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The existence of best proximity points for cyclic quasi-φ-contractions in metric spaces

Akram Safari-Hafshejani[a]

[a] *Department of Pure Mathematics, Payame Noor University (PNU), P. O. Box: 19395-3697, Tehran, Iran*

Abstract. In this paper, we introduce the notion of cyclic quasi-φ-contraction. We prove the existence and uniqueness of best proximity points for this class of mappings on a metric space endowed with ultrametric and *UC* properties. Also, iterative algorithms are furnished to determine such best proximity points. As a result, we establish a fixed point result and a common fixed point theorem. Our results, while generalizing a few existing results in the literature, unify and integrate them.

1. Introduction

Let A and B be nonempty subsets of the metric space (X, d). The self mapping $T : A \cup B \to A \cup B$ is said to be *cyclic* provided that $T(A) \subseteq B$ and $T(B) \subseteq A$. A point $x^* \in A \cup B$ is called a *best proximity point* for T if $d(x^*, Tx^*) = d(A, B)$ where $d(A, B) = \inf\{d(a, b) : \ a \in A, \ b \in B\}$. If $d(A, B) = 0$, x^* is called a *fixed point* of T. In 2006, the cyclic contraction mappings on uniformly convex Banach spaces were introduced and studied by Anthony Eldred and Veeremani [4]. In 2009, *cyclic φ-contraction mappings* on uniformly convex Banach spaces as a generalization of cyclic-contractions, was introduced and studied by Al-Thagafi and Shahzad [3]. Since then, the problems of the existence of best proximity points and fixed points of cyclic mappings, have been extensively studied by many authors; see for instance [1, 2, 5, 6, 8–11, 13–15] and references therein.

In order to extend the obtained best proximity results in uniformly convex Banach spaces to metric spaces, the *UC* property were introduced by Suzuki et al. [15]. They also proved the existence of the best proximity points for cyclic contraction type mappings in metric spaces. In 2022, Safari [12] introduced the geometric concept of the ultrametric property and obtained more general result than Suzuki et al [15].

In this paper, we introduce the notion of cyclic quasi-φ-contraction. We prove the existence and uniqueness of best proximity points for this class of mappings on a metric space endowed with ultrametric and *UC* properties. Also, iterative algorithms are furnished to determine such best proximity points. As a result, we establish a fixed point result and a common fixed point theorem. The presented results extend and improve some recent results in [3, 4, 12, 15] and some other articles.

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Email address: asafari@pnu.ac.ir (Akram Safari-Hafshejani)

2. Preliminaries

Here, we recall some definitions and facts will be used in the next section.

Definition 2.1. *[3] Let A and B be nonempty subsets of the metric space (X, d). The cyclic map $T : A \cup B \to A \cup B$ is said to be* cyclic φ-contraction *if $\varphi : [0, +\infty) \to [0, +\infty)$ is a strictly increasing map and*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B)),$$

for all $x \in A$ and $y \in B$.

Theorem 2.2. *[3, Theorem 8] Let A and B be nonempty convex subsets of a uniformly convex Banach space X such that A is closed. Let $T : A \cup B \to A \cup B$ be a cyclic φ-contraction map. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then there exists a unique $x \in A$ such that $x_{2n} \to x$, $T^2x = x$ and $d(x, Tx) = d(A, B)$.*

Definition 2.3. *[12, 15] Let A and B be nonempty subsets of the metric space (X, d). Then (A, B) is said to satisfies*

(i) the property UC, if $\{x_n\}$ and $\{x'_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y_n) = d(A, B)$, then $\lim_{n\to\infty} d(x_n, x'_n) = 0$;

(ii) ultrametric property if either $d(A, B) = 0$ or there exists $\epsilon_{(A,B)} > 0$ such that for every $0 < \epsilon \leq \epsilon_{(A,B)}$, $x, x' \in A$ and $y \in B$

$$\max\{d(x, y), d(x', y)\} \leq \epsilon + d(A, B) \Rightarrow d(x, x') \leq \epsilon + d(A, B).$$

Suzuki et al. [15] proved that if A and B are nonempty subsets of a uniformly convex Banach space X such that A is convex, then (A, B) has the property UC. In 2019, Safari et al. [12] proved that if A and B are nonempty subsets of the metric space (X, d) such that (A, B) has the UC property, then (A, B) has the ultrametric property.

Lemma 2.4. *[15] Let A and B be nonempty subsets of the metric space (X, d). Assume that (A, B) has the UC property. Let $\{x_n\}$ and $\{y_n\}$ are sequences in A and B respectively, such that either of the following holds*

$$\lim_{m\to\infty} \sup_{n\geq m} d(x_m, y_n) = d(A, B) \quad \text{or} \quad \lim_{n\to\infty} \sup_{m\geq n} d(x_m, y_n) = d(A, B).$$

Then $\{x_n\}$ is Cauchy.

Theorem 2.5. *[12, Theorems 3.5 and 3.6] Let A and B be nonempty subsets of the metric space (X, d) such that A is complete, (A, B) has the UC property and (B, A) has the ultrametric property. Let $T : A \cup B \to A \cup B$ be a generalized cyclic quasi-contraction, i. e., for which there exists $c \in [0, 1)$ such that*

$$d(Tx, Ty) \leq c \max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2}\right\} + (1 - c)d(A, B), \tag{1}$$

for all $x \in A$ and $y \in B$. Then for every $x_0 \in A$ the sequence $\{T^{2n}x_0\}$ converges to some best proximity point $x^* \in A$. Also, every best proximity point of T in A is a fixed point of T^2. Furthermore, if it is assumed that (A_0, B_0) has the Pythagorean property and (B, A) has the UC property, then T has a unique best proximity point x^* in A.

3. Main results

Let (X, d) be a metric space for every $(x, y) \in X \times X$ define $d^*(x, y) := d(x, y) - d(A, B)$. It is immediately that

$$d^*(x, y) \leq d(x, z) + d^*(z, y)$$

and

$$d^*(x, y) - d(A, B) \leq d^*(x, z) + d^*(z, y),$$

for all $x, y, z \in X$.

Definition 3.1. *Let A and B be nonempty subsets of the metric space (X, d). The cyclic map $T : A \cup B \to A \cup B$ is said to be a cyclic quasi-φ-contraction if there exists a strictly increasing map $\varphi : [0, +\infty) \to [0, +\infty)$ such that $I - \varphi$ is a strictly increasing map and*

$$d^*(Tx, Ty) \leq (I - \varphi)\left(\max\left\{d^*(x, y), d^*(x, Tx), d^*(y, Ty), \frac{d^*(x, Ty) + d^*(Tx, y)}{2}\right\}\right), \tag{2}$$

for all $x \in A$ and $y \in B$.

Remark 3.2. *Note that with the conditions of the previous definition, if we have*

$$\begin{aligned} d(Tx, Ty) \leq &\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2}\right\} \\ &- \varphi\left(\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(Tx, y)}{2}\right\}\right) + \varphi(d(A, B)), \end{aligned}$$

since φ is is strictly increasing, it follows that

$$\begin{aligned} d(Tx, Ty) \leq &\max\left\{d^*(x, y), d^*(x, Tx), d^*(y, Ty), \frac{d^*(x, Ty) + d^*(Tx, y)}{2}\right\} + d(A, B) \\ &- \max\Big\{\varphi(d^*(x, y) + d(A, B)) - \varphi(d(A, B)) \\ &\quad , \varphi(d^*(x, Tx) + d(A, B)) - \varphi(d(A, B)) \\ &\quad , \varphi(d^*(y, Ty) + d(A, B)) - \varphi(d(A, B)) \\ &\quad , \varphi(\frac{d^*(x, Ty) + d^*(Tx, y)}{2} + d(A, B)) - \varphi(d(A, B))\Big\}. \end{aligned} \tag{3}$$

Define $\varphi^* : [0, +\infty) \to [0, +\infty)$ by $\varphi^*(t) = \varphi(t + d(A, B)) - \varphi(d(A, B))$ for all $t \geq 0$. Since φ is a strictly increasing map, then φ^* is a strictly increasing map. Also $(I - \varphi^*)(t) = (I - \varphi)(t + d(A, B)) - (I - \varphi)(d(A, B))$, so as $I - \varphi$ is a strictly increasing map, $I - \varphi^*$ is a strictly increasing map, too. Therefore, from (3)*, we get*

$$\begin{aligned} d(Tx, Ty) = &\max\left\{d^*(x, y), d^*(x, Tx), d^*(y, Ty), \frac{d^*(x, Ty) + d^*(Tx, y)}{2}\right\} + d(A, B) \\ &- \max\left\{\varphi^*(d^*(x, y)), \varphi^*(d^*(x, Tx)), \varphi^*(d^*(y, Ty)), \varphi^*(\frac{d^*(x, Ty) + d^*(Tx, y)}{2})\right\} \\ = &\max\left\{d^*(x, y), d^*(x, Tx), d^*(y, Ty), \frac{d^*(x, Ty) + d^*(Tx, y)}{2}\right\} + d(A, B) \\ &- \varphi^*\left(\max\left\{d^*(x, y), d^*(x, Tx), d^*(y, Ty), \frac{d^*(x, Ty) + d^*(Tx, y)}{2}\right\}\right), \end{aligned}$$

hence

$$d^*(Tx, Ty) \leq (I - \varphi^*)\left(\max\left\{d^*(x, y), d^*(x, Tx), d^*(y, Ty), \frac{d^*(x, Ty) + d^*(Tx, y)}{2}\right\}\right).$$

Example 3.3. *Let $T : A \cup B \to A \cup B$ be a cyclic φ-contraction that is*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B)),$$

for all $x \in A$ and $y \in B$, then

$$d^*(Tx, Ty) \leq (I - \varphi^*)(d^*(x, y)),$$

for all $x \in A$ and $y \in B$. Therefore as $I - \varphi$ is a strictly increasing map, a cyclic φ-contraction map is cyclic quasi-φ-contraction map.

Example 3.4. *A cyclic contraction map in the sense of Suzuki et al. [15], is a cyclic quasi-φ-contraction with $\varphi(t) = (1 - c)t$ for $t \geq 0$ and $c \in [0, 1)$.*

Example 3.5. *A generalized cyclic quasi-contraction in Theorem 2.5, is cyclic quasi-φ-contraction with $\varphi(t) = (1-c)t$ for $t \geq 0$ and $c \in [0, 1)$.*

Example 3.6. *Let $X := \mathbb{R}$ with the usual metric. For $A = B = [0, 1]$, define $T : A \cup B \to A \cup B$ by $Tx := \frac{x}{1+x}$ and $\varphi(t) = \frac{t^2}{1+t}$ for $t \geq 0$. Note that $I - \varphi$ is a strictly increasing map, then from Example 2 of [3] T is a cyclic φ-contraction map, so from Example 3.3 it is a cyclic quasi-φ-contraction map. Suppose that for all $x \in A$ and $y \in B$ and some $c \in [0, 1)$, T obey in relation (1). Then we have*

$$\begin{aligned}
|Tx - T0| &= \frac{x}{1 + x} \\
&\leq c \max\left\{|x - 0|, |x - \frac{x}{1 + x}|, |0 - T0|, \frac{|x - T0| + |\frac{x}{1+x} - 0|}{2}\right\} + (1 - c)d(A, B) \\
&= c \max\left\{x, \frac{x^2}{1 + x}, 0, \frac{2x + x^2}{2(1 + x)}\right\} = cx,
\end{aligned}$$

for all $x \in A$. So $\frac{1}{1+x} \leq c$ for all $x \in (0, 1)$. Then $c \geq 1$ is a contradiction. Hence T is not a generalized cyclic quasi-contraction map.

Lemma 3.7. *Let A and B be nonempty subsets of the metric space (X, d) and let $T : A \cup B \to A \cup B$ be a cyclic quasi-φ-contraction. For $x_0 \in A \cup B$ define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then $d^*(x_n, x_{n+1}) \to 0$ as $n \to \infty$.*

Proof. From (2), for every $n \in \mathbb{N}$ we have

$$\begin{aligned}
d^*(x_n, x_{n+1}) =& d^*(Tx_{n-1}, Tx_n) \\
\leq& (I - \varphi)\left(\max\left\{d^*(x_{n-1}, x_n), d^*(x_n, x_{n+1}), \frac{d^*(x_{n-1}, x_{n+1}) + d^*(x_n, x_n)}{2}\right\}\right) \\
=& (I - \varphi)\left(\max\left\{d^*(x_{n-1}, x_n), d^*(x_n, x_{n+1}), \frac{d^*(x_{n-1}, x_{n+1}) - d(A, B)}{2}\right\}\right) \\
\leq& (I - \varphi)\left(\max\left\{d^*(x_{n-1}, x_n), d^*(x_n, x_{n+1}), \frac{d^*(x_{n-1}, x_n) + d^*(x_n, x_{n+1})}{2}\right\}\right) \\
=& (I - \varphi)\left(\max\left\{d^*(x_{n-1}, x_n), d^*(x_n, x_{n+1})\right\}\right).
\end{aligned} \tag{4}$$

Assume that for some $n_0 \in \mathbb{N}$,

$$\max\{d^*(x_{n_0-1}, x_{n_0}), d^*(x_{n_0}, x_{n_0+1})\} = d^*(x_{n_0}, x_{n_0+1}),$$

so by (4) we get $\varphi(d^*(x_{n_0}, x_{n_0+1})) = 0$. As φ is strictly increasing, we have $d^*(x_{n_0}, x_{n_0+1}) = 0$ and so

$$d^*(x_{n_0-1}, x_{n_0}) = d^*(x_{n_0}, x_{n_0+1}) = 0,$$

hence

$$\max\{d^*(x_{n_0-1}, x_{n_0}), d^*(x_{n_0}, x_{n_0+1})\} = d^*(x_{n_0-1}, x_{n_0}).$$

Thus, we may assume that for each $n \in \mathbb{N}$,

$$\max\{d^*(x_{n-1}, x_n), d^*(x_n, x_{n+1})\} = d^*(x_{n-1}, x_n).$$

Hence, from (4) for every $n \in \mathbb{N}$, we obtain

$$d^*(x_n, x_{n+1}) \leq (I - \varphi)(d^*(x_{n-1}, x_n)). \tag{5}$$

Let $d_n^* := d^*(x_{n-1}, x_n)$ for every $n \in \mathbb{N}$. From (5) for every $n \in \mathbb{N}$, we obtain $d_{n+1}^* \leq d_n^*$. So $\{d_n^*\}$ is decreasing. Also, $\{d_n^*\}$ is bounded below by 0, thus $\lim_{n\to\infty} d_n^* = t_0$ for some $t_0 \geq 0$. If $d_{n_0}^* = 0$ for some $n_0 \geq 1$, there is nothing to prove. So assume that $d_n^* > 0$ for each $n \in \mathbb{N}$. Since (5), we have

$$d_{n+1}^* \leq d_n^* - \varphi(d_n^*)$$

and hence

$$0 \leq \varphi(d_n^*) \leq d_n^* - d_{n+1}^*, \tag{6}$$

for each $n \geq 1$. Since φ is strictly increasing and $d_n^* \geq t_0 \geq 0$ for each $n \geq 1$, it follows from (6) that

$$0 = \lim_{n\to\infty} \varphi(d_n^*) \geq \varphi(t_0) \geq \varphi(0) \geq 0, \tag{7}$$

so $\varphi(t_0) = \varphi(0)$. As φ is strictly increasing, we get $t_0 = 0$. □

Lemma 3.8. *Let A and B be nonempty subsets of the metric space (X, d) such that (A, B) has the UC property. Let $T : A \cup B \to A \cup B$ be a cyclic quasi-φ-contraction map. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then $d(x_{2n}, x_{2n+2}) \to 0$ as $n \to \infty$.*

Proof. From Lemma 3.7, we get

$$d(x_{2n}, x_{2n+1}) \to d(A, B) \quad \text{and} \quad d(x_{2n+2}, x_{2n+1}) \to d(A, B),$$

as $n \to \infty$. Because (A, B) has the *UC* property, we get $d(x_{2n}, x_{2n+2}) \to 0$ as $n \to \infty$. □

Lemma 3.9. *Let A and B be nonempty subsets of a metric space (X, d) and let $T : A \cup B \to A \cup B$ be a cyclic quasi-φ-contraction map. Then*

(a) $\varphi(0) = 0$;

(b) $(I - \varphi)(t) \geq 0$ *for all* $t \geq 0$;

(c) for every $t > 0$ *we have* $\varphi(t) > 0$;

(d) for every $t > 0$ *we have* $(I - \varphi)(t) < t$;

(e) φ *and* $I - \varphi$ *are continuous.*

Proof. (a) follows from (7). (b) Since $I - \varphi$ is strictly increasing we get $(I - \varphi)(t) \geq (I - \varphi)(0) = 0$. (c) If $t > 0$ and $\varphi(t) = 0$ then $0 \leq \varphi(\frac{t}{2}) < \varphi(t) = 0$ leads to a contradiction. (d) It follows directly from (c). (e) Let $t_1 < t_2$. Since $I - \varphi$ is strictly increasing, we get $t_1 - \varphi(t_1) < t_2 - \varphi(t_2)$ so $\varphi(t_2) - \varphi(t_1) < t_2 - t_1$. Hence φ and $I - \varphi$ are continuous. □

Lemma 3.10. *Let A and B be nonempty subsets of the metric space (X, d) such that $d(A, B) = 0$. Let $T : A\cup B \to A\cup B$ be a cyclic quasi-φ-contraction map. For $x_0 \in A$ define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then for each $\epsilon > 0$ there exists a positive integer N_0 such that for all $m > n \geq N_0$*

$$d(x_{2m}, x_{2n+1}) < \epsilon.$$

Proof. Suppose the cotrary, then there exists $\epsilon_0 > 0$ such that for each $k \geq 1$, there is $m_k > n_k \geq k$ satisfying

$$d(x_{2m_k}, x_{2n_k+1}) \geq \epsilon_0 \tag{8}$$

and

$$d(x_{2(m_k-1)}, x_{2n_k+1}) < \epsilon_0. \tag{9}$$

It follows from (8), the triangle inequality and (9) that

$$\begin{aligned}\epsilon_0 &\leq d(x_{2m_k}, x_{2n_k+1})\\ &\leq d(x_{2m_k}, x_{2m_k-2}) + d(x_{2(m_k-1)}, x_{2n_k+1})\\ &\leq d(x_{2m_k}, x_{2m_k-2}) + \epsilon_0,\end{aligned}$$

letting $k \to \infty$, Lemma 3.8 implies

$$\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+1}) = \epsilon_0. \tag{10}$$

Applying the triangle inequality, we obtain

$$\begin{aligned}d(x_{2m_k}, x_{2n_k+1}) &\leq d(x_{2m_k}, x_{2m_k+1}) + d(x_{2m_k+1}, x_{2n_k+2}) + d(x_{2n_k+2}, x_{2n_k+1})\\ &\leq 2d(x_{2m_k}, x_{2m_k+1}) + d(x_{2m_k}, x_{2n_k+1}) + 2d(x_{2n_k+2}, x_{2n_k+1}),\end{aligned}$$

so from Lemma 3.7

$$\lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+1}) = \lim_{k\to\infty} d(x_{2m_k+1}, x_{2n_k+2}). \tag{11}$$

On the other hand from Lemma 3.8 and triangle inequality, we have

$$\begin{aligned}&\lim_{k\to\infty} \frac{d(x_{2m_k}, x_{2n_k+2}) + d(x_{2m_k+1}, x_{2n_k+1})}{2}\\ &\leq \lim_{k\to\infty} \frac{2d(x_{2m_k}, x_{2n_k+1}) + d(x_{2n_k+1}, x_{2n_k+2}) + d(x_{2m_k+1}, x_{2m_k})}{2}\\ &= \lim_{k\to\infty} d(x_{2m_k}, x_{2n_k+1}).\end{aligned} \tag{12}$$

Now, by using (2), we get

$$\begin{aligned}d(x_{2m_k+1}, x_{2n_k+2}) \leq \max\Big\{&d(x_{2m_k}, x_{2n_k+1}) - \varphi(d(x_{2m_k}, x_{2n_k+1})), d(x_{2m_k}, x_{2m_k+1})\\ &, d(x_{2n_k+1}, x_{2n_k+2}), \frac{d(x_{2m_k}, x_{2n_k+2}) + d(x_{2m_k+1}, x_{2n_k+1})}{2}\\ &- \varphi\left(\frac{d(x_{2m_k}, x_{2n_k+2}) + d(x_{2m_k+1}, x_{2n_k+1})}{2}\right)\Big\}\\ \leq \max\Big\{&d(x_{2m_k}, x_{2n_k+1}), d(x_{2m_k}, x_{2m_k+1}), d(x_{2n_k+1}, x_{2n_k+2})\\ &, \frac{d(x_{2m_k}, x_{2n_k+2}) + d(x_{2m_k+1}, x_{2n_k+1})}{2}\Big\}.\end{aligned} \tag{13}$$

Letting $k \to \infty$ in (13) and using (10), (11) and (12), since φ is continuous, we get

$$\epsilon_0 \le \epsilon_0 - \lim_{k\to\infty} \varphi(d(x_{2m_k}, x_{2n_k+1})) \le \epsilon_0$$

and hence

$$\lim_{k\to\infty} \varphi(d(x_{2m_k}, x_{2n_k+1})) = 0. \tag{14}$$

Since φ is strictly increasing, it follows from (8) and (14) that

$$\varphi(\epsilon_0) \le \lim_{k\to\infty} \varphi(d(x_{2m_k}, x_{2n_k+1})) = 0 < \varphi(\epsilon_0),$$

a contradiction. □

Theorem 3.11. *Let A and B be nonempty subsets of the metric space (X, d) such that A is complete, (A, B) has the UC property and (B, A) has the ultrametric property. Let $T : A \cup B \to A \cup B$ be a cyclic quasi-φ-contraction. Then for every $x_0 \in A$ the sequence $\{T^{2n}x_0\}$ converges to some best proximity point $x^* \in A$. Furthermore, every best proximity point of T in A is a fixed point of T^2.*

Proof. Take $x_0 \in A$ and consider the sequence $\{x_n\}$ given by $x_{n+1} := Tx_n$ for $n \ge 0$. First, we show that $\{x_{2n}\}$ is a Cauchy sequence. When $d(A, B) = 0$ the claim follows from Lemma 3.10. To prove the claim, it is enough to assume that $d(A, B) > 0$. From Lemma 3.7 and 3.8 we have

$$\lim_{n\to\infty} d^*(x_n, x_{n+1}) = 0 \quad \textit{and} \quad \lim_{n\to\infty} d(x_{2n}, x_{2n+2}) = 0.$$

Fix $\epsilon > 0$ such that $\epsilon < \min\{\epsilon_{(A,B)}, \epsilon_{(B,A)}\}$. $(I - \varphi)$ is strictly increasing and continuous, therefore there exists its inverse $(I - \varphi)^{-1}$, which is strictly increasing and since $(I - \varphi)(\epsilon) < \epsilon$

$$\epsilon = (I - \varphi)^{-1} ((I - \varphi)(\epsilon)) < (I - \varphi)^{-1}(\epsilon),$$

so $\epsilon' := (I - \varphi)^{-1}(\epsilon) - \epsilon > 0$. We choose $L \in \mathbb{N}$ satisfying

$$d^*(x_n, x_{n+1}) < \epsilon \quad \textit{and} \quad d(x_{2n}, x_{2n+2}) < \epsilon' \tag{15}$$

for all $n \ge L$. Fix $n \in \mathbb{N}$ with $n \ge L$. We shall show that

$$d^*(x_{2n+1}, x_{2p}) < \epsilon, \tag{16}$$

for all $p \ge n$. We assume that

$$d^*(x_{2n+1}, x_{2m}) < \epsilon, \tag{17}$$

holds for some $m \ge n$. Then since $d^*(x_{2m+1}, x_{2m}) < \epsilon$ and (B, A) has the ultrametric property, we obtain

$$d^*(x_{2n+1}, x_{2m+1}) < \epsilon \tag{18}$$

and since $d^*(x_{2n+1}, x_{2n+2}) < \epsilon$ and (A, B) has the ultrmetric property we get

$$d^*(x_{2n+2}, x_{2m}) < \epsilon. \tag{19}$$

Also, we have

$$\begin{aligned} d^*(x_{2n+2}, x_{2m+1}) \le &(I - \varphi)\Big(\max\Big\{d^*(x_{2n+1}, x_{2m}), d^*(x_{2n+1}, x_{2n+2}), d^*(x_{2m}, x_{2m+1}) \\ &, \frac{d^*(x_{2n+1}, x_{2m+1}) + d^*(x_{2n+2}, x_{2m})}{2}\Big\}\Big). \end{aligned}$$

Now, by relations (15), (17), (18) and (19) we obtain

$$d^*(x_{2n+2}, x_{2m+1}) < (I-\varphi)(\epsilon) < \epsilon. \tag{20}$$

Since $d^*(x_{2m+2}, x_{2m+1}) < \epsilon$ and (A, B) has the ultrmetric property we obtain

$$d^*(x_{2n+2}, x_{2m+2}) < \epsilon. \tag{21}$$

Hence, we have

$$\begin{aligned} d^*(x_{2n+1}, x_{2m+2}) \leq &(I-\varphi)\Big(\max\Big\{d^*(x_{2n}, x_{2m+1}), d^*(x_{2n}, x_{2n+1}), d^*(x_{2m+1}, x_{2m+2}) \\ &, \frac{d^*(x_{2n}, x_{2m+2}) + d^*(x_{2n+1}, x_{2m+1})}{2}\Big\}\Big) \\ \leq &(I-\varphi)\Big(\max\Big\{d(x_{2n}, x_{2n+2}) + d^*(x_{2n+2}, x_{2m+1}), d^*(x_{2n}, x_{2n+1}) \\ &, d^*(x_{2m+1}, x_{2m+2}) \\ &, \frac{d(x_{2n}, x_{2n+2}) + d^*(x_{2n+2}, x_{2m+2}) + d^*(x_{2n+1}, x_{2m+1})}{2}\Big\}\Big). \end{aligned}$$

Now, by relations (15), (18) and (20) and (21) we obtain

$$d^*(x_{2n+1}, x_{2m+2}) \leq (I-\varphi)\left(\max\left\{\epsilon' + \epsilon, \epsilon, \frac{\epsilon' + 2\epsilon}{2}\right\}\right),$$

where $\epsilon' = (I-\varphi)^{-1}(\epsilon) - \epsilon$, so we have

$$d^*(x_{2n+1}, x_{2m+2}) < (I-\varphi)(\epsilon' + \epsilon) = (I-\varphi)((I-\varphi)^{-1}(\epsilon) - \epsilon + \epsilon) = \epsilon.$$

By induction, we obtain (16) holds for all $p \geq n$ and so we get

$$\lim_{n\to\infty} \sup_{p\geq n} d^*(x_{2n+1}, x_{2p}) = 0 \quad or \quad \lim_{n\to\infty} \sup_{p\geq n} d(x_{2n+1}, x_{2p}) = d(A, B),$$

that by using the UC property of (A, B) and Lemma 2.4 imply $\{x_{2n}\}$ is a Cauchy sequence.

Hence, in both cases $d(A, B) = 0$ and $d(A, B) \neq 0$, we get the sequence $\{x_{2n}\}$ is Cauchy and so convergent to some $x^* \in A$. But we have

$$\begin{aligned} d^*(Tx^*, x_{2n}) \leq &(I-\varphi)\Big(\max\Big\{d^*(x^*, x_{2n-1}), d^*(x^*, Tx^*), d^*(x_{2n-1}, x_{2n}) \\ &, \frac{d^*(x^*, x_{2n}) + d^*(x_{2n-1}, Tx^*)}{2}\Big\}\Big) \\ \leq &(I-\varphi)\Big(\max\Big\{d^*(x^*, x_{2n-1}), d^*(x^*, Tx^*), d^*(x_{2n-1}, x_{2n}) \\ &, \frac{d^*(x^*, x_{2n}) + d(x_{2n-1}, x^*) + d^*(x^*, Tx^*)}{2}\Big\}\Big). \end{aligned}$$

Letting $n \to \infty$ and taking lim sup, because $I - \varphi$ is continuous, we obtain

$$d^*(x^*, Tx^*) \leq (I-\varphi)(d^*(x^*, Tx^*)),$$

so $\varphi(d^*(x^*, Tx^*)) = 0$ and from Lemma 3.9(c) we obtain $d^*(x^*, Tx^*) = 0$. Therefore $d(x^*, Tx^*) = d(A, B)$.

Furthermore, if z^* be an arbitrary best proximity point of T in A then we have

$$\begin{aligned}
d^*(T^2z^*, Tz^*) \leq& (I-\varphi)\left(\max\left\{d^*(Tz^*, z^*), d^*(Tz^*, T^2z^*), \frac{d^*(Tz^*, Tz^*) + d^*(z^*, T^2z^*)}{2}\right\}\right) \\
\leq& (I-\varphi)\left(\max\left\{d^*(Tz^*, z^*), d^*(Tz^*, T^2z^*), \frac{-d(A,B) + d^*(z^*, T^2z^*)}{2}\right\}\right) \\
\leq& (I-\varphi)\left(\max\left\{d^*(z^*, Tz^*), d^*(Tz^*, T^2z^*), \frac{d^*(z^*, Tz^*) + d^*(Tz^*, T^2z^*)}{2}\right\}\right) \\
=& (I-\varphi)\left(\max\left\{d^*(z^*, Tz^*), d^*(Tz^*, T^2z^*)\right\}\right) \\
=& (I-\varphi)\left(d^*(Tz^*, T^2z^*)\right).
\end{aligned}$$

since φ is strictly increasing, from Lemma 3.9(c) we obtain $d^*(T^2z^*, Tz^*) = 0$ and so $d(T^2z^*, Tz^*) = d(A, B)$. Because $d(z^*, Tz^*) = d(A, B)$ and (A, B) has the UC property, we get $T^2z^* = z^*$. □

Let $A_0 := \{x \in A : \quad d(x, y) = d(A, B) \quad for\ some\ y \in B\}$ and $B_0 := \{y \in B : \quad d(x, y) = d(A, B) \quad for\ some\ x \in A\}$. Exactly similar to Theorem 3.6 of [12], it can be proved that if (A_0, B_0) has the Pythagorean property [6] and (B, A) has the UC property, then the best proximity point of T in A is unique, which we omit to prove it here. The Example 3.7 of [12] shows that the Pythagorean property of the pair (A_0, B_0) is necessary to guarantee the uniqueness of best proximity of T. Also, it shows that Theorem 3.11 is stronger than Theorem 2 of [15].

Example 3.12. *Let $X := \mathbb{R}$ with the usual metric. For $A = [1, 2]$ and $B = [-2, -1]$, define $T : A \cup B \to A \cup B$ by*

$$T(x) = \begin{cases} -2 + \frac{1}{x} & \text{if } x \in A, \\ 2 + \frac{1}{x} & \text{if } x \in B. \end{cases}$$

If $\varphi(t) = \frac{t^2}{2+2t}$ for $t \geq 0$. Then for all $x \in A$ and $y \in B$, we have

$$\begin{aligned}
d^*(Tx, Ty) &= 2 + \frac{1}{y} - \frac{1}{x} \\
&= \frac{-2xy - x + y}{-xy} \\
&\leq \frac{-2xy - x + y}{x - y - 1} \\
&= \frac{(x-y-2) - 2(x-1)(y+1)}{x - y - 1} \\
&\leq \frac{(x-y-2) + \frac{(x-1-y-1)^2}{2}}{x - y - 1} \\
&= \frac{2(x-y-2) + (x-y-2)^2}{2 + 2(x-y-2)} \\
&= \frac{2d^*(x,y) + d^*(x,y)^2}{2 + 2d^*(x,y)} \\
&= (I-\varphi)(d^*(x,y)).
\end{aligned}$$

Hence T is a cyclic quasi-φ-contraction map. So, all conditions of Theorem 3.11 are satisfied and $x = 1$ is unique best proximity point T in A and for every $x_0 \in A$ the sequence $T^{2n}x_0$ converges to it as $n \to \infty$. Note that T is not a cyclic φ-contraction map, because

$$d(T(\frac{3}{2}), T(\frac{-3}{2})) = \frac{8}{3} > \frac{29}{24} = 3 - \frac{9}{8} + \frac{4}{6} = d(\frac{3}{2}, \frac{-3}{2}) - \varphi(d(\frac{3}{2}, \frac{-3}{2})) + \varphi(d(A, B)).$$

Corollary 3.13. *Let A and B be nonempty, closed and convex subsets of a uniformly convex Banach space X. Let T be a cyclic mapping on $A \cup B$ such that*

$$\|Tx - Ty\|^* \leq (I - \varphi)\left(\max\left\{\|x - y\|^*, \|x - Tx\|^*, \|y - Ty\|^*, \frac{\|x - Ty\|^* + \|Tx - y\|^*}{2}\right\}\right),$$

for all $x \in A$ and $y \in B$ where $c \in [0, 1)$ and $\|Tx - Ty\|^* := \|Tx - Ty\| - d(A, B)$. Then T has at least a best proximity point x^* in A that is a fixed point of T^2.

Note that when $d(A, B) = 0$, then the pairs (A, B) and (B, A) have the UC property, and (A_0, B_0) has the Pythagorean property. So as a result of Theorems 3.11 we get the following theorem that is the extention of Corollaries 2.3 and 2.10 in [7].

Theorem 3.14. *Let A and B be nonempty and closed subsets of a complete metric space (X, d). Let T be a cyclic mapping on $A \cup B$ such that*

$$d(Tx, Ty) \leq (I - \varphi)\left(\max\left\{d(x, y), d(x, Tx), d(y, Ty), \frac{d(x, Ty) + d(y, Tx)}{2}\right\}\right),$$

for all $x \in A$ and $y \in B$. Then T has a unique fixed point x^* in $A \cap B$ such that the Picard iteration $\{x_n\}$, defined by $x_{n+1} := Tx_n$ for each $n \geq 0$, converges to x^* for any starting point $x_0 \in A \cup B$.

Proof. It can be proved exactly like the proof of Lemma 3.7 $d(x_n, x_{n+1}) \to 0$, since

$$0 \leq d(A, B) = \inf\{d(a, b) : \ a \in A, \ b \in B\} \leq \inf\{d(x_n, x_{n+1}) : \ n \in \mathbb{N}\} = 0,$$

then $d(A, B) = 0$. So from Lemma 3.10, $\{x_n\}$ is a Cauchy sequence and thus there exists $x^* \in A \cup B$ such that $x_n \to x^*$. Now $\{x_{2n}\}$ is a sequence in A and $\{x_{2n+1}\}$ is a sequence in B and both converges to x^*. Since A and B are closed $x^* \in A \cap B$ and by the proof of Theorem 3.11 x^* is a fixed point of T. Since $d(A, B) = 0$, fixed point of T in A and so in $A \cap B$ is unique. □

From Theorem 3.14, we obtain the following common fixed point result which is the extention of Corollary 3.11 in [13], immediatelly.

Corollary 3.15. *Let (X, d) be a complete metric space and let $T : X \to X$ and $S : X \to X$ be two mappings satisfying*

$$d(Tx, Sy) \leq (I - \varphi)\left(\max\left\{d(x, y), d(x, Tx), d(y, Sy), \frac{d(x, Sy) + d(y, Tx)}{2}\right\}\right),$$

for all $x, y \in X$. Then T and S have a unique common fixed point in X.

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