Filomat 38:5 (2024), 1513–1529
https://doi.org/10.2298/FIL2405513K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generating functions for divisor sums and totative sums arising from combinatorial Simsek numbers

Irem Kucukoglu[a]

[a] *Department of Engineering Fundamental Sciences, Faculty of Engineering Alanya Alaaddin Keykubat University, TR-07425 Antalya, Turkey*

Abstract. The main objective of this paper is to introduce and investigate new number families derived from finite sums running over divisors and totatives and containing higher powers of binomial coefficients. Especially, by making decomposition on the generating functions for a kind of combinatorial number families recently introduced by Simsek [29], we also construct generating functions for the newly introduced number families. For symbolic computation of the newly introduced number families and their generating functions, we also give computational implementations in the Wolfram language. By these implementations, some tables of both these number families and their generating functions have been presented for some arbitrarily chosen special cases. Additionally, we provide some applications regarding the Thacker's (totient) function. In particular, by making summation on all totatives of a positive integer, we investigate some special finite sums containing both the Thacker's (totient) function and higher powers of binomial coefficients. By this investigation, some of the problems regarding these finite sums have been partially answered accompanied by some remarks. Furthermore, we propose an open problem regarding a potential relation between one of these number families and a formula involving the Möbius function. Finally, the paper have been concluded by providing an overview on the results of this paper and their potential usage areas, and by making suggestions regarding future studies able to be made.

1. Introduction and Preliminaries

The families of combinatorial numbers and polynomials are both important tools frequently used in almost all areas of mathematics, physics, computational sciences, cryptology and engineering. Therefore, it is very important to be able to classify numbers and polynomials for determining in which areas they can be used. There are many studies in the literature that serve this purpose. Especially, in recent years, Simsek [26–31] has carried out very important studies for the classification of combinatorial numbers and he indexed these numbers according to the order of their description. The starting point for the classification, conducted by Simsek, was the introduction of the combinatorial numbers $y_1(m, n; \lambda)$ by the following formula:

$$y_1(m, n; \lambda) = \frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j} j^m \lambda^j,$$

2020 *Mathematics Subject Classification*. Primary: 05A15, 05A19, 11A25, 11B65, 11B83, Secondary: 11A05, 11L20.

Keywords. Binomial coefficients; Computational implementations; Divisor sums; Generalized Franel numbers; Generating function; Special numbers; Combinatorial Simsek numbers; Number-theoretic (arithmetical) functions; Möbius function; Euler's totient function; Thacker's (totient) function; Totatives.

Received: 21 July 2023; Accepted: 30 August 2023

Communicated by Dragan S. Djordjević

This paper is dedicated to Professor Yilmaz Simsek on the occasion of his 60th birthday.

Email address: irem.kucukoglu@alanya.edu.tr (Irem Kucukoglu)

and the generating functions for these numbers were constructed by Simsek [28] as follows:

$$\frac{(\lambda e^t+1)^n}{n!}=\sum_{m=0}^{\infty} y_1(m,n;\lambda)\frac{t^m}{m!}, \tag{1}$$

(see, for details, [28]; and also [26, 29]).

In [28], which is considered to be the beginning of the aforementioned classification, Simsek also proposed an open problem associated with the recurrence relation of a number family that is very closely related to the numbers $y_1(m,n;\lambda)$. Lately Xu [39] and Goubi [11] succeeded in answering the aforementioned problem. What matters for the literature is that the numbers $y_1(m,n;\lambda)$ were called as *Simsek numbers* by Goubi in his papers [11] and [12].

Just after defining the numbers $y_1(m,n;\lambda)$, Simsek [26–31] also introduced other combinatorial numbers with different indices according to the order in which they were defined. Therefore, in this study the numbers $y_1(m,n;\lambda)$ will be referred as *combinatorial Simsek numbers of the first kind* because of its index.

Among the other studies conducted by Simsek [26–31] for the classification of the combinatorial numbers, the focus of the present study is [29] in which Simsek introduced the combinatorial numbers $y_6(m,n;\lambda,r)$ as follows, for $\lambda\in\mathbb{R}$ (or $\mathbb{C}$), $n,r\in\mathbb{N}=\{1,2,\dots\}$ and $m\in\mathbb{N}\cup\{0\}$:

$$\begin{aligned} F_{y_6}(t,n;\lambda,r) &:= \frac{1}{n!}\,{}_rF_{r-1}\left[\begin{matrix}-n,-n,\dots,-n\\ 1,1,\dots,1\end{matrix};(-1)^r\lambda e^t\right] \qquad (2)\\ &= \sum_{m=0}^{\infty} y_6(m,n;\lambda,r)\frac{t^m}{m!}, \qquad (3)\end{aligned}$$

in which ${}_rF_{r-1}$ denotes the generalized hypergeometric function defined by

$${}_pF_q\left[\begin{matrix}\alpha_1,\dots,\alpha_p\\ \beta_1,\dots,\beta_q\end{matrix};z\right]=\sum_{m=0}^{\infty}\left(\frac{\prod_{j=1}^{p}(\alpha_j)_m}{\prod_{j=1}^{q}(\beta_j)_m}\right)\frac{z^m}{m!},$$

where $(\alpha)_m$ denotes the Pochhammer symbol defined by $(\alpha)_m=\alpha(\alpha+1)\dots(\alpha+m-1)$ with $(\alpha)_0=1$, such that the above series converges for all z if $p<q+1$, and for $|z|<1$ if $p=q+1$. For this series one can assumed that all parameters have real or complex values, except for the β_j, $j=1,2,\dots,q$ none of which is equal to zero or to a negative integer (See, for details, [29, 35]).

By using hypergeometric series techniques in (2), the function $F_{y_6}(t,n;\lambda,r)$ can also be written as follows:

$$F_{y_6}(t,n;\lambda,r)=\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}^r\lambda^j e^{tj}, \tag{4}$$

(*cf.* [29, p. 1329])

Conspicuously, the combination of (3) and (4) implies that the numbers $y_6(m,n;\lambda,r)$ can be expressed explicitly by the following finite sum:

$$y_6(m,n;\lambda,r)=\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}^r j^m\lambda^j, \tag{5}$$

where $\lambda\in\mathbb{R}$ (or $\mathbb{C}$), $n,r\in\mathbb{N}=\{1,2,\dots\}$ and $m\in\mathbb{N}\cup\{0\}$ (*cf.* [29, p. 1347]).

Note that the numbers $y_6(m,n;\lambda,r)$ will be referred as *combinatorial Simsek numbers of the sixth kind* in this study because of its index.

The most obvious relationship between the first kind and sixth kind combinatorial Simsek numbers is as follows:

$$y_6(m,n;\lambda,1)=y_1(m,n;\lambda) \tag{6}$$

(*cf.* [28, 29]).

It should also be noted here that the combinatorial Simsek numbers $y_6(m,n;\lambda,r)$ of the sixth kind are also reduced to the sums $M_{m,r}(n)$ and $S^{(r)}_{n,m}$, when $\lambda = 1$:

$$y_6(m,n;1,r) = \frac{M_{m,r}(n)}{n!} = \frac{S^{(r)}_{n,m}}{n!}, \tag{7}$$

where $M_{m,r}(n)$ denotes the Moment sums (*cf.* [23, Eq. (5.3.1), p. 167]) and $S^{(r)}_{n,m}$ denotes a generalization of the Franel sums (*cf.* [5, Eq. (4), p. 79]) defined by

$$M_{m,r}(n) = S^{(r)}_{n,m} = \sum_{j=0}^{n} \binom{n}{j}^r j^m \tag{8}$$

(*cf.* [29]; and see also [5, 8–10], [23, Eq. (5.1.1), p. 159], [33, p. 67]).

Table 1-Table 3 provide some values of the numbers $y_6(m,n;\lambda,r)$ for some arbitrarily chosen special cases.

	n=1	n=2	n=3	n=4
$y_6(0,n;\lambda,0)$	$1+\lambda$	$\frac{1}{2}(1+\lambda+\lambda^2)$	$\frac{1}{6}(1+\lambda+\lambda^2+\lambda^3)$	$\frac{1}{24}(1+\lambda+\lambda^2+\lambda^3+\lambda^4)$
$y_6(0,n;\lambda,1)$	$1+\lambda$	$\frac{1}{2}(1+2\lambda+\lambda^2)$	$\frac{1}{6}(1+3\lambda+3\lambda^2+\lambda^3)$	$\frac{1}{24}(1+4\lambda+6\lambda^2+4\lambda^3+\lambda^4)$
$y_6(0,n;\lambda,2)$	$1+\lambda$	$\frac{1}{2}(1+4\lambda+\lambda^2)$	$\frac{1}{6}(1+9\lambda+9\lambda^2+\lambda^3)$	$\frac{1}{24}(1+16\lambda+36\lambda^2+16\lambda^3+\lambda^4)$
$y_6(0,n;\lambda,3)$	$1+\lambda$	$\frac{1}{2}(1+8\lambda+\lambda^2)$	$\frac{1}{6}(1+27\lambda+27\lambda^2+\lambda^3)$	$\frac{1}{24}(1+64\lambda+216\lambda^2+64\lambda^3+\lambda^4)$
$y_6(0,n;\lambda,4)$	$1+\lambda$	$\frac{1}{2}(1+16\lambda+\lambda^2)$	$\frac{1}{6}(1+81\lambda+81\lambda^2+\lambda^3)$	$\frac{1}{24}(1+256\lambda+1296\lambda^2+256\lambda^3+\lambda^4)$
$y_6(0,n;\lambda,5)$	$1+\lambda$	$\frac{1}{2}(1+32\lambda+\lambda^2)$	$\frac{1}{6}(1+243\lambda+243\lambda^2+\lambda^3)$	$\frac{1}{24}(1+1024\lambda+7776\lambda^2+1024\lambda^3+\lambda^4)$

Table 1: Some values of the numbers $y_6(m,n;\lambda,r)$ in their special cases when $m = 0$, $n \in \{1,2,3,4\}$ and $r \in \{0,1,2,3,4,5\}$.

	n=1	n=2	n=3	n=4
$y_6(1,n;\lambda,0)$	λ	$\frac{1}{2}(\lambda+2\lambda^2)$	$\frac{1}{6}(\lambda+2\lambda^2+3\lambda^3)$	$\frac{1}{24}(\lambda+2\lambda^2+3\lambda^3+4\lambda^4)$
$y_6(1,n;\lambda,1)$	λ	$\frac{1}{2}(2\lambda+2\lambda^2)$	$\frac{1}{6}(3\lambda+6\lambda^2+3\lambda^3)$	$\frac{1}{24}(4\lambda+12\lambda^2+12\lambda^3+4\lambda^4)$
$y_6(1,n;\lambda,2)$	λ	$\frac{1}{2}(4\lambda+2\lambda^2)$	$\frac{1}{6}(9\lambda+18\lambda^2+3\lambda^3)$	$\frac{1}{24}(16\lambda+72\lambda^2+48\lambda^3+4\lambda^4)$
$y_6(1,n;\lambda,3)$	λ	$\frac{1}{2}(8\lambda+2\lambda^2)$	$\frac{1}{6}(27\lambda+54\lambda^2+3\lambda^3)$	$\frac{1}{24}(64\lambda+432\lambda^2+192\lambda^3+4\lambda^4)$
$y_6(1,n;\lambda,4)$	λ	$\frac{1}{2}(16\lambda+2\lambda^2)$	$\frac{1}{6}(81\lambda+162\lambda^2+3\lambda^3)$	$\frac{1}{24}(256\lambda+2592\lambda^2+768\lambda^3+4\lambda^4)$
$y_6(1,n;\lambda,5)$	λ	$\frac{1}{2}(32\lambda+2\lambda^2)$	$\frac{1}{6}(243\lambda+486\lambda^2+3\lambda^3)$	$\frac{1}{24}(1024\lambda+15\,552\lambda^2+3072\lambda^3+4\lambda^4)$

Table 2: Some values of the numbers $y_6(m,n;\lambda,r)$ in their special cases when $m = 1$, $n \in \{1,2,3,4\}$ and $r \in \{0,1,2,3,4,5\}$.

	n=1	n=2	n=3	n=4
$y_6(2,n;\lambda,0)$	λ	$\frac{1}{2}\left(\lambda+4\lambda^2\right)$	$\frac{1}{6}\left(\lambda+4\lambda^2+9\lambda^3\right)$	$\frac{1}{24}\left(\lambda+4\lambda^2+9\lambda^3+16\lambda^4\right)$
$y_6(2,n;\lambda,1)$	λ	$\frac{1}{2}\left(2\lambda+4\lambda^2\right)$	$\frac{1}{6}\left(3\lambda+12\lambda^2+9\lambda^3\right)$	$\frac{1}{24}\left(4\lambda+24\lambda^2+36\lambda^3+16\lambda^4\right)$
$y_6(2,n;\lambda,2)$	λ	$\frac{1}{2}\left(4\lambda+4\lambda^2\right)$	$\frac{1}{6}\left(9\lambda+36\lambda^2+9\lambda^3\right)$	$\frac{1}{24}\left(16\lambda+144\lambda^2+144\lambda^3+16\lambda^4\right)$
$y_6(2,n;\lambda,3)$	λ	$\frac{1}{2}\left(8\lambda+4\lambda^2\right)$	$\frac{1}{6}\left(27\lambda+108\lambda^2+9\lambda^3\right)$	$\frac{1}{24}\left(64\lambda+864\lambda^2+576\lambda^3+16\lambda^4\right)$
$y_6(2,n;\lambda,4)$	λ	$\frac{1}{2}\left(16\lambda+4\lambda^2\right)$	$\frac{1}{6}\left(81\lambda+324\lambda^2+9\lambda^3\right)$	$\frac{1}{24}\left(256\lambda+5184\lambda^2+2304\lambda^3+16\lambda^4\right)$
$y_6(2,n;\lambda,5)$	λ	$\frac{1}{2}\left(32\lambda+4\lambda^2\right)$	$\frac{1}{6}\left(243\lambda+972\lambda^2+9\lambda^3\right)$	$\frac{1}{24}\left(1024\lambda+31104\lambda^2+9216\lambda^3+16\lambda^4\right)$

Table 3: Some values of the numbers $y_6(m,n;\lambda,r)$ in their special cases when $m=2$, $n\in\{1,2,3,4\}$ and $r\in\{0,1,2,3,4,5\}$.

For other identities satisfied by the numbers $y_6(m,n;\lambda,r)$ and their special values, the reader may refer to the paper of Simsek [29]. Also, to see the other relationships of the numbers $y_1(m,n;\lambda)$ and $y_6(m,n;\lambda,r)$ with other special numbers and polynomials, the reader may also refer to [11, 12, 14, 17, 19–21, 26, 28, 32, 39].

As for the present paper, its main motivation is to construct generating functions for new number families by decomposition of the function $n!F_{y_6}(t,n;\lambda,r)$ into finite sums running over divisors and totatives. Moreover, we aim to investigate some properties of these new number families by the techniques of generating functions and some concepts from the analytic number theory.

The most important assumption to be referred throughout this study is as follows:

$$0^n=\begin{cases}1, & n=0\\ 0, & n\in\mathbb{N}\end{cases}\tag{9}$$

whose implementation in Wolfram language (*cf.* [38]) is given by Implementation 1.

Implementation 1: The following code snippet is written in Wolfram language so that 0^n can be defined as in the equation (9). See, for details, the documentations supplied by [38].

```
Unprotect[Power];
Power[0,0]=1;
Protect[Power];
```

As for the organization and content of the present paper, the next sections are summarized as follows:

In Section 2, by considering a decomposition of the function $n!F_{y_6}(t,n;\lambda,r)$ into two sums separately over all divisors and non-divisors of n, we define a family of numbers, denoted by $\mathcal{K}(n,m;\lambda,r)$, by their generating functions. We also investigate the numbers $\mathcal{K}(n,m;\lambda,r)$ and their generating functions, and we evaluate their several values for some special cases when n is a prime number and prime power. In addition, we provide computational implementations in Wolfram language for symbolic computation of the numbers $\mathcal{K}(n,m;\lambda,r)$ and their generating functions. With these implementations, we also present tables of both the numbers $\mathcal{K}(n,m;\lambda,r)$ and their generating functions, for some arbitrarily chosen special cases.

In Section 3, we provide some applications of the numbers $\mathcal{K}(n,m;\lambda,r)$ and $y_6(m,n;\lambda,r)$ regarding the Thacker's (totient) function. Especially, by summing these numbers over all totatives of n, we investigate some special finite sums that contains both the Thacker's (totient) function and higher powers of binomial coefficients. Moreover, some of the problems, regarding these finite sums, have been partially answered.

In Section 4, as a result of the decomposition of the function $n!F_{y_6}(t,n;\lambda,r)$ into two sums separately over all totatives and cototatives of n, we define another family of numbers, denoted by $\mathcal{W}(n,m;\lambda,r)$, by their generating functions. We also investigate the numbers $\mathcal{W}(n,m;\lambda,r)$ and their generating functions, and we evaluate their several values for some special cases when n is a prime number and prime power. Besides, we provide computational implementations in Wolfram language for symbolic computation of the numbers $\mathcal{W}(n,m;\lambda,r)$ and their generating functions. With these implementations, we also present tables of both the numbers $\mathcal{W}(n,m;\lambda,r)$ and their generating functions, for some arbitrarily chosen special cases. At the end of the Section 4, we propose an open problem regarding a potential relation between the numbers $\mathcal{W}(n,m;\lambda,r)$ and a formula involving the Möbius function.

In Section 5, we conclude the paper by providing an overview of the results of this article, their potential usage areas, and comments for potential future studies.

2. Decomposition into finite sums over divisors and non-divisors

In this section, we decompose the function $n!F_{y_6}(t,n;\lambda,r)$ into two sums separately over all divisors and non-divisors of n, as in the following form:

$$n!F_{y_6}(t,n;\lambda,r) = \sum_{d|n} \binom{n}{d}^r \lambda^d e^{dt} + \sum_{d\nmid n} \binom{n}{d}^r \lambda^d e^{dt}. \tag{10}$$

By setting

$$F_{\mathcal{K}}(t,n;\lambda,r) := n!F_{y_6}(t,n;\lambda,r) - \sum_{d\nmid n} \binom{n}{d}^r \lambda^d e^{dt}, \tag{11}$$

we introduce the numbers $\mathcal{K}(n,m;\lambda,r)$ by the following definition:

Definition 2.1. *The numbers $\mathcal{K}(n,m;\lambda,r)$ are defined by means of the following generating functions:*

$$F_{\mathcal{K}}(t,n;\lambda,r) = \sum_{m=0}^{\infty} \mathcal{K}(n,m;\lambda,r)\frac{t^m}{m!} \tag{12}$$

where $n \in \mathbb{N}$, $m,r \in \mathbb{N}\cup\{0\}$ and $\lambda,t \in \mathbb{R}$ (or $\mathbb{C}$).

For the purpose of investigating some properties of the numbers $\mathcal{K}(n,m;\lambda,r)$ and their generating functions, we first combine (12) with the Taylor expansion of the function e^{dt}. Thus, we get

$$\sum_{m=0}^{\infty} \mathcal{K}(n,m;\lambda,r)\frac{t^m}{m!} = \sum_{d|n} \binom{n}{d}^r \lambda^d \left(\sum_{m=0}^{\infty} d^m \frac{t^m}{m!}\right)$$

which, by comparing the coefficients of $\frac{t^m}{m!}$ on its both sides, yields a formula for the numbers $\mathcal{K}(n,m;\lambda,r)$ as in the following theorem:

Theorem 2.2. *Let $n \in \mathbb{N}$, $m,r \in \mathbb{N}\cup\{0\}$, $\lambda \in \mathbb{R}$ (or $\mathbb{C}$). Then we have*

$$\mathcal{K}(n,m;\lambda,r) = \sum_{d|n} \binom{n}{d}^r \lambda^d d^m. \tag{13}$$

By (13), some special cases of the numbers $\mathcal{K}(n,m;\lambda,r)$ are listed as follows:

Case of n being a prime number: Let p be a prime number. Since the divisors of p are given by $\{1,p\}$, setting $n=p$ in (13) yields

$$\begin{aligned}\mathcal{K}(p,m;\lambda,r) &= \sum_{d|p} \binom{p}{d}^r \lambda^d d^m \\ &= p^r\lambda + \lambda^p p^m.\end{aligned}$$

Case of n being a prime power: Let p be a prime number and k be a positive integer. Since the divisors of p^k are $\{1, p, p^2, \ldots, p^k\}$, setting $n = p^k$ in (13) yields

$$\begin{aligned}
\mathcal{K}\left(p^k, m; \lambda, r\right) &= \sum_{d|p^k} \binom{p^k}{d}^r \lambda^d d^m \\
&= \sum_{j=0}^{k} \binom{p^k}{p^j}^r \lambda^{p^j} p^{jm}.
\end{aligned}$$

For example, if we set $k = 2$ in the above equation, then we have

$$\mathcal{K}\left(p^2, m; \lambda, r\right) = p^{2r}\lambda + \binom{p^2}{p}\lambda^p p^m + \lambda^{p^2} p^{2m}.$$

Next, for the purpose of calculating the values of the numbers $\mathcal{K}(n, m; \lambda, r)$, we implement the formula, given by (13), in the Wolfram language (see: Implementation 2).

Implementation 2: The following code, written in Wolfram language by the aid of the formula (13), includes the procedure `KNum` which returns symbolically the values of the numbers $\mathcal{K}(n, m; \lambda, r)$.

```
KNum[n_,m_,\[Lambda]_,r_]:=Sum[((Binomial[n,d])^r)*(\[Lambda]^d)*(d^m), {d, Divisors[n]}]
```

By the Implementation 2, we compute some values of the numbers $\mathcal{K}(n, m; \lambda, r)$ in their special cases when $n \in \{1, 2, 3, 4, 5\}$ and $r \in \{0, 1, 2, 3, 4, 5\}$, and we present these values in Table 4.

	n=1	n=2	n=3	n=4	n=5
$\mathcal{K}(n, m; \lambda, 0)$	λ	$\lambda + 2^m \lambda^2$	$\lambda + 3^m \lambda^3$	$\lambda + 2^m \lambda^2 + 4^m \lambda^4$	$\lambda + 5^m \lambda^5$
$\mathcal{K}(n, m; \lambda, 1)$	λ	$2\lambda + 2^m \lambda^2$	$3\lambda + 3^m \lambda^3$	$4\lambda + 3 \times 2^{1+m} \lambda^2 + 4^m \lambda^4$	$5\lambda + 5^m \lambda^5$
$\mathcal{K}(n, m; \lambda, 2)$	λ	$4\lambda + 2^m \lambda^2$	$9\lambda + 3^m \lambda^3$	$16\lambda + 9 \times 2^{2+m} \lambda^2 + 4^m \lambda^4$	$25\lambda + 5^m \lambda^5$
$\mathcal{K}(n, m; \lambda, 3)$	λ	$8\lambda + 2^m \lambda^2$	$27\lambda + 3^m \lambda^3$	$64\lambda + 27 \times 2^{3+m} \lambda^2 + 4^m \lambda^4$	$125\lambda + 5^m \lambda^5$
$\mathcal{K}(n, m; \lambda, 4)$	λ	$16\lambda + 2^m \lambda^2$	$81\lambda + 3^m \lambda^3$	$256\lambda + 81 \times 2^{4+m} \lambda^2 + 4^m \lambda^4$	$625\lambda + 5^m \lambda^5$
$\mathcal{K}(n, m; \lambda, 5)$	λ	$32\lambda + 2^m \lambda^2$	$243\lambda + 3^m \lambda^3$	$1024\lambda + 243 \times 2^{5+m} \lambda^2 + 4^m \lambda^4$	$3125\lambda + 5^m \lambda^5$

Table 4: Some values of the numbers $\mathcal{K}(n, m; \lambda, r)$ in their special cases when $n \in \{1, 2, 3, 4, 5\}$ and $r \in \{0, 1, 2, 3, 4, 5\}$.

Next, by (11) we also give some special cases of the functions $F_{\mathcal{K}}(t, n; \lambda, r)$ as follows:

Case of n being a prime number: Let p be a prime number. Since the divisors of p are given by $\{1, p\}$, setting $n = p$ in (11) yields

$$\begin{aligned}
F_{\mathcal{K}}(t, p; \lambda, r) &= \sum_{d|p} \binom{p}{d}^r \lambda^d e^{dt} \\
&= p^r \lambda e^t + \left(\lambda e^t\right)^p.
\end{aligned}$$

Case of n being a prime power: Let p be a prime number and k be a positive integer. Since the divisors of

p^k are $\{1, p, p^2, \ldots, p^k\}$, setting $n = p^k$ in (11) yields

$$\begin{aligned} F_{\mathcal{K}}\left(t, p^k; \lambda, r\right) &= \sum_{d|p^k} \binom{p^k}{d}^r \lambda^d e^{dt} \\ &= \sum_{j=0}^{k} \binom{p^k}{p^j}^r \lambda^{p^j} e^{p^j t}. \end{aligned}$$

For example; if we set $k = 2$ in the above equation, then we have

$$F_{\mathcal{K}}\left(t, p^2; \lambda, r\right) = p^{2r} \lambda e^t + \binom{p^2}{p}\left(\lambda e^t\right)^p p^m + \left(\lambda e^t\right)^{p^2} p^{2m}.$$

For the purpose of calculating the values of the functions $F_{\mathcal{K}}(t, n; \lambda, r)$, we implement the formula, given by (11), in the Wolfram language (see: Implementation 3).

Implementation 3: The following code, written in Wolfram language by the aid of the formula (11), includes the procedure `GenFuncKNum` which returns symbolically the values of the funtions $F_{\mathcal{K}}(t, n; \lambda, r)$.

```
GenFuncKNum[t_,n_,\[Lambda]_,r_]:=Sum[((Binomial[n,d])^r)*(\[Lambda]^d)*Exp[d*t], {d,Divisors[n]}]
```

By the Implementation 3, we compute some values the funtions $F_{\mathcal{K}}(t, n; \lambda, r)$ in their special cases when $n \in \{1, 2, 3, 4, 5\}$ and $r \in \{0, 1, 2, 3, 4, 5\}$, and we present these values in Table 5.

	n=1	n=2	n=3	n=4	n=5
$F_{\mathcal{K}}(t, n; \lambda, 0)$	$e^t \lambda$	$e^t \lambda + e^{2t} \lambda^2$	$e^t \lambda + e^{3t} \lambda^3$	$e^t \lambda + e^{2t} \lambda^2 + e^{4t} \lambda^4$	$e^t \lambda + e^{5t} \lambda^5$
$F_{\mathcal{K}}(t, n; \lambda, 1)$	$e^t \lambda$	$2 e^t \lambda + e^{2t} \lambda^2$	$3 e^t \lambda + e^{3t} \lambda^3$	$4 e^t \lambda + 6 e^{2t} \lambda^2 + e^{4t} \lambda^4$	$5 e^t \lambda + e^{5t} \lambda^5$
$F_{\mathcal{K}}(t, n; \lambda, 2)$	$e^t \lambda$	$4 e^t \lambda + e^{2t} \lambda^2$	$9 e^t \lambda + e^{3t} \lambda^3$	$16 e^t \lambda + 36 e^{2t} \lambda^2 + e^{4t} \lambda^4$	$25 e^t \lambda + e^{5t} \lambda^5$
$F_{\mathcal{K}}(t, n; \lambda, 3)$	$e^t \lambda$	$8 e^t \lambda + e^{2t} \lambda^2$	$27 e^t \lambda + e^{3t} \lambda^3$	$64 e^t \lambda + 216 e^{2t} \lambda^2 + e^{4t} \lambda^4$	$125 e^t \lambda + e^{5t} \lambda^5$
$F_{\mathcal{K}}(t, n; \lambda, 4)$	$e^t \lambda$	$16 e^t \lambda + e^{2t} \lambda^2$	$81 e^t \lambda + e^{3t} \lambda^3$	$256 e^t \lambda + 1296 e^{2t} \lambda^2 + e^{4t} \lambda^4$	$625 e^t \lambda + e^{5t} \lambda^5$
$F_{\mathcal{K}}(t, n; \lambda, 5)$	$e^t \lambda$	$32 e^t \lambda + e^{2t} \lambda^2$	$243 e^t \lambda + e^{3t} \lambda^3$	$1024 e^t \lambda + 7776 e^{2t} \lambda^2 + e^{4t} \lambda^4$	$3125 e^t \lambda + e^{5t} \lambda^5$

Table 5: Some values of the functions $F_{\mathcal{K}}(t, n; \lambda, r)$ in their special cases when $n \in \{1, 2, 3, 4, 5\}$ and $r \in \{0, 1, 2, 3, 4, 5\}$.

Differentiating the equation (11) m times with respect to t, and combining the final equation with (13) yields the following corollary:

Corollary 2.3. *Let $m \in \mathbb{N}$. Then we have*

$$\mathcal{K}(n, m; \lambda, r) = \frac{\partial^m}{\partial t^m} \left\{F_{\mathcal{K}}(t, n; \lambda, r)\right\}\Big|_{t=0}. \tag{14}$$

3. Some applications of the numbers $\mathcal{K}(n, m; \lambda, r)$ and $y_6(m, n; \lambda, r)$ regarding the Thacker's (totient) function

Towards the end of the 19th century, J. J. Sylvester [36] introduced the totatives (or totitives) of a positive integer n to be the positive integers that are less than n and relatively prime (coprime) to n. From then on to the present, the totatives of a given positive integer and finite sums constructed accordingly have been used as auxiliary tools in a wide variety of fields such as mathematics, mathematical physics, computational sciences, cryptology and engineering (see, for details, [2, 6, 13, 25]).

The main aim of this section is to give some applications of the numbers $\mathcal{K}(n,m;\lambda,r)$ and $y_6(m,n;\lambda,r)$ regarding the Thacker's (totient) function denoted by $\varphi_k(n)$ which is a generalization of the Euler's totient function, introduced by Thacker in 1850 as the summation of kth powers of the totatives of the positive integer n, namely:

$$\varphi_k(n) = \sum_{\lambda \in \mathcal{T}_n} \lambda^k \tag{15}$$

where $k \in \mathbb{N}$ and $\mathcal{T}_n$ denotes the set of all totatives of the positive integer n, namely:

$$\mathcal{T}_n = \left\{ j \in \mathbb{N} : 1 \leq j < n,\ \gcd(j,n) = 1 \right\},$$

(*cf.* [1, 6, 25]).

On the other hand, we prefer to denote the set of all cototatives of the positive integer n by $\widehat{\mathcal{T}_n}$ whose definition is given as follows:

$$\widehat{\mathcal{T}_n} = \left\{ j \in \mathbb{N} : 1 \leq j < n,\ \gcd(j,n) > 1 \right\}.$$

Observe that in the case when $k = 0$, the Thacker's (totient) function $\varphi_k(n)$ reduces to the Euler's totient function $\varphi(n)$, namely:

$$\varphi_0(n) = \varphi(n) = \sum_{\substack{j=1 \\ \gcd(j,n)=1}}^{n} 1, \tag{16}$$

which counts of the number of all totatives of the positive integer n. See, for details, [6], [25]; and see also [2] and [13].

The Thacker's (totient) function $\varphi_k(n)$ satisfies the following recurrence relation:

$$\varphi_k(n) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} n^{k-j} \varphi_j(n), \tag{17}$$

(*cf.* [6, p. 142] and the references cited therein).

It follows from (16) and (17) that

$$\varphi_1(n) = \frac{n\varphi(n)}{2} \tag{18}$$

which denotes the sum of all totatives of the positive integer n (*cf.* [6, p. 142], [24], and the references cited therein).

It should be also noted here that in the case when $n = p$ (prime number), the Thacker's (totient) function $\varphi_k(n)$ reduces to the Faulhaber formula (*cf.* [1]):

$$\varphi_k(p) = S_k(p); \quad \text{if } p \text{ is a prime number} \tag{19}$$

where

$$S_k(n) = 1^k + 2^k + \cdots + (n-1)^k \tag{20}$$

which denotes the sum of the kth powers of the consecutive first $(n-1)$ positive integers, and can be traced back to roughly 400 years ago. Back then J. Faulhaber [7] addressed to derive computational formulas for the sum $S_k(n)$ of powers of consecutive integers. There are many formulas in the literature that correspond to the sum $S_k(n)$, but the most remarkable and well-known formulas among others are given as follows:

$$S_k(n+1) = \frac{B_{k+1}(n+1) - B_{k+1}}{k+1}$$

and

$$S_k(n+1) = \frac{1}{k+1}\sum_{j=0}^{k}(-1)^j\binom{k+1}{j}n^{k+1-j}B_j$$

where B_k and $B_k(x)$ stand for the kth Bernoulli numbers and polynomials, respectively (*cf.* [7, 15, 16, 18]).

Remark 3.1. *As a generalization of the Gauss' formula (cf.* [2]*):*

$$\sum_{d|n}\varphi(d) = n, \tag{21}$$

Liouville [22] *gave the following identity for the Thacker's (totient) function $\varphi_k(n)$:*

$$\sum_{d|n}\left(\frac{n}{d}\right)^r\varphi_r(d) = S_r(n+1) \tag{22}$$

which was also handled by Bruckman and Lossers [4, p. 435] *for finding the Dirichlet series of the function $\varphi_k(n)$. But much later, claiming to be the result of Liouville* [22], *some researchers such as Andjić and Meštrovic* [1, p. 4], *Dickson* [6, p. 142] *and Sándor and Crstici* [25, p. 243] *mentioned the following formula with a typo (fraction possibly mistakenly written as a binomial coefficient):*

$$\sum_{d|n}\binom{n}{d}^r\varphi_r(d) \stackrel{?}{=} S_r(n+1), \tag{23}$$

which actually should have been as given in the equation (22)*. Incidentally it should be pointed out here that the identity in* (23) *is not true! Just implementing the right- and the left-hand sides of* (23) *in the Wolfram language (see: Implementation* 4*), we perceive that the equality* (23) *does not hold true.*

Implementation 4: The following code snippet, written in Wolfram language, includes the procedure `ThackerFunc` corresponding to the Thacker's (totient) function given by (15), the procedure `FaulhaberFormula` corresponding to the Faulhaber formula given by (20) and the procedure `TypoL` corresponding to the left-hand side of (23). The following code snippets indicate that the assertion of (23) does not hold true for example when $n = 4$.

```
ThackerFunc[n_, r_]:=Sum[If[CoprimeQ[n, j] == True, j^r,  0],  {j,  n}]
FaulhaberFormula[n_, r_]:=Sum[j^r,  {j ,1,n-1}]
TypoL[n_,  r_]:=Sum[((Binomial[n,d])^r)*ThackerFunc[d, r], {d,  Divisors[n]}]
TypoL[4, 3]=308
FaulhaberFormula[5, 3]=100
```

After detecting the typo mentioned above, the following problems come to mind in this stage:

Open Problem 1: Is there any explicit formula for the following divisor sum:

$$\sum_{d|n}\binom{n}{d}^r\varphi_r(d). \tag{24}$$

Open Problem 2: Is there any explicit formula for the following divisor sum:

$$\sum_{d|n}\binom{n}{d}^r d^m\varphi_d(n). \tag{25}$$

Next, we are motivated to partially answer the above problems and derive some formulas for the above type divisor sums by the aid of the numbers $\mathcal{K}(n, m; \lambda, r)$ and $y_6(m, n; \lambda, r)$.

Theorem 3.2. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{\lambda \in \mathcal{T}_n} \mathcal{K}(n,m;\lambda,r) = \sum_{d|n} \binom{n}{d}^r d^m \varphi_d(n). \tag{26}$$

Proof. Summing $\mathcal{K}(n,m;\lambda,r)$ over the set $\mathcal{T}_n$ of all totatives of the positive integer n, we get

$$\sum_{\lambda \in \mathcal{T}_n} \mathcal{K}(n,m;\lambda,r) = \sum_{\lambda \in \mathcal{T}_n} \sum_{d|n} \binom{n}{d}^r \lambda^d d^m,$$

which yields

$$\sum_{\lambda \in \mathcal{T}_n} \mathcal{K}(n,m;\lambda,r) = \sum_{d|n} \binom{n}{d}^r d^m \sum_{\lambda \in \mathcal{T}_n} \lambda^d,$$

By blending (15) with the equation just above, we arrive at the assertion of Theorem 3.2. □

Setting $n = p$ (prime number) in (26), and using (18), (19) and (20) in the final equation, we arrive at the following corollary:

Corollary 3.3. *Let p be a prime number. Then we have*

$$\sum_{\lambda \in \mathcal{T}_p} \mathcal{K}(p,m;\lambda,r) = \frac{p^{r+1}(p-1)}{2} + p^m \left(1 + 2^p + \cdots + (p-1)^p\right). \tag{27}$$

Next, we also investigate some relations between the Thacker's (totient) function $\varphi_k(n)$ and the numbers $y_6(m,n;\lambda,r)$ as follows:

Theorem 3.4. *Let $n \in \mathbb{N}$. Then we have*

$$\sum_{\lambda \in \mathcal{T}_n} y_6(m,n;\lambda,r) = \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j}^r j^m \varphi_j(n). \tag{28}$$

Proof. Summing the numbers $y_6(m,n;\lambda,r)$ over the set $\mathcal{T}_n$ of all totatives of the positive integer n, we get

$$\sum_{\lambda \in \mathcal{T}_n} y_6(m,n;\lambda,r) = \frac{1}{n!} \sum_{\lambda \in \mathcal{T}_n} \sum_{j=0}^{n} \binom{n}{j}^r \lambda^j j^m,$$

which yields

$$\sum_{\lambda \in \mathcal{T}_n} y_6(m,n;\lambda,r) = \frac{1}{n!} \sum_{j=0}^{n} \binom{n}{j}^r j^m \sum_{\lambda \in \mathcal{T}_n} \lambda^j.$$

By blending (15) with the equation just above, we arrive at the desired result. □

Setting $n = p$ (prime number) in (28), and using (19) and (20) in the final equation, we arrive at the following identity:

$$\sum_{\lambda \in \mathcal{T}_p} y_6(m,p;\lambda,r) = \frac{1}{p!} \sum_{j=0}^{p} \binom{p}{j}^r j^m S_j(p). \tag{29}$$

Combining (29) with (5) yields the following corollary:

Corollary 3.5. *Let p be a prime number. Then we have*

$$\sum_{\lambda\in\mathcal{T}_p} y_6\left(m,p;\lambda,r\right)=\sum_{v=1}^{p-1} y_6\left(m,p;v,r\right). \tag{30}$$

Theorem 3.6. *Let $m\in\mathbb{N}$. Then we have*

$$\sum_{\lambda\in\mathcal{T}_m} y_6\left(m,n;\lambda,r\right)=\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}^{r} j^{m}\varphi_j\left(m\right). \tag{31}$$

Proof. Summing the numbers $y_6\left(m,n;\lambda,r\right)$ over the set $\mathcal{T}_m$ of all totatives of the positive integer m, we get

$$\sum_{\lambda\in\mathcal{T}_m} y_6\left(m,n;\lambda,r\right)=\frac{1}{n!}\sum_{\lambda\in\mathcal{T}_m}\sum_{j=0}^{n}\binom{n}{j}^{r}\lambda^{j} j^{m},$$

which yields

$$\sum_{\lambda\in\mathcal{T}_m} y_6\left(m,n;\lambda,r\right)=\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}^{r} j^{m}\sum_{\lambda\in\mathcal{T}_m}\lambda^{j}.$$

By blending (15) with the equation just above, we arrive at the desired result. □

Setting $m=p$ (prime number) in (31), and using (19) and (20) in the final equation, we get

$$\sum_{\lambda\in\mathcal{T}_p} y_6\left(p,n;\lambda,r\right)=\frac{1}{n!}\sum_{j=0}^{n}\binom{n}{j}^{r} j^{p} S_j\left(p\right). \tag{32}$$

Combining (32) with (5) yields the following corollary:

Corollary 3.7. *Let p be a prime number. Then we have*

$$\sum_{\lambda\in\mathcal{T}_p} y_6\left(p,n;\lambda,r\right)=\sum_{v=1}^{p-1} y_6\left(p,n;v,r\right). \tag{33}$$

4. Decomposition into finite sums over totatives and cototatives

In this section, by pursuing a method that of Simsek [33], we also decompose the function $n!F_{y_6}\left(t,n;\lambda,r\right)$ into two sums separately over all totatives and cototatives of n, as in the following form:

$$n!F_{y_6}\left(t,n;\lambda,r\right)=\sum_{\substack{j=1\\ \gcd(j,n)=1}}^{n}\binom{n}{j}^{r}\lambda^{j}e^{jt}+\sum_{\substack{j=1\\ \gcd(j,n)>1}}^{n}\binom{n}{j}^{r}\lambda^{j}e^{jt} \tag{34}$$

where gcd denotes the usual greatest common divisor.

Remark 4.1. *By considering the definition of the set $\mathcal{T}_n$ of all totatives and the set $\widehat{\mathcal{T}_n}$ of all cototatives of the positive integer n, the equation* (34) *can also be expressed equally as follows:*

$$n!F_{y_6}\left(t,n;\lambda,r\right)=\sum_{j\in\mathcal{T}_n}\binom{n}{j}^{r}\lambda^{j}e^{jt}+\sum_{j\in\widehat{\mathcal{T}_n}}\binom{n}{j}^{r}\lambda^{j}e^{jt}. \tag{35}$$

By setting

$$F_{\mathcal{W}}(t,n;\lambda,r) := n!F_{y_6}(t,n;\lambda,r) - \sum_{\substack{j=1 \\ \gcd(j,n)>1}}^{n} \binom{n}{j}^r \lambda^j e^{jt}, \tag{36}$$

we introduce the numbers $\mathcal{W}(n,m;\lambda,r)$ by the following definition:

Definition 4.2. *The numbers $\mathcal{W}(n,m;\lambda,r)$ are defined by means of the following generating functions:*

$$F_{\mathcal{W}}(t,n;\lambda,r) = \sum_{m=0}^{\infty} \mathcal{W}(n,m;\lambda,r)\frac{t^m}{m!} \tag{37}$$

where $n \in \mathbb{N}$, $m,r \in \mathbb{N} \cup \{0\}$ and $\lambda, t \in \mathbb{R}$ (or $\mathbb{C}$).

For the purpose of investigating some properties of the numbers $\mathcal{W}(n,m;\lambda,r)$ and their generating functions, we first combine (37) with the Taylor expansion of the function e^{jt}. Thus, we have

$$\sum_{m=0}^{\infty} \mathcal{W}(n,m;\lambda,r)\frac{t^m}{m!} = \sum_{\substack{j=1 \\ \gcd(j,n)=1}}^{n} \binom{n}{j}^r \lambda^j \left(\sum_{m=0}^{\infty} j^m \frac{t^m}{m!}\right)$$

which, by comparing the coefficients of $\frac{t^m}{m!}$ on its both sides, yields a formula for the numbers $\mathcal{W}(n,m;\lambda,r)$ as in the following theorem:

Theorem 4.3. *Let $n \in \mathbb{N}$, $m,r \in \mathbb{N} \cup \{0\}$, $\lambda \in \mathbb{R}$ (or $\mathbb{C}$). Then we have*

$$\mathcal{W}(n,m;\lambda,r) = \sum_{\substack{j=1 \\ \gcd(j,n)=1}}^{n} \binom{n}{j}^r \lambda^j j^m. \tag{38}$$

Remark 4.4. *By considering the definition of the set $\mathcal{T}_n$ of all totatives of the positive integer n, the equation* (38) *can also be expressed equally as follows:*

$$\mathcal{W}(n,m;\lambda,r) = \sum_{j \in \mathcal{T}_n} \binom{n}{j}^r \lambda^j j^m. \tag{39}$$

By (38), some special cases of the numbers $\mathcal{W}(n,m;\lambda,r)$ are listed as follows:

Case of n being a prime number: Let p be a prime number. Then, setting $n = p$ in (38) yields

$$\begin{aligned} \mathcal{W}(p,m;\lambda,r) &= \sum_{\substack{j=1 \\ \gcd(j,p)=1}}^{p} \binom{p}{j}^r \lambda^j j^m \\ &= \sum_{j=1}^{p-1} \binom{p}{j}^r \lambda^j j^m \end{aligned}$$

which, by (5), implies the following corollary:

Corollary 4.5. *Let p be a prime number. Then we have*

$$\mathcal{W}(p,m;\lambda,r) = p!y_6(m,p;\lambda,r) - \lambda^p p^m.$$

Remark 4.6. *By setting*

$$F_r(m,n;\lambda) = n! y_6(m,n;\lambda,r), \tag{40}$$

Simsek [29] *also introduced the generalized rth order Franel numbers* $F_r(m,n;\lambda)$, *which is a generalization of the Franel sums (cf.* [8, 9]*) and Franel-type sums (cf.* [5, 23]*). Thus, the Corollary* 4.5 *can also be expressed, in terms of the generalized rth order the Franel numbers, as follows:*

$$\mathcal{W}(p,m;\lambda,r) = F_r(m,p;\lambda) - \lambda^p p^m$$

where p is a prime number.

Case of n being a prime power: Let p be a prime number and k be a positive integer. Then, setting $n = p^k$ in (38) yields

$$\begin{aligned}\mathcal{W}\left(p^k,m;\lambda,r\right) &= \sum_{\substack{j=1\\ \gcd\left(j,p^k\right)=1}}^{p^k} \binom{p^k}{j}^r \lambda^j j^m \\ &= \sum_{\substack{j=1\\ j\not\equiv 0\,(\mathrm{mod}\ p)}}^{p^k} \binom{p^k}{j}^r \lambda^j j^m.\end{aligned}$$

For example; if we set $p = 2$ and $k = 3$ in the above equation, then we have

$$\mathcal{W}(8,m;\lambda,r) = 8^r\lambda + 3^m(56)^r\lambda^3 + 5^m(56)^r\lambda^5 + 7^m 8^r\lambda^7.$$

For the purpose of calculating the values of the numbers $\mathcal{W}(n,m;\lambda,r)$, we implement the formula, given by (38), in the Wolfram language (see: Implementation 5).

Implementation 5: The following code, written in Wolfram language by the aid of the formula (13), includes the procedure `WNum` which returns symbolically the values of the numbers $\mathcal{W}(n,m;\lambda,r)$.

```
WNum[n_,m_,\[Lambda]_,r_]:=Sum[If[CoprimeQ[j, n] == True, (Binomial[n,j]^r)*(\[Lambda]^j)*(j^m),
0], {j,1,n}]
```

By the Implementation 5, we compute some of the numbers $\mathcal{W}(n,m;\lambda,r)$ in their special cases when $n \in \{1,2,3,4,5\}$ and $r \in \{0,1,2,3,4,5\}$, and we present these values in Table 6.

	n=1	n=2	n=3	n=4	n=5
$\mathcal{W}(n,m;\lambda,0)$	λ	λ	$\lambda + 2^m\lambda^2$	$\lambda + 3^m\lambda^3$	$\lambda + 2^m\lambda^2 + 3^m\lambda^3 + 4^m\lambda^4$
$\mathcal{W}(n,m;\lambda,1)$	λ	2λ	$3\lambda + 3\times 2^m\lambda^2$	$4\lambda + 4\times 3^m\lambda^3$	$5\lambda + 5\times 2^{1+m}\lambda^2 + 10\times 3^m\lambda^3 + 5\times 4^m\lambda^4$
$\mathcal{W}(n,m;\lambda,2)$	λ	4λ	$9\lambda + 9\times 2^m\lambda^2$	$16\lambda + 16\times 3^m\lambda^3$	$25\lambda + 25\times 2^{2+m}\lambda^2 + 100\times 3^m\lambda^3 + 25\times 4^m\lambda^4$
$\mathcal{W}(n,m;\lambda,3)$	λ	8λ	$27\lambda + 27\times 2^m\lambda^2$	$64\lambda + 64\times 3^m\lambda^3$	$125\lambda + 125\times 2^{3+m}\lambda^2 + 1000\times 3^m\lambda^3 + 125\times 4^m\lambda^4$
$\mathcal{W}(n,m;\lambda,4)$	λ	16λ	$81\lambda + 81\times 2^m\lambda^2$	$256\lambda + 256\times 3^m\lambda^3$	$625\lambda + 625\times 2^{4+m}\lambda^2 + 10\,000\times 3^m\lambda^3 + 625\times 4^m\lambda^4$
$\mathcal{W}(n,m;\lambda,5)$	λ	32λ	$243\lambda + 243\times 2^m\lambda^2$	$1024\lambda + 1024\times 3^m\lambda^3$	$3125\lambda + 3125\times 2^{5+m}\lambda^2 + 100\,000\times 3^m\lambda^3 + 3125\times 4^m\lambda^4$

Table 6: Some values of the numbers $\mathcal{W}(n,m;\lambda,r)$ in their special cases when $n \in \{1,2,3,4,5\}$ and $r \in \{0,1,2,3,4,5\}$.

Next, for the purpose of calculating the values of the functions $F_{\mathcal{W}}(t,n;\lambda,r)$, we implement the formula, given by (37), in the Wolfram language (see: Implementation 6).

Implementation 6: The following code, written in Wolfram language by the aid of the formula (36), includes the procedure `GenFuncWNum`, returns symbolically the values of the generating funtions $F_{\mathcal{W}}(t,n;\lambda,r)$.

```
GenFuncWNum[t_,n_,\[Lambda]_,r_]:=Sum[If[CoprimeQ[j, n]==True, (Binomial[n,j]^r)*(\[Lambda]^j)*
  Exp[j*t], 0], {j ,1,n}]
```

By the Implementation 6, we compute some of the functions $F_{\mathcal{W}}(t,n;\lambda,r)$ in their special cases when $n \in \{1,2,3,4,5\}$ and $r \in \{0,1,2,3,4,5\}$, and we present these values in Table 7.

	n=1	n=2	n=3	n=4	n=5
$F_{\mathcal{W}}(t,n;\lambda,0)$	$e^{t}\lambda$	$e^{t}\lambda$	$e^{t}\lambda+e^{2t}\lambda^2$	$e^{t}\lambda+e^{3t}\lambda^3$	$e^{t}\lambda+e^{2t}\lambda^2+e^{3t}\lambda^3+e^{4t}\lambda^4$
$F_{\mathcal{W}}(t,n;\lambda,1)$	$e^{t}\lambda$	$2e^{t}\lambda$	$3e^{t}\lambda+3e^{2t}\lambda^2$	$4e^{t}\lambda+4e^{3t}\lambda^3$	$5e^{t}\lambda+10e^{2t}\lambda^2+10e^{3t}\lambda^3+5e^{4t}\lambda^4$
$F_{\mathcal{W}}(t,n;\lambda,2)$	$e^{t}\lambda$	$4e^{t}\lambda$	$9e^{t}\lambda+9e^{2t}\lambda^2$	$16e^{t}\lambda+16e^{3t}\lambda^3$	$25e^{t}\lambda+100e^{2t}\lambda^2+100e^{3t}\lambda^3+25e^{4t}\lambda^4$
$F_{\mathcal{W}}(t,n;\lambda,3)$	$e^{t}\lambda$	$8e^{t}\lambda$	$27e^{t}\lambda+27e^{2t}\lambda^2$	$64e^{t}\lambda+64e^{3t}\lambda^3$	$125e^{t}\lambda+1000e^{2t}\lambda^2+1000e^{3t}\lambda^3+125e^{4t}\lambda^4$
$F_{\mathcal{W}}(t,n;\lambda,4)$	$e^{t}\lambda$	$16e^{t}\lambda$	$81e^{t}\lambda+81e^{2t}\lambda^2$	$256e^{t}\lambda+256e^{3t}\lambda^3$	$625e^{t}\lambda+10\,000e^{2t}\lambda^2+10\,000e^{3t}\lambda^3+625e^{4t}\lambda^4$
$F_{\mathcal{W}}(t,n;\lambda,5)$	$e^{t}\lambda$	$32e^{t}\lambda$	$243e^{t}\lambda+243e^{2t}\lambda^2$	$1024e^{t}\lambda+1024e^{3t}\lambda^3$	$3125e^{t}\lambda+100\,000e^{2t}\lambda^2+100\,000e^{3t}\lambda^3+3125e^{4t}\lambda^4$

Table 7: Some values of the functions $F_{\mathcal{W}}(t,n;\lambda,r)$ in their special cases when $n \in \{1,2,3,4,5\}$ and $r \in \{0,1,2,3,4,5\}$.

Next, by (36) we also give some special cases of the functions $F_{\mathcal{W}}(t,n;\lambda,r)$ as follows:

Case of n being a prime number: Let p be a prime number. Then, setting $n = p$ in (37) yields

$$\begin{aligned} F_{\mathcal{W}}(t,p;\lambda,r) &= \sum_{\substack{j=1 \\ \gcd(j,p)=1}}^{p} \binom{p}{j}^r \lambda^j e^{tj} \\ &= \sum_{j=1}^{p-1} \binom{p}{j}^r \lambda^j e^{tj} \end{aligned}$$

which, by (4), implies the following corollary:

Corollary 4.7. *Let p be a prime number. Then we have*

$$F_{\mathcal{W}}(t,p;\lambda,r) = p! F_{y_6}(t,p;\lambda,r) - \lambda^p e^{pt}.$$

Case of n being a prime power: Let p be a prime number and k be a positive integer. Then, setting $n = p^k$ in (37) yields

$$\begin{aligned} F_{\mathcal{W}}\left(t,p^k;\lambda,r\right) &= \sum_{\substack{j=1 \\ \gcd(j,p^k)=1}}^{p^k} \binom{p^k}{j}^r \lambda^j e^{tj} \\ &= \sum_{\substack{j=1 \\ j\not\equiv 0 \pmod p}}^{p^k} \binom{p^k}{j}^r \lambda^j e^{tj}. \end{aligned}$$

For example, if we set $p = 2$ and $k = 3$ in the above equation, then we have

$$F_{\mathcal{W}}(t,8;\lambda,r) = 8^r e^t \lambda + (56)^r e^{3t}\lambda^3 + (56)^r e^{5t}\lambda^5 + 8^r e^{7t}\lambda^7.$$

Differentiating the equation (36) m times with respect to t, and combining the final equation with (38) yields the following corollary:

Corollary 4.8. *Let $m \in \mathbb{N}$. Then we have*

$$\mathcal{W}(n,m;\lambda,r) = \frac{\partial^m}{\partial t^m}\{F_{\mathcal{W}}(t,n;\lambda,r)\}\Big|_{t=0}. \tag{41}$$

Recall that the Möbius function $\mu(n)$, which is one of the most frequently used number-theoretic (arithmetical) functions, is defined by (*cf.* [2]):

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ (-1)^m & \text{if } n \text{ is a square-free integer with } m \text{ distinct prime factors,} \\ 0 & \text{if } n \text{ has a squared prime factor.} \end{cases}$$

One of the property of the Möbius function $\mu(n)$ is given by

$$\sum_{\substack{j=1 \\ \gcd(j,b)=1}}^{ab} f(j) = \sum_{d|b} \mu(d) \sum_{j=1}^{\frac{ab}{d}} f(dj) \tag{42}$$

(*cf.* [3, p. 124, Eq. (17)]).

If we substitute

$$f(j) = \binom{n}{j}^r \lambda^j j^m$$

into (42), then we have

$$\sum_{\substack{j=1 \\ \gcd(j,b)=1}}^{ab} \binom{n}{j}^r \lambda^j j^m = \sum_{d|b} \mu(d) \sum_{j=1}^{\frac{ab}{d}} \binom{n}{dj}^r \left(\lambda^d\right)^j d^m j^m, \tag{43}$$

by which, the following open problem comes to mind:

Open problem 3: Is there any relation between the equation (43) and the numbers $\mathcal{W}(n,m;\lambda,r)$?

Remark 4.9. *Substituting $\lambda = 1$, $r = 1$, $m = 0$ into* (38) *and $\lambda = 1$, $r = 1$, $t = 0$ into* (36) *yields the following another case:*

$$\mathcal{W}(n,0;1,1) = F_{\mathcal{W}}(0,n;1,1) = \sum_{\substack{j=1 \\ \gcd(j,n)=1}}^{n} \binom{n}{j} = 2^n \sum_{d|n} \frac{\mu(d)}{d} \sum_{k=1}^{d} (-1)^k \cos^n\left(\frac{k\pi}{d}\right) \tag{44}$$

(cf. [37, Eq. (28), p. 7]*), such that the above formula corresponds to the sequence* `A056188` *appearing in the Sloane's On-Line Encyclopedia of Integer Sequences (OEIS). See, for details,* [34]. *Due to the above observation, we can conclude that the numbers $\mathcal{W}(n,m;\lambda,r)$ and their generating functions $F_{\mathcal{W}}(t,n;\lambda,r)$ unify the above sequence.*

5. Conclusion

In this study, by separating the function $n!F_{y_6}(t,n;\lambda,r)$ into sums running over divisors and totatives of the positive integer n, two new number families have been introduced with their generating functions, and these numbers have been examined to find out some of their features. Furthermore, some applications of the numbers $\mathcal{K}(n,m;\lambda,r)$ and $y_6(m,n;\lambda,r)$ regarding the Thacker's (totient) function have been provided in addition to some remarks with some open problems. In conclusion, we show that the decomposition of the generating functions, for combinatorial number families involving higher powers of binomial coefficients, into finite sums running over divisors, non-divisors, totatives and cototatives reveals interesting number families and number-theoretic (arithmetical) functions which have potentially find an application in a wide variety of fields such as mathematics, mathematical physics, computational sciences, cryptology and engineering. Especially, many more interesting results may be obtained when these number families are further examined with the methods of analytical number theory. Therefore, the results of this paper open up new fields of work by getting attention of the researchers interested in related fields.

Acknowledgments

I would like to dedicate this paper to my PhD advisor Professor Yilmaz SIMSEK for celebrating his 60th birthday. Endless thanks to him since he continues to inspire the young researchers and increase their academic motivation with an endless energy despite being sixty years old. I wish him many more healthy days with his family and loved ones.

References

[1] M. Andjić, R. Meštrovic, *An Identity in Commutative Rings with Unity with Applications to Various Sums of Powers*, Discrete Dyn. Nat. Soc. **2017** (2017), Article ID 9092515; doi:10.1155/2017/9092515.
[2] T. M. Apostol, Introduction to Analytic Number Theory, *Springer-Verlag*, New York-Heidelberg-Berlin, 1976.
[3] M. Beck, *Dedekind Cotangent Sums*, Acta Arith. **109**(2) (2003), 109–130.
[4] P. S. Bruckman, *The Generalized Totient Function*, (Problem #6446, solution by O. P. Lossers), Amer. Math. Monthly **92**(6) (1985), 434–435.
[5] T. W. Cusick, *Recurrences for Sums of Powers of Binomials*, J. Combin. Theory Ser. A **52** (1989), 77–83.
[6] L. E. Dickson, *History of the Theory of Numbers*, (Volume 1: Divisibility and Primality), Chelsea Publ. Comp., New York, 1952.
[7] J. Faulhaber, *Academia Algebrae*, Augspurg, bey Johann Ulrich Schönigs, 1631.
[8] J. Franel, *On a Question of Laisant*, L'Intermédiaire des Math. **1** (1894), 45–47.
[9] J. Franel, *On a Question of J. Franel*, L'Intermédiaire des Math. **2** (1895), 33–35.
[10] R. Golombek, D. Marburg, *Aufgabe 1088, Summen mit Quadraten von Binomialkoeffizienten*, El. Math. **50** (1995), 125–131.
[11] M. Goubi, *An Affirmative Answer to Two Questions Concerning Special Case of Simsek Numbers and Open Problems*, Appl. Anal. Discrete Math. **14** (2020), 94–105.
[12] M. Goubi, *Generating Functions for Generalization Simsek Numbers and Their Applications*, Appl. Anal. Discrete Math. **17** (2023), 262–272.
[13] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics* (2nd. ed.), Addison-Wesley, 1994.
[14] N. Kilar, *Asymptotic Expressions and Formulas for Finite Sums of Powers of Binomial Coefficients Involving Special Numbers and Polynomials*, J. Inequal. Spec. Funct. **14**(1) (2023), 51–67.
[15] T. Kim, D. S. Kim, J. Kwon, *Analogues of Faulhaber's Formula for Poly-Bernoulli and Type 2 Poly-Bernoulli Polynomials*, Montes Taurus J. Pure Appl. Math. **3**(1) (2021), 1–6.
[16] D. Knuth, *Johann Faulhaber and Sums of Powers*, Math. Comp. **61** (1993), 277–294.
[17] I. Kucukoglu, *Some Relationships Between the Numbers of Lyndon Words and a Certain Class of Combinatorial Numbers Containing Powers of Binomial Coefficients*, Adv. Stud. Contemp. Math. **30**(4) (2020), 529–538.
[18] I. Kucukoglu, *Computational and Implementational Analysis of Generating Functions for Higher Order Combinatorial Numbers and Polynomials Attached to Dirichlet Characters*, Math. Meth. Appl. Sci. **45** (2022), 5043–5066.
[19] I. Kucukoglu, *Computational Implementations for Analyzing the q-analogues of a Combinatorial Number Family by Their Derivative Formulas, Interpolation Functions and p-adic q-integrals*, Authorea, 2023; doi: 10.22541/au.168435038.81682594/v1.
[20] I. Kucukoglu, Y. Simsek, *Observations on Identities and Relations for Interpolation Functions and Special Numbers*, Adv. Stud. Contemp. Math. **28**(1) (2018), 41–56.
[21] I. Kucukoglu, Y. Simsek, *Identities and Derivative Formulas for the Combinatorial and Apostol-Euler Type Numbers by Their Generating Functions*, Filomat **32**(20) (2018), 6879–6891.
[22] J. Liouville, *Sur l'expression $\varphi(n)$, qui marque combien la suite $1, 2, 3, \ldots, n$ contient de nombres premiers á n*, J. de Math. **2** (1857), 110–112.
[23] V. H. Moll, *Numbers and Functions: from a Classical-experimental Mathematician's Point of View*, Student Mathematical Library (Volume 65), American Mathematical Society, Providence, Rhode Island, 2012.
[24] J. Sándor, *A Note Concerning the Euler Totient*, Available at `https://rgmia.org/papers/v12n3/art51.pdf`, (Accession date: 14 July, 2023).
[25] J. Sándor, B. Crstici, *Handbook of Number Theory* II, (Chapter 3: The many facets of Euler's totient), Kluwer Academic Publishers, Dordrecht, 2004.
[26] Y. Simsek, *Computation Methods for Combinatorial Sums and Euler-type Numbers Related to New Families of Numbers*, Math. Methods Appl. Sci. **40**(7) (2017), 2347–2361.
[27] Y. Simsek, *Identities and Relations Related to Combinatorial Numbers and Polynomials*, Proc. Jangjeon Math. Soc. **20**(1) (2017), 127–135.

[28] Y. Simsek, *New Families of Special Numbers for Computing Negative Order Euler Numbers and Related Numbers and Polynomials*, Appl. Anal. Discrete Math. **12**(1) (2018), 1–35.
[29] Y. Simsek, *Generating Functions for Finite Sums Involving Higher Powers of Binomial Coefficients: Analysis of Hypergeometric Functions Including New Families of Polynomials and Numbers*, J. Math. Anal. Appl. **477**(2) (2019), 1328–1352.
[30] Y. Simsek, *Some New Families of Special Polynomials and Numbers Associated with Finite Operators*, Symmetry **12**(2) (2020), Article ID: 237; doi:10.3390/sym12020237
[31] Y. Simsek, *On Boole-type Combinatorial Numbers and Polynomials*, Filomat **34**(2) (2020), 559–565.
[32] Y. Simsek, *Relations Between a Certain Combinatorial Numbers and Fibonacci Numbers*, Proceedings Book of MICOPAM 2020-2021, ISBN: 978-625-00-0397-8; pp. 23–25, 2021.
[33] Y. Simsek, *Some Classes of Finite Sums Related to the Generalized Harmonic Functions and Special Numbers and Polynomials*, Montes Taurus J. Pure Appl. Math. **4**(3) (2022), 61–79.
[34] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, Sequence: OEIS: A056188.
[35] H. M. Srivastava, J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.
[36] J. J. Sylvester, *On Certain Ternary Cubic-Form Equations*, (Excursus A. On the Divisors of Cyclotomic Functions), Amer. J. Math. **2**(4) (1879), 357–393.
[37] L. Tóth, *Weighted Gcd-Sum Functions*, J. Integer Seq. **14** (2011), Article 11.7.7.
[38] Wolfram Research, Inc., *Wolfram Cloud*, Champaign, IL, 2022; `https://www.wolframcloud.com`.
[39] A. Xu, *On an Open Problem of Simsek Concerning the Computation of a Family of Special Numbers*, Appl. Anal. Discrete Math. **13**(1) (2019), 061–072.