta), \mathfrak{B}^{\#-1}(e^i)\rangle\\
&\quad = \langle\alpha, e_i\rangle\langle \mathfrak{B}^{\#-1*}N^*(\beta), e^i\rangle = \langle\alpha, \mathfrak{B}^{\#-1*}N^*(\beta)\rangle = \langle N\mathfrak{B}^{\#-1}(\alpha), \beta\rangle.
\end{aligned}$$

By Remark 3.10, $\mathfrak{B}^{\#-1}N^* = N\mathfrak{B}^{\#-1}$, we have $N(r^1)\otimes r^2 = r^1\otimes N(r^2)$, which implies that Eq.(43) is equivalent to Eq.(37).

For $a,b\in C$, set $a^* = \mathfrak{B}^\#(a)$ and $b^* = \mathfrak{B}^\#(b)$, by Eq.(38), for $c^*\in C^*$, we have

$$\begin{aligned}
&\langle c^*, \mathfrak{B}(N(a)_1, b)N(a)_2 + N(b)_1\mathfrak{B}(N(b)_2, a) - N(b_1)\mathfrak{B}(b_2, a) - \mathfrak{B}(a_1, b)N(a_2)\rangle\\
&\quad = -\langle c^*, N(a)_2\rangle\mathfrak{B}(b, N(a)_1) - \langle c^*, N(b)_1\rangle\mathfrak{B}(a, N(b)_2) + \langle c^*, N(b_1)\rangle\mathfrak{B}(a, b_2)\\
&\qquad +\langle c^*, N(a_2)\rangle\mathfrak{B}(b, a_1)\\
&\quad \overset{(35)}{=} -\langle c^*, N(a)_2\rangle\langle\mathfrak{B}^\#(b), N(a)_1\rangle - \langle c^*, N(b)_1\rangle\langle\mathfrak{B}^\#(a), N(b)_2\rangle + \langle c^*, N(b_1)\rangle\langle\mathfrak{B}^\#(a), b_2\rangle\\
&\qquad +\langle c^*, N(a_2)\rangle\langle\mathfrak{B}^\#(b), a_1\rangle\\
&\quad = -\langle c^*, N(a)_2\rangle\langle b^*, N(a)_1\rangle - \langle c^*, N(b)_1\rangle\langle a^*, N(b)_2\rangle + \langle c^*, N(b_1)\rangle\langle a^*, b_2\rangle + \langle c^*, N(a_2)\rangle\langle b^*, a_1\rangle\\
&\quad \overset{(42)}{=} -\langle b^*\otimes c^*, N(r^2)_1\otimes N(r^2)_2\rangle\langle a^*, r^1\rangle - \langle c^*\otimes a^*, N(r^2)_1\otimes N(r^2)_2\rangle\langle b^*, r^1\rangle\\
&\qquad +\langle c^*\otimes a^*, N(r^2{}_1)\otimes r^2{}_2\rangle\langle b^*, r^1\rangle + \langle b^*\otimes c^*, r^2{}_1\otimes N(r^2{}_2)\rangle\langle a^*, r^1\rangle\\
&\quad \overset{(39)}{=} \langle a^*\otimes c^*\otimes b^*, r^2\otimes N(r^1)_2\otimes N(r^1)_1 + N(r^1)_2\otimes N(r^1)_1\otimes r^2 + r^1{}_1\otimes N(r^2)\otimes r^1{}_2\rangle\\
&\quad \overset{(43)}{=} \langle a^*\otimes c^*\otimes b^*, N(r^2)\otimes r^1{}_2\otimes r^1{}_1 + r^1{}_2\otimes r^1{}_1\otimes N(r^2) + r^1{}_1\otimes N(r^2)\otimes r^1{}_2\rangle,
\end{aligned}$$

which implies that Eq.(38) is equivalent to Eq.(44). □

Proposition 3.17. *Let (C, N) be a Nijenhuis coalgebra and $r \in C \otimes C$ be nondegenerate and anti-symmetric. If r is Nijenhuis 2-cycle on (C, N), then $r^{\#-1} + r^{\#-1}N$ is an $\mathcal{O}$-operator on the bicomodule $(C^*, \Delta_R{}^*, \Delta_L{}^*)$ over C.*

Proof. $r^{\#-1} + r^{\#-1}N$ is an $\mathcal{O}$-operator on the bicomodule $(C^*, \Delta_R{}^*, \Delta_L{}^*)$ over C if and only if

$$\begin{aligned} = \; & r^{\#-1}(c_1) \otimes r^{\#-1}(c_2) + r^{\#-1}(c_1) \otimes r^{\#-1}N(c_2) + r^{\#-1}N(c_1) \otimes r^{\#-1}(c_2) + r^{\#-1}N(c_1) \otimes r^{\#-1}N(c_2) \\ & -(r^{\#-1} \otimes id)\Delta_R{}^*(r^{\#-1}(c)) - (r^{\#-1} \otimes id)\Delta_R{}^*(r^{\#-1}N(c)) - (r^{\#-1}N \otimes id)\Delta_R{}^*(r^{\#-1}(c)) \\ & -(r^{\#-1}N \otimes id)\Delta_R{}^*(r^{\#-1}N(c)) - (id \otimes r^{\#-1})\Delta_L{}^*(r^{\#-1}(c)) - (id \otimes r^{\#-1})\Delta_L{}^*(r^{\#-1}N(c)) \\ & -(id \otimes r^{\#-1}N)\Delta_L{}^*(r^{\#-1}(c)) - (id \otimes r^{\#-1}N)\Delta_L{}^*(r^{\#-1}N(c)). \end{aligned} \tag{45}$$

That $r^{\#-1}N$ is an $\mathcal{O}$-operator on the bicomodule $(C^*, \Delta_R{}^*, \Delta_L{}^*)$ over C can be checked as follows.
$r^{\#-1}N$ is an $\mathcal{O}$-operator if and only if

$$\begin{aligned} 0 &= r^{\#-1}N(c_1) \otimes r^{\#-1}N(c_2) - (r^{\#-1}N \otimes id)\Delta_R{}^*(r^{\#-1}N(c)) - (id \otimes r^{\#-1}N)\Delta_L{}^*(r^{\#-1}N(c)) \\ &= r^{\#-1}N(c_1) \otimes r^{\#-1}N(c_2) - r^{\#-1}N(r^{\#-1}N(c)^{(-1)}) \otimes r^{\#-1}N(c)^{(0)} - r^{\#-1}N(c)^{[0]} \otimes r^{\#-1}N(r^{\#-1}N(c)^{[1]}). \end{aligned}$$

Applying $r^{\#} \otimes r^{\#}$ to the above equation, we get

$$0 = N(c_1) \otimes N(c_2) - N(r^{\#-1}N(c)^{(-1)}) \otimes r^{\#}(r^{\#-1}N(c)^{(0)}) - r^{\#}(r^{\#-1}N(c)^{[0]}) \otimes N(r^{\#-1}N(c)^{[1]}).$$

Set $c = r^{\#}(c^*)$. By Eq.(43), $r^{\#-1}N = N^* r^{\#-1}$, then for all $a^*, b^* \in C^*$, we obtain

$$\begin{aligned} & \langle b^* \otimes a^*, N(c_1) \otimes N(c_2) - N(r^{\#-1}N(c)^{(-1)}) \otimes r^{\#}(r^{\#-1}N(c)^{(0)}) - r^{\#}(r^{\#-1}N(c)^{[0]}) \otimes N(r^{\#-1}N(c)^{[1]})\rangle \\ \overset{(1)}{=}\; & \langle b^* \otimes a^*, (N \otimes id)\Delta(c) + (id \otimes N)\Delta(c) - \Delta N^2(c) - (N \otimes r^{\#})(N^* r^{\#-1}(c)^{(-1)} \otimes N^* r^{\#-1}(c)^{(0)}) \\ & -(r^{\#} \otimes N)(N^* r^{\#-1}(c)^{[0]} \otimes N^* r^{\#-1}(c)^{[1]})\rangle \\ \overset{(12)}{=}\; & \langle b^* \otimes a^*, N(N(r^2)_1) \otimes N(r^2)_2\rangle\langle c^*, r^1\rangle + \langle b^* \otimes a^*, N(r^2)_1 \otimes N(N(r^2)_2)\rangle\langle c^*, r^1\rangle \\ & -\langle b^* \otimes a^*, N^2(r^2)_1 \otimes N^2(r^2)_2\rangle\langle c^*, r^1\rangle - \langle b^* \otimes a^*, N(e_{i2}) \otimes r^{\#}(e^i)\rangle\langle N^*(c^*), e_{i1}\rangle \\ & -\langle b^* \otimes a^*, r^{\#}(e^i) \otimes N(e_{i1})\rangle\langle N^*(c^*), e_{i2}\rangle \\ \overset{(43)}{=}\; & -\langle b^* \otimes a^* \otimes c^*, N(r^1{}_1) \otimes r^1{}_2 \otimes N(r^2)\rangle - \langle b^* \otimes a^* \otimes c^*, r^1{}_1 \otimes N(r^1{}_2) \otimes N(r^2)\rangle \\ & +\langle b^* \otimes a^* \otimes c^*, r^1{}_1 \otimes r^1{}_2 \otimes N^2(r^2)\rangle - \langle b^* \otimes a^* \otimes c^*, N(r^1{}_2) \otimes r^2 \otimes N(r^1{}_1)\rangle \\ & -\langle b^* \otimes a^* \otimes c^*, r^2 \otimes N(r^1{}_1) \otimes N(r^1{}_2)\rangle \\ \overset{(39)}{=}\; & \langle b^* \otimes a^* \otimes c^*, N(r^2) \otimes r^1{}_1 \otimes N(r^1{}_2) + r^1{}_2 \otimes N(r^2) \otimes N(r^1{}_1) + r^1{}_1 \otimes r^1{}_2 \otimes N^2(r^2)\rangle \\ \overset{(44)}{=}\; & 0. \end{aligned}$$

Thus $r^{\#-1}N$ is an $\mathcal{O}$-operator.
By Lemma 3.13, $r^{\#-1}$ is an $\mathcal{O}$-operator.
Then Eq.(45) is equivalent to

$$\begin{aligned} 0 \;=\; & r^{\#-1}(c_1) \otimes r^{\#-1}N(c_2) + r^{\#-1}N(c_1) \otimes r^{\#-1}(c_2) - (r^{\#-1} \otimes id)\Delta_R{}^*(r^{\#-1}N(c)) \\ & -(r^{\#-1}N \otimes id)\Delta_R{}^*(r^{\#-1}(c)) - (id \otimes r^{\#-1})\Delta_L{}^*(r^{\#-1}N(c)) - (id \otimes r^{\#-1}N)\Delta_L{}^*(r^{\#-1}(c)). \end{aligned}$$

Applying $r^{\#} \otimes r^{\#}$ to the above equation, we have

$$\begin{aligned} 0 \;=\; & c_1 \otimes N(c_2) + N(c_1) \otimes c_2 - (id \otimes r^{\#})\Delta_R{}^*(r^{\#-1}N(c)) - (N \otimes r^{\#})\Delta_R{}^*(r^{\#-1}(c)) \\ & -(r^{\#} \otimes id)\Delta_L{}^*(r^{\#-1}N(c)) - (r^{\#} \otimes N)\Delta_L{}^*(r^{\#-1}(c)). \end{aligned}$$

Let $c^* = r^{\#-1}(c)$ and by $r^{\#-1}N = N^* r^{\#-1}$, one can obtain

$$\begin{aligned} 0 \;=\; & c_1 \otimes N(c_2) + N(c_1) \otimes c_2 - (id \otimes r^{\#})\Delta_R{}^*(N^*(c^*)) - (N \otimes r^{\#})\Delta_R{}^*(c^*) \\ & -(r^{\#} \otimes id)\Delta_L{}^*(N^*(c^*)) - (r^{\#} \otimes N)\Delta_L{}^*(c^*). \end{aligned} \tag{46}$$

Next we only need to prove that Eq.(46) holds. For all $a^*, b^* \in C^*$, one calculates

$$\begin{aligned}
&\langle b^*\otimes a^*, c_1\otimes N(c_2)+N(c_1)\otimes c_2-(id\otimes r^\#)\Delta_R{}^*(N^*(c^*))-(N\otimes r^\#)\Delta_R{}^*(c^*)\\
&\quad -(r^\#\otimes id)\Delta_L{}^*(N^*(c^*))-(r^\#\otimes N)\Delta_L{}^*(c^*)\rangle\\
\overset{(12)}{=}\ &\langle b^*\otimes a^*, (id\otimes N)\triangle r^\#(c^*)\rangle+\langle b^*\otimes a^*, (N\otimes id)\triangle r^\#(c^*)\rangle\\
&-\langle b^*\otimes a^*, (id\otimes r^\#)(N^*(c^*)^{(-1)}\otimes N^*(c^*)^{(0)})\rangle-\langle b^*\otimes a^*, (N\otimes r^\#)(c^{*(-1)}\otimes c^{*(0)})\rangle\\
&-\langle b^*\otimes a^*, (r^\#\otimes id)(N^*(c^*)^{[0]}\otimes N^*(c^*)^{[1]})\rangle-\langle b^*\otimes a^*, (r^\#\otimes N)(c^{*[0]}\otimes c^{*[1]})\rangle\\
=\ &\langle b^*\otimes a^*, r^2{}_1\otimes N(r^2{}_2)\rangle\langle c^*, r^1\rangle+\langle b^*\otimes a^*, N(r^2{}_1)\otimes r^2{}_2\rangle\langle c^*, r^1\rangle\\
&-\langle b^*\otimes a^*, e_{i2}\otimes r^2\rangle\langle e^i, r^1\rangle\langle c^*, N(e_{i1})\rangle-\langle b^*\otimes a^*, N(e_{i2})\otimes r^2\rangle\langle e^i, r^1\rangle\langle c^*, e_{i1}\rangle\\
&-\langle b^*\otimes a^*, r^2\otimes e_{i1}\rangle\langle e^i, r^1\rangle\langle c^*, N(e_{i2})\rangle-\langle b^*\otimes a^*, r^2\otimes N(e_{i1})\rangle\langle e^i, r^1\rangle\langle c^*, e_{i2}\rangle\\
=\ &\langle b^*\otimes a^*\otimes c^*, -r^1{}_1\otimes N(r^1{}_2)\otimes r^2-r^2\otimes N(r^1{}_1)\otimes r^1{}_2-N(r^1{}_1)\otimes r^1{}_2\otimes r^2\\
&-N(r^1{}_2)\otimes r^2\otimes r^1{}_1-r^1{}_2\otimes r^2\otimes N(r^1{}_1)-r^2\otimes r^1{}_1\otimes N(r^1{}_2)\rangle\\
\overset{(39)}{=}\ &\langle b^*\otimes a^*\otimes c^*, r^1{}_2\otimes N(r^2)\otimes r^1{}_1+N(r^2)\otimes r^1{}_1\otimes r^1{}_2+r^1{}_1\otimes r^1{}_2\otimes N(r^2)\rangle\\
\overset{(44)}{=}\ &0,
\end{aligned}$$

finishing the proof. □

Theorem 3.18. *Let C be a coalgebra, r be a nondegenerate anti-symmetric 2-cycle on C and $\mathfrak{B}$ be a skew-symmetric solution of cYBe in C. If $\mathfrak{B}+r^{-1}$ is a skew-symmetric solution of cYBe in C, then (C,N) is a Nijenhuis coalgebra, where r^{-1} is defined by $r^{-1}(a,b)=\langle r^{\#-1}(a), b\rangle$ and*

$$N = r^\#\mathfrak{B}^\#. \tag{47}$$

Proof. Since $\mathfrak{B}+r^{-1}$ is a skew-symmetric solution of cYBe, by Lemma 3.8, we know that $\mathfrak{B}^\#+r^{\#-1}$ is an $\mathcal{O}$-operator on the bicomodule $(C^*, \Delta_R{}^*, \Delta_L{}^*)$ over C. By the assumption that r is a nondegenerate 2-cycle on C and $\mathfrak{B}$ is a skew-symmetric solution of cYBe in C, by Lemmas 3.8 and 3.13, we know that $\mathfrak{B}^\#$ and $r^{\#-1}$ are $\mathcal{O}$-operators on the bicomodule $(C^*, \Delta_R{}^*, \Delta_L{}^*)$ over C. Thus we can obtain

$$\begin{aligned}
0=\ &r^{\#-1}(c_1)\otimes\mathfrak{B}^\#(c_2)+\mathfrak{B}^\#(c_1)\otimes r^{\#-1}(c_2)-r^{\#-1}(\mathfrak{B}^\#(c)^{(-1)})\otimes\mathfrak{B}^\#(c)^{(0)}\\
&-\mathfrak{B}^\#(r^{\#-1}(c)^{(-1)})\otimes r^{\#-1}(c)^{(0)}-\mathfrak{B}^\#(c)^{[0]}\otimes r^{\#-1}(\mathfrak{B}^\#(c)^{[1]})-r^{\#-1}(c)^{[0]}\otimes\mathfrak{B}^\#(r^{\#-1}(c)^{[1]}).
\end{aligned}$$

Set $c=r^\#(c^*)$, one has

$$\begin{aligned}
0=\ &r^{\#-1}(r^\#(c^*)_1)\otimes\mathfrak{B}^\#(r^\#(c^*)_2)+\mathfrak{B}^\#(r^\#(c^*)_1)\otimes r^{\#-1}(r^\#(c^*)_2)-r^{\#-1}(\mathfrak{B}^\#(r^\#(c^*))^{(-1)})\otimes\mathfrak{B}^\# r^\#(c^*)^{(0)}\\
&-\mathfrak{B}^\#(c^{*(-1)})\otimes c^{*(0)}-\mathfrak{B}^\# r^\#(c^*)^{[0]}\otimes r^{\#-1}(\mathfrak{B}^\# r^\#(c^*)^{[1]})-c^{*[0]}\otimes\mathfrak{B}^\#(c^{*[1]}).
\end{aligned}$$

Applying $r^\#\otimes r^\#$ to the above equation, we obtain

$$\begin{aligned}
0=\ &r^\#(c^*)_1\otimes r^\#\mathfrak{B}^\#(r^\#(c^*)_2)+r^\#\mathfrak{B}^\#(r^\#(c^*)_1)\otimes r^\#(c^*)_2-\mathfrak{B}^\# r^\#(c^*)^{(-1)}\otimes r^\#(\mathfrak{B}^\# r^\#(c^*)^{(0)})\\
&-r^\#\mathfrak{B}^\#(c^{*(-1)})\otimes r^\#(c^{*(0)})-r^\#(\mathfrak{B}^\# r^\#(c^*)^{[0]})\otimes\mathfrak{B}^\# r^\#(c^*)^{[1]}-r^\#(c^{*[0]})\otimes r^\#\mathfrak{B}^\#(c^{*[1]}).
\end{aligned}$$

Therefore, for all $a^*, b^*\in C^*$, we have

$$\begin{aligned}
0\ =\ &\langle a^*\otimes b^*, r^\#(c^*)_1\otimes r^\#\mathfrak{B}^\#(r^\#(c^*)_2)+r^\#\mathfrak{B}^\#(r^\#(c^*)_1)\otimes r^\#(c^*)_2-\mathfrak{B}^\# r^\#(c^*)^{(-1)}\otimes r^\#(\mathfrak{B}^\# r^\#(c^*)^{(0)})\\
&-r^\#\mathfrak{B}^\#(c^{*(-1)})\otimes r^\#(c^{*(0)})-r^\#(\mathfrak{B}^\# r^\#(c^*)^{[0]})\otimes\mathfrak{B}^\# r^\#(c^*)^{[1]}-r^\#(c^{*[0]})\otimes r^\#\mathfrak{B}^\#(c^{*[1]})\rangle\\
\overset{(12)}{=}\ &\langle a^*, r^2{}_1\rangle\langle b^*, r^\#\mathfrak{B}^\#(r^2{}_2)\rangle\langle c^*, e_i\rangle\langle e^i, r^1\rangle+\langle a^*, r^\#\mathfrak{B}^\#(r^2{}_1)\rangle\langle b^*, r^2{}_1\rangle\langle c^*, e_i\rangle\langle e^i, r^1\rangle-\langle a^*, e_{i2}\rangle\\
&\langle b^*, r^\#(e^i)\rangle\langle\mathfrak{B}^\# r^\#(c^*), e_{i1}\rangle-\langle a^*, r^\#\mathfrak{B}^\#(e_{i2})\rangle\langle b^*, r^\#(e^i)\rangle\langle c^*, e_{i1}\rangle-\langle a^*, r^\#(e^i)\rangle\langle b^*, e_{i1}\rangle\\
&\langle\mathfrak{B}^\# r^\#(c^*), e_{i2}\rangle-\langle a^*, r^\#(e^i)\rangle\langle b^*, r^\#\mathfrak{B}^\#(e_{i1})\rangle\langle c^*, e_{i2}\rangle\\
\overset{(35)}{=}\ &\langle a^*, r^2{}_1\rangle\langle b^*, \bar{r}^2\rangle\mathfrak{B}(r^2{}_2, \bar{r}^1)\langle c^*, r^1\rangle+\langle a^*, \bar{r}^2\rangle\langle b^*, r^2{}_2\rangle\mathfrak{B}(r^2{}_1, \bar{r}^1)\langle c^*, r^1\rangle-\langle a^*, r^1{}_2\rangle\langle b^*, r^2\rangle
\end{aligned}$$

$$\mathfrak{B}(\bar{r}^2, r^1{}_1)\langle c^*, \bar{r}^1\rangle - \langle a^*, \bar{r}^2\rangle\mathfrak{B}(r^1{}_2, \bar{r}^1)\langle b^*, r^2\rangle\langle c^*, r^1{}_1\rangle - \langle a^*, r^2\rangle\langle b^*, r^1{}_1\rangle\mathfrak{B}(\bar{r}^2, r^1{}_2)\langle c^*, \bar{r}^1\rangle$$
$$-\langle a^*, r^2\rangle\langle b^*, \bar{r}^2\rangle\mathfrak{B}(r^1{}_1, \bar{r}^1)\langle c^*, r^1{}_2\rangle$$
$$-\bar{r}^2 \otimes r^2 \otimes r^1{}_1\mathfrak{B}(r^1{}_2, \bar{r}^1) - r^1{}_2 \otimes r^2 \otimes \bar{r}^1\mathfrak{B}(\bar{r}^2, r^1{}_1) - r^2 \otimes r^1{}_1 \otimes \bar{r}^1\mathfrak{B}(\bar{r}^2, r^1{}_2))\rangle$$
$$\overset{(39)}{=} \langle a^* \otimes b^* \otimes c^*, r^1{}_2 \otimes \bar{r}^2 \otimes r^1{}_1\mathfrak{B}(r^2, \bar{r}^1) + \bar{r}^2 \otimes r^1{}_1 \otimes r^1{}_2\mathfrak{B}(r^2, \bar{r}^1) + r^1{}_1 \otimes r^1{}_2 \otimes \bar{r}^1\mathfrak{B}(\bar{r}^2, r^2)\rangle.$$

So

$$r^1{}_2 \otimes \bar{r}^2 \otimes r^1{}_1\mathfrak{B}(r^2, \bar{r}^1) + \bar{r}^2 \otimes r^1{}_1 \otimes r^1{}_2\mathfrak{B}(r^2, \bar{r}^1) + r^1{}_1 \otimes r^1{}_2 \otimes \bar{r}^1\mathfrak{B}(\bar{r}^2, r^2) = 0. \tag{48}$$

Then for all $c \in C$,

$$N(c_1) \otimes N(c_2) - N(N(c)_1) \otimes N(c)_2 - N(c)_1 \otimes N(N(c)_2) + N^2(c)_1 \otimes N^2(c)_2$$
$$\overset{(47)}{=} r^2\langle \mathfrak{B}^\#(c_1), r_1\rangle \otimes \bar{r}^2\langle \mathfrak{B}^\#(c_2), \bar{r}^1\rangle - \langle \mathfrak{B}^\#(c), r_1\rangle\langle \mathfrak{B}^\#(r^2{}_1), \bar{r}^1\rangle\bar{r}^2 \otimes r^2{}_2$$
$$-r^2{}_1\langle \mathfrak{B}^\#(c), r_1\rangle \otimes \bar{r}^2\langle \mathfrak{B}^\#(r^2{}_2), \bar{r}^1\rangle + \langle \mathfrak{B}^\#(c), r_1\rangle\bar{r}^2{}_1 \otimes \bar{r}^2{}_2\langle \mathfrak{B}^\#(r^2), \bar{r}^1\rangle$$
$$\overset{(35)}{=} r^2\mathfrak{B}(c_1, r^1) \otimes \bar{r}^2\mathfrak{B}(c_2, \bar{r}^1) - \bar{r}^2\mathfrak{B}(c, r^1) \otimes r^2{}_2\mathfrak{B}(r^2{}_1, \bar{r}^1) - r^2{}_1\mathfrak{B}(c, r^1) \otimes \bar{r}^2\mathfrak{B}(r^2{}_2, \bar{r}^1)$$
$$+\bar{r}^2{}_1\mathfrak{B}(c, r_1) \otimes \bar{r}^2{}_2\mathfrak{B}(r^2, \bar{r}^1)$$
$$\overset{(39)}{=} r^2\mathfrak{B}(c_1, r^1) \otimes \bar{r}^2\mathfrak{B}(c_2, \bar{r}^1) - \bar{r}^2\mathfrak{B}(c, r^1{}_2) \otimes r^1{}_1\mathfrak{B}(r^2, \bar{r}^1) - \bar{r}^2\mathfrak{B}(c, r^1{}_1) \otimes r^2\mathfrak{B}(r^1{}_2, \bar{r}^1)$$
$$-r^2\mathfrak{B}(c, r^1{}_2) \otimes \bar{r}^2\mathfrak{B}(r^1{}_1, \bar{r}^1) - r^1{}_2\mathfrak{B}(c, r^1{}_1) \otimes \bar{r}^2\mathfrak{B}(r^2, \bar{r}^1) - r^1{}_1\mathfrak{B}(c, \bar{r}^1) \otimes r^1{}_2\mathfrak{B}(\bar{r}^2, r^2)$$
$$\overset{(48)}{=} r^2\mathfrak{B}(c_1, r^1) \otimes \bar{r}^2\mathfrak{B}(c_2, \bar{r}^1) - \bar{r}^2\mathfrak{B}(c, r^1{}_1) \otimes r^2\mathfrak{B}(r^1{}_2, \bar{r}^1) - r^2\mathfrak{B}(c, r^1{}_2) \otimes \bar{r}^2\mathfrak{B}(r^1{}_1, \bar{r}^1)$$
$$= r^2\mathfrak{B}(r^1, c_1) \otimes \bar{r}^2\mathfrak{B}(\bar{r}^1, c_2) - r^2\mathfrak{B}(c, \bar{r}^1{}_1) \otimes \bar{r}^2\mathfrak{B}(\bar{r}^1{}_2, r^1) + r^2\mathfrak{B}(r^1{}_2, c) \otimes \bar{r}^2\mathfrak{B}(r^1{}_1, \bar{r}^1)$$
$$\overset{(36)}{=} 0.$$

Thus, (C, N) is a Nijenhuis coalgebra. □

Corollary 3.19. *Let C be a coalgebra, $\mathfrak{B}$ be a skew-symmetric solution of cYBe in C and r a nondegenerate anti-symmetric 2-cycle on C. If $\mathfrak{B} + r^{-1}$ is a skew-symmetric solution of cYBe in C, then r is a Nijenhuis 2-cycle on (C, N) and $\mathfrak{B}$ is a solution of the Nijenhuis admissible cYBe in (C, N), where the Nijenhuis operator is defined by Eq.(47).*

Proof. By Theorem 3.18, we know that $N = r^\#\mathfrak{B}^\#$ is a Nijenhuis operator over C. Since

$$N(r^1) \otimes r^2 = \langle \mathfrak{B}^\#(r^1), \bar{r}^1\rangle\bar{r}^2 \otimes r^2 = \mathfrak{B}(r^1, \bar{r}^1)\bar{r}^2 \otimes r^2,$$
$$r^1 \otimes N(r^2) = r^1 \otimes \langle \mathfrak{B}^\#(r^2), \bar{r}^1\rangle\bar{r}^2 = r^1 \otimes \mathfrak{B}(r^2, \bar{r}^1)\bar{r}^2$$

and

$$\mathfrak{B}(r^1, \bar{r}^1)\bar{r}^2 \otimes r^2 = -\mathfrak{B}(\bar{r}^1, r^2)r^1 \otimes \bar{r}^2 = r^1 \otimes \mathfrak{B}(r^2, \bar{r}^1)\bar{r}^2,$$

then

$$N(r^1) \otimes r^2 = r^1 \otimes N(r^2).$$

$$r^1{}_1 \otimes r^1{}_2 \otimes N(r^2) + N(r^2) \otimes r^1{}_1 \otimes r^1{}_2 + r^1{}_2 \otimes N(r^2) \otimes r^1{}_1$$
$$= r^1{}_1 \otimes r^1{}_2 \otimes r^\#\mathfrak{B}^\#(r^2) + r^\#\mathfrak{B}^\#(r^2) \otimes r^1{}_1 \otimes r^1{}_2 + r^1{}_2 \otimes r^\#\mathfrak{B}^\#(r^2) \otimes r^1{}_1$$
$$\overset{(35)}{=} r^1{}_1 \otimes r^1{}_2 \otimes \mathfrak{B}(r^2, \bar{r}^1)\bar{r}^2 + \mathfrak{B}(r^2, \bar{r}^1)\bar{r}^2 \otimes r^1{}_1 \otimes r^1{}_2 + r^1{}_2 \otimes \mathfrak{B}(r^2, \bar{r}^1)\bar{r}^2 \otimes r^1{}_1$$
$$= r^1{}_1 \otimes r^1{}_2 \otimes \mathfrak{B}(\bar{r}^2, r^2)\bar{r}^1 + \mathfrak{B}(r^2, \bar{r}^1)\bar{r}^2 \otimes r^1{}_1 \otimes r^1{}_2 + r^1{}_2 \otimes \mathfrak{B}(r^2, \bar{r}^1)\bar{r}^2 \otimes r^1{}_1$$
$$\overset{(48)}{=} 0.$$

Thus r is a Nijenhuis 2-cycle on (C, N).

In what follows, we prove that $\mathfrak{B}$ is a solution of the Nijenhuis admissible cYBe in (C, N).

For all $a, c \in C$,

$$\mathfrak{B}(a, N(c)) = \langle \mathfrak{B}^\#(a), N(c)\rangle = \langle \mathfrak{B}^\#(a), r^\#\mathfrak{B}^\#(c)\rangle = \langle \mathfrak{B}^\#(a), r^2\rangle\langle \mathfrak{B}^\#(c), r^1\rangle = \mathfrak{B}(a, r^2)\mathfrak{B}(c, r^1),$$
$$\mathfrak{B}(N(a), c) = \langle \mathfrak{B}^\# N(a), c\rangle = \langle \mathfrak{B}^\# r^\# \mathfrak{B}^\#(a), c\rangle = \langle \mathfrak{B}^\#(a), r^1\rangle\langle \mathfrak{B}^\#(r^2), c\rangle = \mathfrak{B}(a, r^1)\mathfrak{B}(r^2, c)$$

and

$$\mathfrak{B}(a, r^2)\mathfrak{B}(c, r^1) = -\mathfrak{B}(a, r^1)\mathfrak{B}(c, r^2) = \mathfrak{B}(a, r^1)\mathfrak{B}(r^2, c),$$

then

$$\mathfrak{B}(a, N(c)) = \mathfrak{B}(N(a), c).$$

Next we check that Eq.(36) holds, set $a^* = \mathfrak{B}^\#(a)$, $b^* = \mathfrak{B}^\#(b)$ and $c^* = \mathfrak{B}^\#(c)$, we have

$$\begin{aligned}
&\langle c^*, \mathfrak{B}(N(a)_1, b)N(a)_2 + N(b)_1\mathfrak{B}(N(b)_2, a) - N(b_1)\mathfrak{B}(b_2, a) - \mathfrak{B}(a_1, b)N(a_2)\rangle \\
&= -\langle c^*, N(a)_2\rangle\mathfrak{B}(b, N(a)_1) - \langle c^*, N(b)_1\rangle\mathfrak{B}(a, N(b)_2) + \langle \mathfrak{B}^\#(c), N(b_1)\rangle\mathfrak{B}(a, b_2) \\
&\quad +\langle \mathfrak{B}^\#(c), N(a_2)\rangle\mathfrak{B}(b, a_1) \\
&\overset{(35)}{=} -\langle b^* \otimes c^*, r^2{}_1 \otimes r^2{}_2\rangle\langle a^*, r^1\rangle - \langle c^* \otimes a^*, r^2{}_1 \otimes r^2{}_2\rangle\langle b^*, r^1\rangle + \mathfrak{B}(c, r^2)\mathfrak{B}(b_1, r^1)\mathfrak{B}(a, b_2) \\
&\quad +\mathfrak{B}(c, r^2)\mathfrak{B}(a_2, r^1)\mathfrak{B}(b, a_1) \\
&\overset{(36)}{=} -\langle a^* \otimes b^* \otimes c^*, r^1 \otimes r^2{}_1 \otimes r^2{}_2\rangle - \langle a^* \otimes b^* \otimes c^*, r^2{}_2 \otimes r^1 \otimes r^2{}_1\rangle + \mathfrak{B}(c, r^2)\mathfrak{B}(a, r^1{}_1)\mathfrak{B}(b, r^1{}_2) \\
&= \langle a^* \otimes b^* \otimes c^*, r^2 \otimes r^1{}_1 \otimes r^1{}_2 + r^1{}_2 \otimes r^2 \otimes r^1{}_1 + r^1{}_1 \otimes r^1{}_2 \otimes r^2\rangle \\
&\overset{(39)}{=} 0,
\end{aligned}$$

finishing the proof. □

3.5. Admissible Nijenhuis systems

Definition 3.20. *Let C be a coalgebra, $\mathfrak{B}$ be a solution of cYBe in C and r be a 2-cycle on C. Then $(\mathfrak{B}, r)$ is called an* **admissible Nijenhuis system** *if $r_N = N(r^1) \otimes r^2$ is also a 2-cycle, where N is defined by Eq.(47).*

Theorem 3.21. *Let $\mathfrak{B}$ be the solution of cYBe and r be anti-symmetric. Then $(\mathfrak{B}, r)$ is an admissible Nijenhuis system if and only if r is 2-cycle and*

$$\begin{aligned}
&r^1 \otimes \bar r^2 \otimes r^2{}_2\mathfrak{B}(r^2{}_1, \bar r^1) + r^1 \otimes r^2{}_1 \otimes \bar r^2\mathfrak{B}(r^2{}_2, \bar r^1) + r^2{}_2 \otimes r^1 \otimes \bar r^2\mathfrak{B}(r^2{}_1, \bar r^1) \\
&+\bar r^2 \otimes r^1 \otimes r^2{}_1\mathfrak{B}(r^2{}_2, \bar r^1) + \bar r^2 \otimes r^2{}_2 \otimes r^1\mathfrak{B}(r^2{}_1, \bar r^1) + r^2{}_1 \otimes \bar r^2 \otimes r^1\mathfrak{B}(r^2{}_2, \bar r^1) = 0.
\end{aligned} \tag{49}$$

Proof. ($\Longrightarrow$) Since $(\mathfrak{B}, r)$ is an admissible Nijenhuis system, r is a 2-cycle. Then we have

$$\begin{aligned}
&N(r^1{}_1) \otimes r^1{}_2 \otimes r^2 + N(r^2) \otimes r^1{}_1 \otimes r^1{}_2 + N(r^1{}_2) \otimes r^2 \otimes r^1{}_1 + r^1{}_1 \otimes N(r^1{}_2) \otimes r^2 + r^2 \otimes N(r^1{}_1) \otimes r^1{}_2 \\
&+r^1{}_2 \otimes N(r^2) \otimes r^1{}_1 + r^1{}_1 \otimes r^1{}_2 \otimes N(r^2) + r^2 \otimes r^1{}_1 \otimes N(r^1{}_2) + r^1{}_2 \otimes r^2 \otimes N(r^1{}_1) = 0.
\end{aligned} \tag{50}$$

Since r_N is also a 2-cycle, by Eqs.(44) and (50), we have

$$\begin{aligned}
&N(r^1{}_1) \otimes r^1{}_2 \otimes r^2 + N(r^1{}_2) \otimes r^2 \otimes r^1{}_1 + r^1{}_1 \otimes N(r^1{}_2) \otimes r^2 + r^2 \otimes N(r^1{}_1) \otimes r^1{}_2 \\
&\qquad +r^2 \otimes r^1{}_1 \otimes N(r^1{}_2) + r^1{}_2 \otimes r^2 \otimes N(r^1{}_1) = 0.
\end{aligned}$$

Replacing N by $r^\#\mathfrak{B}^\#$, one has

$$\begin{aligned}
0 &= r^\#\mathfrak{B}^\#(r^1{}_1) \otimes r^1{}_2 \otimes r^2 + r^\#\mathfrak{B}^\#(r^1{}_2) \otimes r^2 \otimes r^1{}_1 + r^1{}_1 \otimes r^\#\mathfrak{B}^\#(r^1{}_2) \otimes r^2 + r^2 \otimes r^\#\mathfrak{B}^\#(r^1{}_1) \otimes r^1{}_2 \\
&\quad +r^2 \otimes r^1{}_1 \otimes r^\#\mathfrak{B}^\#(r^1{}_2) + r^1{}_2 \otimes r^2 \otimes r^\#\mathfrak{B}^\#(r^1{}_1) \\
&= \langle \mathfrak{B}^\#(r^1{}_1), \bar r^1\rangle\bar r^2 \otimes r^1{}_2 \otimes r^2 + \langle \mathfrak{B}^\#(r^1{}_2), \bar r^1\rangle\bar r^2 \otimes r^2 \otimes r^1{}_1 + r^1{}_1 \otimes \langle \mathfrak{B}^\#(r^1{}_2), \bar r^1\rangle\bar r^2 \otimes r^2 \\
&\quad +r^2 \otimes \langle \mathfrak{B}^\#(r^1{}_1), \bar r^1\rangle\bar r^2 \otimes r^1{}_2 + r^2 \otimes r^1{}_1 \otimes \langle \mathfrak{B}^\#(r^1{}_2), \bar r^1\rangle\bar r^2 + r^1{}_2 \otimes r^2 \otimes \langle \mathfrak{B}^\#(r^1{}_1), \bar r^1\rangle\bar r^2 \\
&\overset{(35)}{=} \mathfrak{B}(r^1{}_1, \bar r^1)\bar r^2 \otimes r^1{}_2 \otimes r^2 + \mathfrak{B}(r^1{}_2, \bar r^1)\bar r^2 \otimes r^2 \otimes r^1{}_1 + r^1{}_1 \otimes \mathfrak{B}(r^1{}_2, \bar r^1)\bar r^2 \otimes r^2
\end{aligned}$$

$$+r^2\otimes\mathfrak{B}(r^1{}_1,\bar{r}^1)\bar{r}^2\otimes r^1{}_2+r^2\otimes r^1{}_1\otimes\mathfrak{B}(r^1{}_2,\bar{r}^1)\bar{r}^2+r^1{}_2\otimes r^2\otimes\mathfrak{B}(r^1{}_1,\bar{r}^1)\bar{r}^2$$
$$=-[\mathfrak{B}(r^2{}_1,\bar{r}^1)\bar{r}^2\otimes r^2{}_2\otimes r^1+\mathfrak{B}(r^2{}_2,\bar{r}^1)\bar{r}^2\otimes r^1\otimes r^2{}_1+r^2{}_1\otimes\mathfrak{B}(r^2{}_2,\bar{r}^1)\bar{r}^2\otimes r^1$$
$$+r^1\otimes\mathfrak{B}(r^2{}_1,\bar{r}^1)\bar{r}^2\otimes r^2{}_2+r^1\otimes r^2{}_1\otimes\mathfrak{B}(r^2{}_2,\bar{r}^1)\bar{r}^2+r^2{}_2\otimes r^1\otimes\mathfrak{B}(r^2{}_1,\bar{r}^1)\bar{r}^2].$$

($\Longleftarrow$) can be proved similarly. □

The following result is direct.

Corollary 3.22. *Let C be a coalgebra, r be an anti-symmetric 2-cycle such that Eq.(49) holds. Then $(\mathfrak{B}, r)$ is an admissible Nijenhuis system if and only if $\mathfrak{B}$ is a skew-symmetric solution of cYBe in C.*

Corollary 3.23. *Let C be a coalgebra, $\mathfrak{B}$ be a skew-symmetric bilinear form and r be an anti-symmetric element in $C\otimes C$. If $(\mathfrak{B}, r)$ is an admissible Nijenhuis system, then r is a Nijenhuis 2-cycle on (C, N), where N is defined by Eq.(47).*

Proof. By the proof of Corollary 3.19, we have $N(r^1)\otimes r^2 = r^1\otimes N(r^2)$ and the rest is obvious. □

4. Conclusion

In this paper, we investigate some properties of Nijenhuis operators on coalgebras. In [8], we presented a class of bialgebra theory of Nijenhuis algebra based on associative D-bialgebras. Especially, we obtained the compatible conditions between the Nijenhuis operator on the algebra structure and the Nijenhuis operator on the coalgebra structure via dual representation. By replacing the compatibility condition that the comultiplication Δ is a derivation in an associative D-bialgebra by the condition that Δ is an algebra map in the usual bialgebra, we can get the notion of Nijenhuis bialgebra in the Hopf algebra theory.

Definition 4.1. *Let (H,μ,Δ,T) be a Hopf algebra (maybe without unit and counit). Assume that F is a Nijenhuis operator on the algebra (H,μ) and N is a Nijenhuis operator on the coalgebra (H,Δ) such that*

$$N(F(a)b)+aN^2(b)=F(a)N(b)+N(aN(b)), \tag{51}$$
$$N(aF(b))+N^2(a)b=N(a)F(b)+N(N(a)b), \tag{52}$$
$$(id\otimes N)\Delta F+(F^2\otimes id)\Delta=(F\otimes N)\Delta+(F\otimes id)\Delta F, \tag{53}$$
$$(N\otimes id)\Delta F+(id\otimes F^2)\Delta=(N\otimes F)\Delta+(id\otimes F)\Delta F \tag{54}$$

Then we call (H,μ,Δ,F,N) a **Nijenhuis bialgebra**. *If further $F\circ T=T\circ F, N\circ T=T\circ N$, then we call (H,μ,Δ,T,F,N) a* **Nijenhuis Hopf algebra**.

We will study this algebraic structure in future.

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