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Numerical solutions of nonlinear quadratic integral equations of Urysohn type on the half-line by using rational Legendre spectral method

Radjai Abir[a], Remili Walid[a], Rahmoune Azedine[a,*]

[a]*Department of Mathematics, University Mohamed El Bachir El Ibrahimi of Bordj Bou Arreridj, El Anasser, 34030, Algeria*

Abstract. A numerical method for solving nonlinear quadratic integral equations of Urysohn type on the half-line is presented. This approach reduces the given equation to a systematic procedure by using a rational Legendre-collocation approximation (RLC). The rate of convergence, error analysis and stability of the RLC method are investigated. Moreover, several numerical examples are carried out to verify the accuracy and reliability of the proposed method.

1. Introduction

The class of nonlinear integral equations known as quadratic integral equations holds great importance due to their numerous significant applications in engineering and sciences. These equations naturally arise in various fields such as radiative transfer theory, kinetic theory of gases, the theory of neutron transport, and traffic theory. (see, e.g., [1–8] and reference therein). Although numerous studies have conducted in recent years to examine the existence and uniqueness of solutions for different types of these equations, limited attention has been given to those defined on unbounded intervals. For instance, Banaś and al. [9] studied the solvability of nonlinear quadratic integral equation of Hammerstein type on an unbounded interval in some Banach space, consisting of all real functions defined, bounded and continuous on $\mathbb{R}_+$. The authors in [10–16], investigated the existence of solutions for the Urysohn integral equation on an unbounded interval. In a related context, Karaoui et al. [17] examined the existence of solutions to nonlinear quadratic integral equations in the Banach space $L^p(\mathbb{R}_+)$. Furthermore, in the second part of their work, they provided a numerical method for solving nonlinear quadratic Volterra integral equations, but over a bounded interval only.

Note that analytically solving nonlinear integral equations on unbounded intervals is not a trivial task in general, thus numerical methods are required. Therefore, some efficient numerical algorithms have been developed by a few authors to solve such problems. For example, the Nyström method for convolution and non-convolution kernels was explored in [18], the finite-section approximation method was presented in [19], projection and multi-projection methods were discussed in [20, 21], Galerkin and

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* Corresponding author: Rahmoune Azedine
Email addresses: abir.radjai@univ-bba.dz (Radjai Abir), walid.remili@univ-bba.dz (Remili Walid), a.rahmoune@univ-bba.dz (Rahmoune Azedine)

multi-Galerkin methods based on Laguerre polynomials were investigated in [22], the Sinc-Nyström method based on single and double exponential transformations was proposed in [23], modified Legendre rational and exponential collocation methods were proposed in [24], and superconvergence results for non-linear Hammerstein integral equations on unbounded domain was discused in [25]. However, to the best of our knowledge, no numerical methods have yet been applied to quadratic integral equations on unbounded intervals.

In the present work we propose an efficient numerical method for the solution of nonlinear quadratic integral equations of Urysohn type, defined on the half-line, namely

$$u(x) = a(x) + f(x, u(x)) \int_0^\infty k(x, t, u(t))dt, \quad x \in [0, \infty), \tag{1}$$

where $k(x, t, .)$, $a(x)$ and $f(x, u(x))$ are given continuous functions and $u(x)$ is the unknown function.

We first derive the so-called rational Legendre functions that can be obtained by combining the classical Legendre polynomials with an appropriate mapping [24, 26], and then we apply the rational Legendre collocation method to solve the given equation.

In the next section we present certain properties of rational Legendre functions and in section 3 we discuss the existence of a unique solution to Eq. (1) in the weighted L^2 space, assuming some natural conditions. In section 4, the Legendre collocation method is presented for the solution of the Urysohn equation (1) in which the integral part is replaced by their operational matrix representations with collocation points. In addition, we discuss the convergence of the approximate solution to the exact solution and address the stability of the proposed method. In order to show the efficiency and demonstrate the accuracy and stability of the proposed method, some numerical examples are presented in section 5.

2. Orthogonal rational Legendre functions for the semi-infinite interval

In this section, we introduce rational Legendre functions and recall some basic properties. Even more, we present function approximations in some weighted L^2-space.

The well-known Legendre polynomials are orthogonal in the interval $I = [-1, 1]$ with respect to the uniform weight function. They can be determined with the help of the following recurrence formula [27]:

$$(n + 1)P_{n+1}(y) = (2n + 1)yP_n(y) - nP_{n-1}(y) \quad n \geq 1. \tag{2}$$

Besides

$$P_0(y) = 1, \quad P_1(y) = y, \quad P_n(1) = 1, \quad P_n(-1) = (-1)^n. \tag{3}$$

The set of Legendre polynomials forms an orthogonal system, namely,

$$\int_{-1}^1 P_n(y)P_m(y)dy = \frac{2}{2n+1}\delta_{n,m}, \tag{4}$$

where $\delta_{n,m}$,is the Kronecker delta function. Furthermore, for any function $U \in L^2(I)$, we write

$$U(y) = \sum_{j=0}^\infty c_jP_j(y) \text{ with } c_j = \frac{2j+1}{2}\int_{-1}^1 U(y)P_j(y)dy. \tag{5}$$

For a given positive integer N, let $\mathcal{P}_N$ denote the space of all algebraic polynomials of degree not exceeding N. we denote the collocation points by $\{\sigma_i^N\}_{i=0}^N$ which is the set of $(N + 1)$ Legendre-Gauss points, and by $\{\omega_i^N\}_{i=0}^N$ the corresponding weights. The associated Gauss-Legendre quadrature formula is defined by :

$$\int_{-1}^1 \phi(y)dy = \sum_{i=0}^N \phi(\sigma_i^N)\omega_i^N, \quad \forall\phi \in \mathcal{P}_{2N+1}. \tag{6}$$

Let us consider the following one to one invertible mapping between $x \in \mathbb{R}_+ = [0, \infty)$ and $y \in I$, with $s > 0$ of the form:

$$y = \eta_s(x) = \frac{x - s}{x + s}, \quad x = \varphi_s(y) = \frac{s(1 + y)}{1 - y}. \tag{7}$$

It is clear that

$$\frac{dy}{dx} = \frac{2s}{(x + s)^2}, \quad \frac{dx}{dy} = \frac{2s}{(1 - y)^2}, \tag{8}$$

where s is a positive scaling factor. The *rational Legendre functions* can be defined by

$$R_{s,n}(x) := P_n\left(\eta_s(x)\right), \quad n = 0, 1, 2, \ldots,. \tag{9}$$

They are orthogonal on the interval $\mathbb{R}_+$ with respect to the weight function

$$\rho_s(x) = \frac{dy}{dx} = \frac{2s}{(x + s)^2}, \tag{10}$$

equivalently

$$\int_0^\infty R_{s,n}(x) R_{s,m}(x) \rho_s(x) dx = \frac{2}{2n + 1} \delta_{n,m}. \tag{11}$$

Let us define

$$L^2_{\rho_s}(\mathbb{R}_+) = \left\{ u : \mathbb{R}_+ \to \mathbb{R} \mid u \text{ is measurable and } \|u\|_{\rho_s} < \infty \right\},$$

where

$$\|u\|_{\rho_s} = \int_0^\infty |u(x)|^2 \rho_s(x) dx,$$

is the norm induced by the inner product of the space $L^2_{\rho_s}(\mathbb{R}_+)$,

$$\langle u, v \rangle_{\rho_s} = \int_0^\infty u(x) v(x) \rho_s(x) dx. \tag{12}$$

It is not hard to show that $\{R_{s,j}\}_{j=0}^\infty$ forms a complete basis in $L^2_{\rho_s}(\mathbb{R}_+)$. For any function $u \in L^2_{\rho_s}(\mathbb{R}_+)$, the following expansion holds

$$u(x) = \sum_{j=0}^\infty \hat{u}_{s,j} R_{s,j}(x) \text{ with } \hat{u}_{s,j} = \frac{2j + 1}{2} \int_0^\infty u(x) R_{s,j}(x) \rho_s(x) dx. \tag{13}$$

In the sequel, $k(x, t, u(t))$ in Eq. (1) will be assumed, so that $k_s : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ defined by

$$k_s(x, t, u(t)) = \frac{k(x, t, u(t))}{\rho_s(t)},$$

is a bounded function. This allows us to rearrange Eq. (1) into the following form:

$$u(x) = a(x) + f(x, u(x)) \int_0^\infty k_s(x, t, u(t)) \rho_s(t) dt, \quad x \in [0, \infty). \tag{14}$$

2.1. Rational Lagrange interpolation

First, we define $\mathbb{X}_N^s$ the finite dimensional approximation subspace spanned for a given positive integer N by the set of rational Legendre functions as

$$\mathbb{X}_N^s := \{v \mid v(x) = \phi(\eta_s(x)), \forall \phi \in \mathcal{P}_N\}. \tag{15}$$

The set of rational Legendre-Gauss $\{\zeta_{N,i}^s\}_{i=0}^N$, which is defined as

$$\zeta_{N,i}^s = \varphi_s(\sigma_i^N), \quad 0 \leqslant i \leqslant N. \tag{16}$$

Applying a mapping (7) to the quadrature formula (6) leads to the rational Legendre-Gauss quadrature:

$$\int_0^\infty v(x)\rho_s(x)dx = \sum_{i=0}^N v(\zeta_{N,i}^s)\omega_i^N, \quad \forall v \in \mathbb{X}_{2N+1}^s. \tag{17}$$

The rational Lagrange basis functions are defined by the following formula:

$$L_{i,s}^N(x) = \prod_{j=0, j\neq i}^N \frac{\eta_s(x) - \eta_s(\zeta_{N,j}^s)}{\eta_s(\zeta_{N,i}^s) - \eta_s(\zeta_{N,j}^s)}, \quad 0 \le i \le N, \tag{18}$$

then it is clear that the functions $L_{i,s}^N(x)$ satisfy

$$L_{i,s}^N(\zeta_{N,j}^s) = \delta_{i,j}. \tag{19}$$

For any $u \in C(\mathbb{R}^+)$, we can define the Lagrange interpolating polynomial $\mathcal{I}_N^s u \in \mathbb{X}_N^s$, satisfying:

$$\mathcal{I}_N^s u \in \mathbb{X}_N^s \text{ such that } \mathcal{I}_N^s u(\zeta_{N,j}^s) = u(\zeta_{N,j}^s), \quad 0 \leqslant j \leqslant N, \tag{20}$$

which can be expanded as

$$\mathcal{I}_N^s u(x) = \sum_{i=0}^N u(\zeta_{N,i}^s) L_{i,s}^N(x). \tag{21}$$

The following estimate quoted from lemma 5.5 of [28].

Lemma 2.1. *Let $\{L_{i,s}^N(x)\}_{i=1}^N$ be the $N-th$ rational Lagrange interpolation functions associated with the rational Legendre collocation points. Then*

$$\|\mathcal{I}_N^s\|_\infty := \sup_{x\in\mathbb{R}^+} \sum_{i=0}^N |L_{i,s}^N(x)| = \mathcal{O}(N^{1/2}). \tag{22}$$

In order to describe the approximation errors, we introduce new differential operators as follows:

$$D_x u = g_s(x)\frac{du}{dx}, \qquad g_s(x) := \frac{dx}{dy}, \tag{23}$$

and an induction argument leads to

$$D_x^m u = g_s(x)\frac{d}{dx}\left(g_s(x)\frac{d}{dx}\left(\cdots\left(g_s(x)\frac{du}{dx}\right)\cdots\right)\right) = \partial_y^m U_s, \quad m = 0, 1, \ldots, \tag{24}$$

where

$$u(x) = u(\varphi_s(y)) := U_s(y). \tag{25}$$

To prove error estimates for the above scheme, we begin by defining the following weighted Hilbert space with some useful lemmas about rational Lagrange interpolation based on the rational Legendre-Gauss points. For a nonnegative integer m, define

$$H^m_{\rho_s}(\mathbb{R}_+) = \{u \mid D^r_x u \in L^2_{\rho_s}(\mathbb{R}_+) \ \ 0 \le r \le m\},$$

related to the following semi-norm and the norm:

$$|u|^s_m = \|D^m_x u\|_{L^2_{\rho_s}(\mathbb{R}_+)},\ \|u\|^s_m = \left(\sum_{r=0}^{m} \|D^r_x u\|^2_{L^2_{\rho_s}(\mathbb{R}_+)}\right)^{1/2}.$$

Also, it is convenient to introduce the semi-norms

$$|u|^{m;N}_{\rho_s} := |u|_{H^{m;N}_{\rho_s}(\mathbb{R}_+)} = \left(\sum_{r=\min(m,N+1)}^{m} \|D^r_x u\|^2_{L^2_{\rho_s}(\mathbb{R}_+)}\right)^{1/2}.$$

In the following, we prove the below lemma, which estimates the error between the approximate and exact solutions.

Lemma 2.2. *Assume that $u \in H^m_{\rho_s}(\mathbb{R}_+)$ we have*

$$\|u - \mathcal{I}^s_N u\|_{L^2_{\rho_s}(\mathbb{R}_+)} \le cN^{-m}|u|^{m;N}_{\rho_s}, \tag{26}$$

$$\|u - \mathcal{I}^s_N u\|_{\infty} \le cN^{1/2-m}|u|^{m;N}_{\rho_s}, \tag{27}$$

where c is a positive constant independent of N and u.

Proof. Let I_N be the Lagrange interpolation operator associated with the Legendre collocation points, we have for the weighted L^2-norm

$$\|u - \mathcal{I}^s_N u\|^2_{L^2_{\rho_s}(\mathbb{R}_+)} = \int_0^\infty |u(x) - \mathcal{I}^s_N u(x)|^2 \rho_s(x)dx = \int_{-1}^{1} |U_s(y) - I_N U_s(y)|^2 dy = \|U_s - I_N U_s\|^2_{L^2(I)}. \tag{28}$$

Next for the infinity norm, we consider

$$\|u - \mathcal{I}^s_N u\|_\infty = \sup_{x\in\mathbb{R}_+} |u(x) - \mathcal{I}^s_N u(x)| = \sup_{y\in I} |U_s(y) - I_N U_s(y)| = \|U_s - I_N U_s\|_\infty. \tag{29}$$

According to Lemma 1 of [29], it is mentioned that for any $U_s \in H^m(I)$ and $m \ge 0$,

$$\|U_s - I_N U_s\|_{L^2(I)} \leqslant cN^{-m}|U_s|^{m;N}, \tag{30}$$

$$\|U_s - I_N U_s\|_\infty \leqslant cN^{1/2-m}|U_s|^{m;N}. \tag{31}$$

From (23), (24) and (25), we have

$$\|\partial^m_y U_s\|^2_{L^2(I)} = \int_{-1}^{1} |\partial^m_y U_s(y)|^2 dy = \int_0^\infty |D^m_x u(x)|^2 \rho_s(x)dx = \|D^m_x u\|^2_{L^2_{\rho_s}(\mathbb{R}_+)}. \tag{32}$$

This implies $|U_s|^{m;N} = |u|^{m;N}_{\rho_s}$. Hence, we obtain the desired estimate, i.e.,

$$\|u - \mathcal{I}^s_N u\|_{L^2_{\rho_s}(\mathbb{R}_+)} \le cN^{-m}|u|^{m;N}_{\rho_s}, \tag{33}$$

$$\|u - \mathcal{I}^s_N u\|_\infty \le cN^{1/2-m}|u|^{m;N}_{\rho_s}. \tag{34}$$

This completes the proof.

3. Existence and uniqueness

In order to consider the Eq. (1) in the weighted space $L^2_{\rho_s}(\mathbb{R}_+)$, we will adopt the following assumptions:

C1. There exist a continuous and bounded function $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|k(x,t,y) - k(x,t,z)| \leq g(x,t)|y - z|, \text{ for all } x,t \in \mathbb{R}_+.$$

C2. For every $x \geq 0$,

$$\int_0^\infty |g(x,t)|^2 (t+s)^2 dt \leq M_1 < \infty \text{ for all } s > 0.$$

C3. For every $x \geq 0$,

$$\int_0^\infty |k(x,t,u(t))| dt \leq M_2 < \infty.$$

From now onwards, we make the following assumptions on the nonlinear function $f(.,u(.))$:

C4. $f(x,z)$ is continuous on $\mathbb{R}_+ \times \mathbb{R}$ and bounded:

$$\sup_{x \in \mathbb{R}_+} |f(x,u(x))| \leq M_3.$$

C5. The functions $f(x,u(x))$ is Lipschitz continuous in u i.e., for any $u,v \in L^2_{\rho_s}(\mathbb{R}_+)$, there exist constants L such that

$$|f(x,u(x)) - f(x,v(x))| \leq L|u(x) - v(x)|.$$

Next, we define the operator $\mathcal{T}$ on $L^2_{\rho_s}(\mathbb{R}_+)$ by

$$\mathcal{T}(u) := F(u)\mathcal{K}(u) + a, \text{ where } \mathcal{K}(u)(x) = \int_0^\infty k(x,t,u(t))dt \text{ and } F(u)(x) = f(x,u(x)),$$

so that Eq. (1) can be written as

$$\mathcal{T}(u) = u. \tag{35}$$

Theorem 3.1. *Assume $\mathcal{K} : L^2_{\rho_s}(\mathbb{R}_+) \to L^2_{\rho_s}(\mathbb{R}_+)$ is bounded, if the following condition holds:*

$$\left(LM_2 + M_3 \sqrt{\frac{M_1}{s}} \right) < 1. \tag{36}$$

Then Eq. (1) *has a unique solution in $L^2_{\rho_s}(\mathbb{R}_+)$ for all $a \in L^2_{\rho_s}(\mathbb{R}_+)$.*

Proof. For all $u,v \in L^2_{\rho_s}(\mathbb{R}_+)$, we have

$$\begin{aligned} |\mathcal{T}(u)(x) - \mathcal{T}(v)(x)| &= |F(u)(x)\mathcal{K}(u)(x) - F(v)(x)\mathcal{K}(v)(x)| \\ &= |F(u)(x)\mathcal{K}(u)(x) - F(v)(x)\mathcal{K}(u)(x) + F(v)(x)\mathcal{K}(u)(x) - F(v)(x)\mathcal{K}(v)(x)| \\ &\leq |F(u)(x) - F(v)(x)||\mathcal{K}(u)(x)| + |F(v)(x)||\mathcal{K}(u)(x) - \mathcal{K}(v)(x)|. \end{aligned} \tag{37}$$

From Lipschitz continuity of f and the assumptions C1, C3 and C4, we get

$$\begin{aligned}|\mathcal{T}(u)(x)-\mathcal{T}(v)(x)| &\le L|u(x)-v(x)||\mathcal{K}(u)(x)|+M_3|\mathcal{K}(u)(x)-\mathcal{K}(v)(x)|\\ &\le LM_2|u(x)-v(x)|+M_3\int_0^\infty |g(x,t)||u(t)-v(t)|dt.\end{aligned} \tag{38}$$

By applying Cauchy-Schwarz inequality and using assumption C2, we obtain

$$\int_0^\infty |g(x,t)||u(t)-v(t)|dt \le \left(\int_0^\infty |g(x,t)|^2\frac{1}{\rho_s(t)}dt\right)^{1/2}\left(\int_0^\infty |u(t)-v(t)|^2\rho_s(t)dt\right)^{1/2} \le \sqrt{\frac{M_1}{2s}}\|u-v\|_{L^2_{\rho_s}(\mathbb{R}_+)}. \tag{39}$$

This implies

$$|\mathcal{T}(u)(x)-\mathcal{T}(v)(x)| \le LM_2|u(x)-v(x)|+M_3\sqrt{\frac{M_1}{2s}}\|u-v\|_{L^2_{\rho_s}(\mathbb{R}_+)}.$$

Hence

$$\|\mathcal{T}(u)-\mathcal{T}(v)\|_{L^2_{\rho_s}(\mathbb{R}_+)} \le \left(LM_2+M_3\sqrt{\frac{M_1}{s}}\right)\|u-v\|_{L^2_{\rho_s}(\mathbb{R}_+)}. \tag{40}$$

It follows that for $\left(LM_2+M_3\sqrt{\frac{M_1}{s}}\right)<1$, $\mathcal{T}$ is a contraction operator, so that it has a unique fixed point and that fixed point is the solution of of Eq. (1).

4. Rational Legendre collocation method

An approximate solution of Eq. (1) may be obtained by simply collocating, that is forcing Eq. (14) to be exact at the rational Legendre-Gauss points $\{\zeta^s_{N,j}\}_{j=0}^N$, namely

$$u(\zeta^s_{N,j}) = a(\zeta^s_{N,j}) + f(\zeta^s_{N,j}, u(\zeta^s_{N,j}))\int_0^\infty k_s(\zeta^s_{N,j}, t, u(t))\rho_s(t)dt, \quad j=0\cdots N. \tag{41}$$

By applying the rational Legendre-Gauss quadrature formula (17) to the above equation, we obtain

$$u(\zeta^s_{N,j}) = a(\zeta^s_{N,j}) + f(\zeta^s_{N,j}, u(\zeta^s_{N,j}))\sum_{i=0}^N k_s(\zeta^s_{N,j}, \zeta^s_{N,i}, u(\zeta^s_{N,i}))\omega_i^N, \quad j=0\cdots N, \tag{42}$$

Using $u^N_{s,j}$, $0\le j\le N$, to approximate the function value $u(\zeta^s_{N,j})$, and use

$$u^N_s(x) = \sum_{j=0}^N u^N_{s,j}L^N_{j,s}(x), \tag{43}$$

to approximate the function $u(x)$, namely

$$u(\zeta^s_{N,j}) \sim u^N_{s,j}, \quad u(x)\sim u^N_s(x). \tag{44}$$

Then, the discrete spectral Legendre-collocation method for solving Eq. (1) leads to the following fully discrete problem

$$u^N_{s,j} = a(\zeta^s_{N,j}) + f(\zeta^s_{N,j}, u^N_{s,j})\sum_{i=0}^N k_s(\zeta^s_{N,j}, \zeta^s_{N,i}, u^N_{s,i})\omega_i^N, \quad j=0\cdots N, \tag{45}$$

which is a nonlinear system of the form

$$\mathrm{u} = \mathbf{H}(\mathrm{u}), \quad \mathbf{H}(\mathrm{u}) = \mathbf{A} + \mathbf{F}(\mathrm{u})\mathbf{M}(\mathrm{u})\mathbf{W}, \tag{46}$$

where $\mathbf{M}, \mathbf{W}, \mathbf{A}$ and $\mathbf{F}$ are given by:

$$\mathbf{M}(\mathrm{u}) = (k_s(\zeta^s_{N,j}, \zeta^s_{N,i}, u^N_{s,j}))_{0\le i,j\le N},\ \mathbf{W} = \mathrm{diag}((\omega^N_i)_{0\le i\le N}),\ \mathbf{A} = (a(\zeta^s_{N,j}))_{0\le j\le N},\ \mathbf{F}(\mathrm{u}) = \mathrm{diag}((f(\zeta^s_{N,j}, u^N_{s,j}))_{0\le j\le N}),$$

and the unknown is the vector $\mathrm{u} \equiv [u^N_{s,0}, u^N_{s,1}, \dots, u^N_{s,N}]^T$.

To achieve a highly accurate numerical solution of (46), we would need to apply the following iterative process

$$\mathrm{u}^{(k)} = \mathbf{H}(\mathrm{u}^{(k-1)}), \tag{47}$$

with the initial value $\mathrm{u}^{(0)} = \mathbf{A}$.

4.1. Error analysis

In this section we provided error analysis for the proposed method to indicate its exponential rate of convergence, provided that a and f are bounded sufficiently smooth functions. In order to do that, the above assumptions are taken into account.

Theorem 4.1. *Let u be the exact solution to Eq.* (1) *and u^N_s be the approximate solution obtained by using the spectral-collocation scheme* (45). *For $m \ge 1$, assume that $D^r_t k_s(x, t, I^s_N u(t)) \in L^2_{\rho_s}(\mathbb{R}_+)$ for $\min(m, N+1) \le r \le m$, If $u \in H^m_{\rho_s}(\mathbb{R}_+)$, then*

$$\|u - u^N_s\|_{L^2_{\rho_s}(\mathbb{R}_+)} = O(N^{-m}), \ \ \|u - u^N_s\|_\infty = O(N^{\frac{1}{2}-m}). \tag{48}$$

Proof. Let the quadratic Urysohn integral equation

$$u(x) = a(x) + f(x, u(x)) \int_0^\infty k_s(x, t, u(t))\rho_s(t)dt, \tag{49}$$

while using the approximate solution, we have

$$u^N_s(x) = \mathcal{I}^s_N a(x) + \mathcal{I}^s_N f(x, \mathcal{I}^s_N u(x)) \int_0^\infty \mathcal{I}^s_{N,N} k_s(x, t, \mathcal{I}^s_N u(t))\rho_s(t)dt. \tag{50}$$

Subtracting (50) from (49), we get the error equation

$$\begin{aligned} u(x) - u^N_s(x) &= a(x) - \mathcal{I}^s_N a(x) + f(x, u(x)) \int_0^\infty (k_s(x, t, u(t)) - k_s(x, t, \mathcal{I}^s_N u(t)))\rho_s(t)dt \\ &+ (f(x, u(x)) - f(x, \mathcal{I}^s_N u(x))) \int_0^\infty k_s(x, t, \mathcal{I}^s_N u(t))\rho_s(t)dt \\ &+ f(x, \mathcal{I}^s_N u(x)) \int_0^\infty (k_s(x, t, \mathcal{I}^s_N u(t)) - \mathcal{I}^s_{N,N} k_s(x, t, \mathcal{I}^s_N u(t)))\rho_s(t)dt \\ &+ (f(x, \mathcal{I}^s_N u(x)) - \mathcal{I}^s_N f(x, \mathcal{I}^s_N u(x))) \int_0^\infty \mathcal{I}^s_{N,N} k_s(x, t, \mathcal{I}^s_N u(t))\rho_s(t)dt \\ &= J_0(x) + J_1(x) + J_2(x) + J_3(x) + J_4(x), \end{aligned} \tag{51}$$

where

$$J_0(x) = a(x) - \mathcal{I}_N^s a(x), \tag{52}$$

$$J_1(x) = f(x,u(x)) \int_0^\infty (k_s(x,t,u(t)) - k_s(x,t,\mathcal{I}_N^s u(t)))\rho_s(t)dt, \tag{53}$$

$$J_2(x) = (f(x,u(x)) - f(x,\mathcal{I}_N^s u(x))) \int_0^\infty k_s(x,t,\mathcal{I}_N^s u(t))\rho_s(t)dt, \tag{54}$$

$$J_3(x) = f(x,\mathcal{I}_N^s u(x)) \int_0^\infty (k_s(x,t,\mathcal{I}_N^s u(t)) - \mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u(t)))\rho_s(t)dt, \tag{55}$$

$$J_4(x) = (f(x,\mathcal{I}_N^s u(x)) - \mathcal{I}_N^s f(x,\mathcal{I}_N^s u(x))) \int_0^\infty \mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u(t))\rho_s(t)dt. \tag{56}$$

By the triangle inequality, we have

$$\|u - u_s^N\|_{L^2_{\rho_s}(\mathbb{R}_+)} \leq \sum_{k=0}^{4} \|J_k\|_{L^2_{\rho_s}(\mathbb{R}_+)}, \quad \|u - u_s^N\|_\infty \leq \sum_{k=0}^{4} \|J_k\|_\infty. \tag{57}$$

It follows immediately from Lemma 2.2 that

$$\|J_0\|_{L^2_{\rho_s}(\mathbb{R}_+)} \leq cN^{-m}|a|_{\rho_s}^{m;N}, \;\; \|J_0\|_\infty \leq cN^{1/2-m}|a|_{\rho_s}^{m;N}. \tag{58}$$

On the other hand, by assumptions C1 and C4, we have

$$\begin{aligned}
|J_1(x)| &= \left| f(x,u(x)) \int_0^\infty (k_s(x,t,u(t)) - k_s(x,t,\mathcal{I}_N^s u(t)))\rho_s(t)dt \right| \\
&\leq M_3 \int_0^\infty |g(x,t)||u(t) - \mathcal{I}_N^s u(t)|dt.
\end{aligned}$$

Using Cauchy Schwarz inequality as in (39), we get

$$\|J_1\|_{L^2_{\rho_s}(\mathbb{R}_+)} \leq M_3 \sqrt{\frac{M_1}{2s}} \|u - \mathcal{I}_N^s u\|_{L^2_{\rho_s}(\mathbb{R}_+)} \left(\int_0^\infty \rho_s(x)dx \right)^{1/2} = M_3 \sqrt{\frac{M_1}{s}} \|u - \mathcal{I}_N^s u\|_{L^2_{\rho_s}(\mathbb{R}_+)}. \tag{59}$$

Next for the infinity norm, by using assumption C2 we obtain

$$\begin{aligned}
|J_1(x)| &\leq M_3 \|u - \mathcal{I}_N^s u\|_\infty \int_0^\infty |g(x,t)|dt \\
&\leq M_3 \|u - \mathcal{I}_N^s u\|_\infty \left(\int_0^\infty |g(x,t)|^2 \frac{1}{\rho_s(t)} dt \right)^{1/2} \left(\int_0^\infty \rho_s(t)dt \right)^{1/2} \\
&\leq M_3 \sqrt{\frac{M_1}{s}} \|u - \mathcal{I}_N^s u\|_\infty.
\end{aligned} \tag{60}$$

Then, according to Lemma 2.2, it follows that:

$$\|J_1\|_{L^2_{\rho_s}(\mathbb{R}_+)} \leq C_1 N^{-m}|u|_{\rho_s}^{m;N}, \;\; \|J_1\|_\infty \leq C_1 N^{1/2-m}|u|_{\rho_s}^{m;N}. \tag{61}$$

Using assumptions C1, C3 and invoking (39), we can write

$$\begin{aligned}
\int_0^\infty |k(x,t,\mathcal{I}_N^s u(t))|dt &\leq \int_0^\infty |k(x,t,\mathcal{I}_N^s u(t)) - k(x,t,u(t)|dt + \int_0^\infty |k(x,t,u(t)|dt \\
&\leq \int_0^\infty |g(x,t)||u(t) - \mathcal{I}_N^s u(t)|dt + M_2 \leq \sqrt{\frac{M_1}{2s}} \|u - \mathcal{I}_N^s u\|_{L^2_{\rho_s}(\mathbb{R}_+)} + M_2,
\end{aligned}$$

so that from C5, we get

$$
\begin{aligned}
|J_2(x)| &\leq |f(x,u(x)-f(x,\mathcal{I}_N^s u(x))| \int_0^\infty |k_s(x,t,\mathcal{I}_N^s u(t))|\rho_s(t)dt \\
&\leq L|u(x)-\mathcal{I}_N^s u(x)| \int_0^\infty |k(x,t,\mathcal{I}_N^s u(t))|dt \\
&\leq L\left(M_2+\sqrt{\frac{M_1}{2s}}\|u-\mathcal{I}_N^s u\|_{L^2_{\rho_s}(\mathbb{R}_+)}\right)|u(x)-\mathcal{I}_N^s u(x)|.
\end{aligned}
$$

Hence by Lemma 2.2, we have

$$
\|J_2\|_{L^2_{\rho_s}(\mathbb{R}_+)} \leq C_2 N^{-m}|u|^{m;N}_{\rho_s}, \ \ \|J_2\|_\infty \leq C_2 N^{1/2-m}|u|^{m;N}_{\rho_s}. \tag{62}
$$

Also, we have

$$
|J_3(x)| \leq |f(x,\mathcal{I}_N^s u(x))| \int_0^\infty |e^s_{N,N}(x,t)|\rho_s(t)dt.
$$

where $e^s_{N,N}(x,t)=k_s(x,t,\mathcal{I}_N^s u(t))-\mathcal{I}^s_{N,N}k_s(x,t,\mathcal{I}_N^s u(t))$. By using assumptions C4 and C5 we get

$$
|f(x,\mathcal{I}_N^s u(x))| \leq |f(x,u(x))|+|f(x,\mathcal{I}_N^s u(x))-f(x,u(x))| \leq M_3+L\|u-\mathcal{I}_N^s u\|_\infty.
$$

Thus

$$
|J_3(x)| \leq (M_3+L\|u-\mathcal{I}_N^s u\|_\infty)\int_0^\infty |e^s_{N,N}(x,t)|\rho_s(t)dt.
$$

Finally, by using Cauchy-Schwarz inequality we write

$$
|J_3(x)| \leq \sqrt{2}(M_3+L\|u-\mathcal{I}_N^s u\|_\infty)\|e^s_{N,N}(x,.)\|_{L^2_{\rho_s}(\mathbb{R}_+)}.
$$

Now, from (26) we have

$$
\|e^s_{N,N}(x,.)\|_{L^2_{\rho_s}(\mathbb{R}_+)} \leq cN^{-m}|k_s(x,.,\mathcal{I}_N^s u(.))|^{m;N}_{\rho_s}.
$$

Hence, we get

$$
\begin{aligned}
\|J_3\|_{L^2_{\rho_s}(\mathbb{R}_+)} &\leq \sqrt{2}(M_3+L\|u-\mathcal{I}_N^s u\|_\infty)cN^{-m}|k_s(x,.,\mathcal{I}_N^s u(.))|^{m;N}_{\rho_s}\left(\int_0^\infty \rho_s(x)dx\right)^{1/2} \\
&\leq 2(M_3+L\|u-\mathcal{I}_N^s u\|_\infty)cN^{-m}|k_s(x,.,\mathcal{I}_N^s u(.))|^{m;N}_{\rho_s}.
\end{aligned}
$$

For the infinity norm, we use (27) to get

$$
\begin{aligned}
\|J_3\|_\infty &\leq (M_3+L\|u-\mathcal{I}_N^s u\|_\infty)\|e^s_{N,N}(x,.)\|_\infty \int_0^\infty \rho_s(t)dt \\
&\leq 2(M_3+L\|u-\mathcal{I}_N^s u\|_\infty)cN^{1/2-m}|k_s(x,.,\mathcal{I}_N^s u(.))|^{m;N}_{\rho_s}.
\end{aligned}
$$

Hence

$$
\|J_3\|_{L^2_{\rho_s}(\mathbb{R}_+)} \leq C_3 N^{-m}|k_s(x,.,\mathcal{I}_N^s u(.))|^{m;N}_{\rho_s}, \tag{63}
$$

$$
\|J_3\|_\infty \leq C_3 N^{1/2-m}|k_s(x,.,\mathcal{I}_N^s u(.))|^{m;N}_{\rho_s}. \tag{64}
$$

To estimate J_4 using Lemma 2.1, we obtain the following expression:

$$\begin{aligned} |J_4(x)| &\le |f(x, \mathcal{I}_N^s u(x)) - \mathcal{I}_N^s f(x, \mathcal{I}_N^s u(x))| \int_0^\infty |\mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u(t))| \rho_s(t)dt \\ &\le |f(x, \mathcal{I}_N^s u(x)) - \mathcal{I}_N^s f(x, \mathcal{I}_N^s u(x))| \sup_{x,t\in\mathbb{R}_+} |\mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u(t))| \int_0^\infty \rho_s(t)dt \\ &\le 2|f(x, \mathcal{I}_N^s u(x)) - \mathcal{I}_N^s f(x, \mathcal{I}_N^s u(x))| \|\mathcal{I}_N^s\|_\infty^2 \sup_{0\le i,j\le N} |k_s(\zeta_{N,j}^s, \zeta_{N,i}^s, \mathcal{I}_N^s u(\zeta_{N,i}^s))|. \end{aligned} \tag{65}$$

Therefore, by using Lemma 2.2, we can derive the following result:

$$\|J_4\|_{L^2_{\rho_s}(\mathbb{R}_+)} \le C_4 N^{-m} \|\mathcal{I}_N^s\|_\infty^2 \sup_{0\le i,j\le N} |k_s(\zeta_{N,j}^s, \zeta_{N,i}^s, \mathcal{I}_N^s u(\zeta_{N,i}^s))| |f|_{\rho_s}^{m;N}, \tag{66}$$

$$\|J_4\|_\infty \le C_4 N^{1/2-m} \|\mathcal{I}_N^s\|_\infty^2 \sup_{0\le i,j\le N} |k_s(\zeta_{N,j}^s, \zeta_{N,i}^s, \mathcal{I}_N^s u(\zeta_{N,i}^s))| |f|_{\rho_s}^{m;N}. \tag{67}$$

Finally, the statement of the theorem follows from the triangle inequality.

4.2. Stability of the RLC method

In this subsection, we focus on discussing the stability of the RLC method for solving integral equation (1). To analyze the stability, we introduce a function $a_\varepsilon(x) = a(x) + \varepsilon$. This addition allows us to use the impulse from theorem 3.1, which guarantees the existence of a solution u_ε. The solution u_ε is obtained by satisfying the following equation:

$$u_\varepsilon(x) = a_\varepsilon(x) + f(x, u_\varepsilon(x)) \int_0^\infty k(x,t,u_\varepsilon(t))dt, \quad x \in [0,\infty). \tag{68}$$

Applying the RLC method to the perturbed problem (68) we can obtain the corresponding scheme

$$u_{s,\varepsilon}^N(x) = \mathcal{I}_N^s a_\varepsilon(x) + \mathcal{I}_N^s f(x, \mathcal{I}_N^s u_\varepsilon(x)) \int_0^\infty \mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u_\varepsilon(t)) \rho_s(t)dt. \tag{69}$$

Definition 4.2. *The numerical method is said to be stable if*

$$\|u_s^N - u_{s,\varepsilon}^N\| \le C\|a - a_\varepsilon\|, \tag{70}$$

where $u_{s,\varepsilon}^N$ is the numerical solution of the perturbed problem.

In the following theorem, we focus solely on demonstrating the maximum norm stability of the RLC method because the same argument can be applied in the $L^2_{\rho_s}$−norm. For notational convenience, let us denote

$$\gamma_s = LM_2 + M_3\sqrt{\frac{M_1}{s}}.$$

Theorem 4.3. *Let $\{u_s^N\}$ and $\{u_{s,\varepsilon}^N\}$ be two sequences of numerical solutions obtained by the RLC schemes* (50) *and* (69)*, respectively. Further assume*

$$\sup_{x,t\in\mathbb{R}_+} g(x,t)(t+s)^2 < \infty. \tag{71}$$

If $\gamma_s < 1$, then we have

$$\|u_s^N - u_{s,\varepsilon}^N\|_\infty \le \|\mathcal{I}_N^s\|_\infty \left(1 + \frac{\tau_s}{1-\gamma_s}\|\mathcal{I}_N^s\|_\infty^3\right)\varepsilon, \tag{72}$$

where

$$\tau_s = 2L \sup_{0\le i,j\le N} |k_s(\zeta_{N,j}^s, \zeta_{N,i}^s, \mathcal{I}_N^s u_\varepsilon(\zeta_{N,i}^s))| + \frac{1}{s} \sup_{0\le j\le N} |f(\zeta_{N,j}^s, \mathcal{I}_N^s u_\varepsilon(\zeta_{N,j}^s))| \sup_{x,t\in\mathbb{R}_+} g(x,t)(t+s)^2.$$

Proof. Note that $\|a - a_\varepsilon\|_\infty = \varepsilon$. By (50) and (69), we have

$$u_s^N(x) - u_{s,\varepsilon}^N(x) = \mathcal{I}_N^s a(x) - \mathcal{I}_N^s a_\varepsilon(x) + (\mathcal{I}_N^s f(x, \mathcal{I}_N^s u(x)) - \mathcal{I}_N^s f(x, \mathcal{I}_N^s u_\varepsilon(x))) \int_0^\infty \mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u_\varepsilon(t))\rho_s(t)dt + \mathcal{I}_N^s f(x, \mathcal{I}_N^s u_\varepsilon(x)) \int_0^\infty (\mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u(t)) - \mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u_\varepsilon(t)))\rho_s(t)dt..$$

Let us denote

$$J_0^\varepsilon(x) = \mathcal{I}_N^s a(x) - \mathcal{I}_N^s a_\varepsilon(x), \tag{73}$$

$$J_1^\varepsilon(x) = (\mathcal{I}_N^s f(x, \mathcal{I}_N^s u(x)) - \mathcal{I}_N^s f(x, \mathcal{I}_N^s u_\varepsilon(x))) \int_0^\infty \mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u_\varepsilon(t))\rho_s(t)dt, \tag{74}$$

$$J_2^\varepsilon(x) = \mathcal{I}_N^s f(x, \mathcal{I}_N^s u_\varepsilon(x)) \int_0^\infty (\mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u(t)) - \mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u_\varepsilon(t)))\rho_s(t)dt. \tag{75}$$

where

$$u_s^N(x) - u_{s,\varepsilon}^N(x) = J_0^\varepsilon(x) + J_1^\varepsilon(x) + J_2^\varepsilon(x).$$

For J_0^ε, it is easy to get

$$\|J_0^\varepsilon\|_\infty \le \|\mathcal{I}_N^s\|_\infty \|a - a_\varepsilon\|_\infty \le \|\mathcal{I}_N^s\|_\infty \varepsilon. \tag{76}$$

Before starting the estimation of J_1^ε and J_2^ε, we need to estimate $u - u_\varepsilon$ in the infinity norm, where u is the exact solution of equation (1) and u_ε is the exact solution of the perturbed problem (69), under the above assumptions as follows.

$$|u(x) - u_\varepsilon(x)| \le LM_2|u(x) - u_\varepsilon(x)| + M_3 \int_0^\infty |g(x,t)||u(t) - u_\varepsilon(t)|dt + |a(x) - a_\varepsilon(x)|.$$

By applying Cauchy-Schwarz inequality, we get

$$\|u - u_\varepsilon\|_\infty \le \gamma_s \|u - u_\varepsilon\|_\infty + \|a - a_\varepsilon\|_\infty. \tag{77}$$

This implies

$$\|u - u_\varepsilon\|_\infty \le \frac{\|a - a_\varepsilon\|_\infty}{(1-\gamma_s)} \le \frac{\varepsilon}{(1-\gamma_s)}. \tag{78}$$

To estimate J_1^ε, we can write

$$\begin{aligned} |J_1^\varepsilon(x)| &\le |\mathcal{I}_N^s f(x, \mathcal{I}_N^s u(x)) - \mathcal{I}_N^s f(x, \mathcal{I}_N^s u_\varepsilon(x))| \int_0^\infty |\mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u_\varepsilon(t))|\rho_s(t)dt \\ &\le |\mathcal{I}_N^s f(x, \mathcal{I}_N^s u(x)) - \mathcal{I}_N^s f(x, \mathcal{I}_N^s u_\varepsilon(x))| \sup_{x,t\in\mathbb{R}_+} |\mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u_\varepsilon(t))| \int_0^\infty \rho_s(t)dt \\ &\le 2\|\mathcal{I}_N^s\|_\infty^2 \sup_{0\le i,j\le N} |k_s(\zeta_{N,j}^s, \zeta_{N,i}^s, \mathcal{I}_N^s u_\varepsilon(\zeta_{N,i}^s))| |\mathcal{I}_N^s(f(x, \mathcal{I}_N^s u(x)) - f(x, \mathcal{I}_N^s u_\varepsilon(x)))|. \end{aligned}$$

Using the assumption C5, we obtain

$$\|J_1^\varepsilon\|_\infty \le 2L\|\mathcal{I}_N^s\|_\infty^4 \sup_{0\le i,j\le N} |k_s(\zeta_{N,j}^s, \zeta_{N,i}^s, \mathcal{I}_N^s u_\varepsilon(\zeta_{N,i}^s))| \|u - u_\varepsilon\|_\infty. \tag{79}$$

Hence, by (78), we get

$$\|J_1^{\varepsilon}\|_{\infty} \leq 2L\|\mathcal{I}_N^s\|_{\infty}^4 \sup_{0\leq i,j\leq N} |k_s(\zeta_{N,j}^s, \zeta_{N,i}^s, \mathcal{I}_N^s u_{\varepsilon}(\zeta_{N,i}^s))| \frac{\varepsilon}{(1-\gamma_s)}. \tag{80}$$

In order to estimate J_2^{ε}, we use assumption C1 and C4, which allows us to derive that

$$\begin{aligned}
|J_2^{\varepsilon}(x)| &\leq |\mathcal{I}_N^s f(x, \mathcal{I}_N^s u_{\varepsilon}(x))| \int_0^{\infty} |\mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u(t)) - \mathcal{I}_{N,N}^s k_s(x,t,\mathcal{I}_N^s u_{\varepsilon}(t))| \rho_s(t)dt \\
&\leq \|\mathcal{I}_N^s\|_{\infty} \sup_{0\leq j\leq N} |f(\zeta_{N,j}^s, \mathcal{I}_N^s u_{\varepsilon}(\zeta_{N,j}^s))| \int_0^{\infty} |\mathcal{I}_{N,N}^s (k_s(x,t,\mathcal{I}_N^s u(t)) - k_s(x,t,\mathcal{I}_N^s u_{\varepsilon}(t))| \rho_s(t)dt \\
&\leq \|\mathcal{I}_N^s\|_{\infty}^3 \sup_{0\leq j\leq N} |f(\zeta_{N,j}^s, \mathcal{I}_N^s u_{\varepsilon}(\zeta_{N,j}^s))| \sup_{x,t\in\mathbb{R}_+} |k_s(x,t,\mathcal{I}_N^s u(t)) - k_s(x,t,\mathcal{I}_N^s u_{\varepsilon}(t))| \int_0^{\infty} \rho_s(t)dt \\
&\leq \frac{1}{s}\|\mathcal{I}_N^s\|_{\infty}^3 \sup_{0\leq j\leq N} |f(\zeta_{N,j}^s, \mathcal{I}_N^s u_{\varepsilon}(\zeta_{N,j}^s))| \sup_{x,t\in\mathbb{R}_+} g(x,t)(t+s)^2 |\mathcal{I}_N^s(u(t) - u_{\varepsilon}(t))| \\
&\leq \frac{1}{s}\|\mathcal{I}_N^s\|_{\infty}^4 \sup_{0\leq j\leq N} |f(\zeta_{N,j}^s, \mathcal{I}_N^s u_{\varepsilon}(\zeta_{N,j}^s))| \sup_{x,t\in\mathbb{R}_+} g(x,t)(t+s)^2 \|u - u_{\varepsilon}\|_{\infty}.
\end{aligned} \tag{81}$$

Then, we have

$$\|J_2^{\varepsilon}\|_{\infty} \leq \frac{1}{s}\|\mathcal{I}_N^s\|_{\infty}^4 \sup_{0\leq j\leq N} |f(\zeta_{N,j}^s, \mathcal{I}_N^s u_{\varepsilon}(\zeta_{N,j}^s))| \sup_{x,t\in\mathbb{R}_+} g(x,t)(t+s)^2 \|u - u_{\varepsilon}\|_{\infty}, \tag{82}$$

Hence, using (78), we get

$$\|J_2^{\varepsilon}\|_{\infty} \leq \frac{1}{s}\|\mathcal{I}_N^s\|_{\infty}^4 \sup_{0\leq j\leq N} |f(\zeta_{N,j}^s, \mathcal{I}_N^s u_{\varepsilon}(\zeta_{N,j}^s))| \sup_{x,t\in\mathbb{R}_+} g(x,t)(t+s)^2 \frac{\varepsilon}{(1-\gamma_s)}. \tag{83}$$

Finally, by using the triangle inequality, we obtain the desired result.

5. Illustrative examples

In this section, we provide numerical examples to illustrate the practical application of our theoretical results. All computations were carried out using Matlab. For the subsequent part of our analysis, we investigate the stability of the system at specific points by considering various values of the perturbation parameter ε. This examination allows us to assess the system's robustness and sensitivity to perturbations. Furthermore, we introduce the notation $e_s^N = u - u_s^N$ to represent the error between the exact solution u and the numerical solution u_s^N obtained using our proposed method.

Example 5.1. Let us consider the quadratic Urysohn integral equation

$$u(x) = \frac{x+2}{x+3}e^{-x} + u(x)\int_0^{+\infty} e^{-t(x+1)}u^2(t)dt, \quad x \in [0,\infty). \tag{84}$$

In order to confirm the effectiveness of our method, we present this example with the following known smooth exact solution: $u(x) = e^{-x}$. This choice of exact solution allows us to assess the accuracy and reliability of our approach in solving the quadratic Urysohn integral equation. To evaluate the performance of our approach, we conducted a series of experiments using the RLC method described above. The obtained results are summarized in Table 1. We can observe that they are in good accordance with the theoretical analysis provided by Theorem 4.1.

Table 1: Comparison of errors for Example 5.1.

N	$s=1$		$s=2$		$s=3$	
	$\|\|e_s^N\|\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|\|e_s^N\|\|_\infty$	$\|\|e_s^N\|\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|\|e_s^N\|\|_\infty$	$\|\|e_s^N\|\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|\|e_s^N\|\|_\infty$
4	1.01e−02	3.35e−01	5.06e−03	6.33e−02	5.81e−03	1.54e−01
8	9.99e−04	3.95e−02	3.60e−04	1.19e−02	1.40e−04	4.05e−03
16	2.13e−05	7.90e−04	2.87e−06	8.66e−05	9.64e−07	1.40e−05
32	6.00e−08	2.24e−06	2.26e−09	7.44e−08	2.02e−10	1.13e−09
64	5.04e−12	8.56e−11	2.32e−14	7.16e−13	1.60e−14	2.13e−13

Example 5.2. [10] Let us consider the following quadratic Urysohn integral equation:

$$u(x) = xe^{-4x^2} + \arctan(x + u(x)) \int_0^{+\infty} e^{-t(x+1)} u^2(t)dt, \quad x \in [0, \infty). \tag{85}$$

In Table 2, we present the numerical errors obtained by computing the $L^2_{\rho_s}$ norm and the infinity norm of the difference between u_s^{256} and u_s^N using the RLC scheme for various $s-$parameter. These results indicate that the spectral accuracy is obtained for this problem. Furthermore, Table 3 evaluates u_s^N at various points using the RLC method with $s = 3/2$. Additionally, we plot the numerical solution for $n = 128$ and $s = 3/2$ (refer to Figure 1).

Table 2: Comparison of errors for Example 5.2 with $E_s^N = u_s^{256} - u_s^N$.

N	$s=3/2$		$s=2$		$s=3$	
	$\|\|E_s^N\|\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|\|E_s^N\|\|_\infty$	$\|\|E_s^N\|\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|\|E_s^N\|\|_\infty$	$\|\|E_s^N\|\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|\|E_s^N\|\|_\infty$
4	3.80e−02	6.63e−02	4.40e−02	6.71e−02	6.18e−02	1.80e−01
8	8.84e−03	3.07e−02	8.76e−03	2.52e−02	1.57e−02	5.97e−02
16	3.28e−04	5.52e−04	3.73e−04	6.17e−04	6.51e−04	1.17e−03
32	4.10e−07	2.20e−06	5.46e−07	1.29e−06	1.13e−06	7.09e−06
64	1.72e−12	9.21e−12	8.08e−12	5.89e−12	3.18e−11	6.92e−12
128	8.11e−15	2.22e−15	6.81e−14	6.22e−15	2.71e−13	7.70e−15

Table 3: Some values of $u_s^N(x)$ at selected points for Example 5.2

N	x			
	0.5	5	10	15
4	0.1	0.00	0.0	0.0
8	0.1	0.00	0.00	0.00
16	0.191	0.00	0.001	0.000
32	0.19115	0.0040215	0.001356	0.00057
64	0.19115470283	0.0040215733	0.00135659880	0.000574038487
128	0.1911547028388	0.004021573357	0.0013565988032	0.00057403848735

Example 5.3. [30] Let us consider the following quadratic Urysohn integral equation:

$$u(x) = \frac{x}{x^2+4} + u(x)^4 \int_0^{+\infty} \frac{u(t)}{(1+x+t)^2(1+u^2(t))} dt, \quad x \in [0, \infty). \tag{86}$$

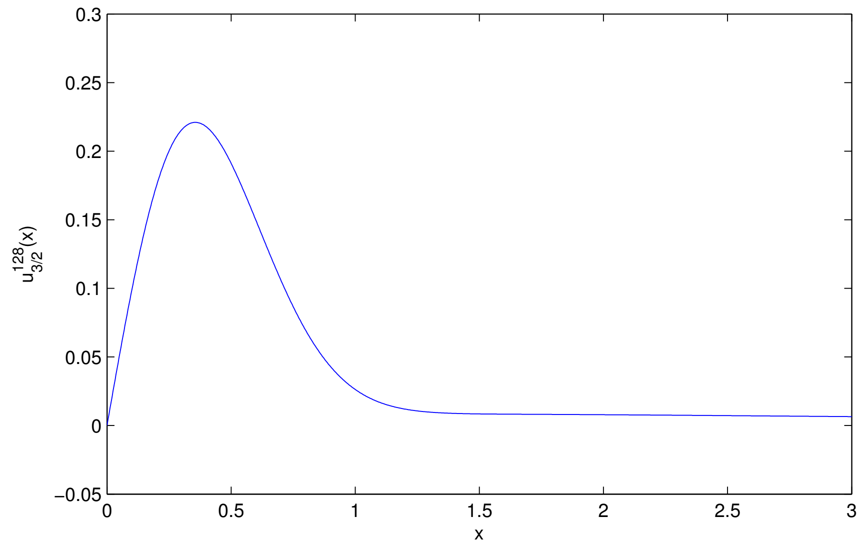

Figure 1: Numerical solution of Example 5.2 with $N = 128$ and $s = 3/2$

The numerical errors for the $L^2_{\rho_s}$ norm and the infinity norm are presented in Table 4. The analysis of the obtained results also demonstrates that the RLC method yields excellent results with all choices of s. To provide additional clarity, we further examine the values of u^N_s at various points using the RLC method with $s = 2$ in Table 5. Moreover, Figure 2 represents the corresponding numerical solution plotted over the interval $[0, 30]$.

Table 4: Comparison of errors for Example 5.3 with $E^N_s = u^{128}_s - u^N_s$.

	$s = 1$		$s = 2$		$s = 3$	
N	$\|\|E^N_s\|\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|\|E^N_s\|\|_\infty$	$\|\|E^N_s\|\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|\|E^N_s\|\|_\infty$	$\|\|E^N_s\|\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|\|E^N_s\|\|_\infty$
4	1.13e−02	3.02e−02	2.35e−02	7.13e−02	2.06e−02	9.21e−02
8	5.58e−04	1.55e−03	1.03e−03	1.31e−03	2.03e−03	1.10e−02
16	3.87e−06	6.29e−06	9.40e−06	1.33e−05	2.46e−05	6.53e−05
32	4.02e−10	1.75e−09	1.96e−09	1.14e−08	1.04e−08	7.67e−08
64	7.40e−16	1.44e−15	6.80e−16	1.22e−15	3.59e−15	3.80e−14

Example 5.4. [12] Let us consider the following quadratic Urysohn integral equation:

$$u(x) = xe^{-x} + \frac{\sqrt{u^2(x) + 1}}{x + 1} \int_0^{+\infty} e^{-(x+t+1)} \sqrt{1 + |u(t)|}dt, \quad x \in [0, \infty). \tag{87}$$

Table 6 presents the $L^2_{\rho_s}$ and infinity errors, indicating the discrepancy between u^{128}_s and u^N_s obtained using the RLC scheme with various s. The results reveal that the numerical errors for $s = 3$ and $s = 4$ exhibit remarkable rapprochement, outperforming those obtained for $s = 2$. Consequently, we perform an evaluation of u^N_s at specific points using the RLC method with $s = 3$, as outlined in Table 7. Furthermore, we provide the absolute values of the RLC coefficients and illustrate the numerical solution in Figure 3.

Table 5: Some values of $u_s^N(x)$ at selected points for Example 5.3

	x			
N	0.5	5	10	15
4	0.4	0.19	0.09	0.06
8	0.40	0.19	0.09	0.06
16	0.4043	0.1923	0.099	0.06637
32	0.40436119	0.19234757	0.099011074518	0.066371815
64	0.404361192947411	0.19234757172204	0.09901107471898	0.066371815933000

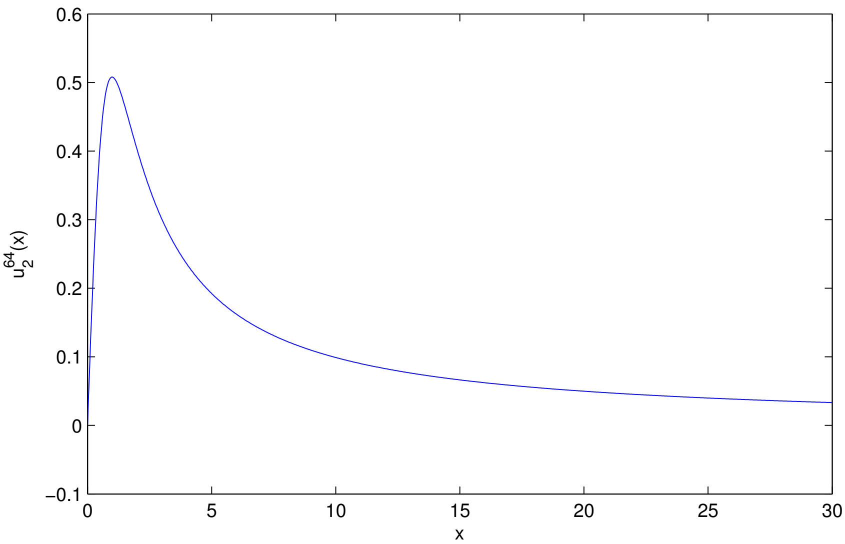

Figure 2: Numerical solution of Example 5.3 with $N = 64$ and $s = 2$

Table 6: Comparison of errors for Example 5.4 with $E_s^N = u_s^{128} - u_s^N$.

	$s = 2$		$s = 3$		$s = 4$	
N	$\|\|E_s^N\|\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|\|E_s^N\|\|_\infty$	$\|\|E_s^N\|\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|\|E_s^N\|\|_\infty$	$\|\|E_s^N\|\|_{L^2_{\rho_s}(\mathbb{R}_+)}$	$\|\|E_s^N\|\|_\infty$
4	2.26e−02	3.11e−02	1.99e−02	5.62e−02	1.64e−02	5.71e−02
8	1.27e−03	3.17e−03	1.31e−03	1.97e−03	7.88e−04	3.15e−03
16	2.68e−05	6.15e−05	9.47e−06	2.00e−05	3.53e−06	7.65e−06
32	2.42e−08	7.78e−08	3.17e−09	7.16e−09	5.81e−10	2.11e−09
64	4.47e−13	1.56e−12	1.65e−14	3.14e−14	1.01e−15	3.42e−14

5.1. *Stability results*

In order to demonstrate the stability of the examples, we investigate the effect of perturbation ε on the non-linear system of algebraic equations (41). We specifically focus on the input perturbation $(\mathbf{A} + \varepsilon)$ and observe that the output of the system undergoes minimal changes. The stability of Example 5.2 is shown in Table 8 for different values of $\varepsilon = 10^{-2}, 10^{-3}$, and 10^{-4}. The same principle is applied to the third example, as presented in Table 10. Remarkably, we observe that the approximate solutions exhibit negligible variation across various values of perturbation ε. To provide further insight, we have included the relative errors for Examples 2 and 3 obtained through the RLC method in Tables 9 and 11 for $\varepsilon = 10^{-1}, 10^{-2}, 10^{-3}$, and 10^{-4},

Table 7: Some values of $u_s^N(x)$ at selected points for Example 5.4

	x			
N	0.5	5	10	15
8	0.501	0.034	0.000	0.000
16	0.5018	0.03418	0.00045	0.00000
32	0.501879827	0.034183019	0.0004558	0.00000459
64	0.5018798271510	0.034183019457366	0.00045581118021	0.000004596928060
128	0.50187982715105	0.0341830194573660	0.000455811180218	0.0000045969280602

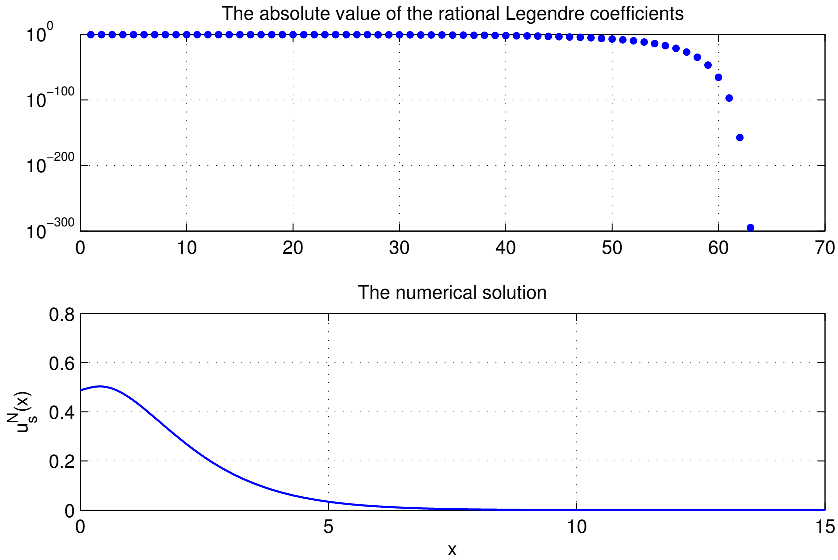

Figure 3: Numerical results of RLC-scheme for Example 5.4 with $N = 64$ and $s = 3$

respectively. Also we denote

$$R^{\varepsilon}_{\rho_s} = \frac{\|u_s^N - u_{s,\varepsilon}^N\|_{L^2_{\rho_s}(\mathbb{R}_+)}}{\|u_s^N\|_{L^2_{\rho_s}(\mathbb{R}_+)}}, \quad R^{\varepsilon}_{\infty} = \frac{\|u_s^N - u_{s,\varepsilon}^N\|_{\infty}}{\|u_s^N\|_{\infty}}.$$

Table 8: Stability results of Example 5.2 with $s = 3/2$

x	u_s^N	$u_s^N(\varepsilon = 10^{-2})$	$u_s^N(\varepsilon = 10^{-3})$	$u_s^N(\varepsilon = 10^{-4})$
0.5	0.1911547	0.2022581	0.1922591	0.1912651
5	0.0040216	0.0146030	0.0050771	0.0041271
10	0.0013566	0.0115974	0.0023793	0.0014588
20	0.0002872	0.0103658	0.0012944	0.0003880
40	0.0000449	0.0100682	0.0010469	0.0001451
80	0.0000061	0.0100131	0.0010066	0.0001061

Table 9: The relative errors for Example 5.2 with $s = 3/2$ and $N = 128$.

ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$R^{\varepsilon}_{\rho_s}$	1.19e−00	1.14e−01	1.13e−02	1.13e−03
R^{ε}_{∞}	5.44e−01	5.13e−02	5.10e−03	5.10e−04

Table 10: Stability results of Example 5.3 with $s = 2$ and $N = 64$.

x	u^N_s	$u^N_s(\varepsilon = 10^{-2})$	$u^N_s(\varepsilon = 10^{-3})$	$u^N_s(\varepsilon = 10^{-4})$
10	0.0990111	0.0990013	0.0990101	0.0990110
20	0.0498753	0.0498741	0.0498752	0.0498753
30	0.0332963	0.0332960	0.0332963	0.0332963
40	0.0249844	0.0249842	0.0249844	0.0249844
50	0.0199920	0.0199919	0.0199920	0.0199920
60	0.0166620	0.0166620	0.0166620	0.0166620
70	0.0142828	0.0142828	0.0142828	0.0142828
80	0.0124980	0.0124980	0.0124980	0.0124980

Table 11: The relative errors for Example 5.3 with $s = 2$ and $N = 64$.

ε	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$R^{\varepsilon}_{\rho_s}$	3.30e−01	3.24e−02	3.24e−03	3.24e−04
R^{ε}_{∞}	2.29e−01	2.17e−02	2.16e−03	2.16e−04

6. Conclusion

In this study, an efficient approach based on rational Legendre functions basis is described for the numerical solution of nonlinear quadratic Urysohn integral equations on the half-line. The error analysis and stability of the method are theoretically investigated. Numerical examples are given to support the theoretical findings. The results show that the rational Legendre approach is very accurate and stable. Furthermore, it is worth noting that the proposed method can be extended to solving similar types of equations as well.

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