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On generalized Milne type inequalities for new conformable fractional integrals

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Abstract. In this study, we first obtained a new identity for differentiable convex functions with the help of new conformable fractional integrals. Then, using this identity, we proved new Milne-type inequalities for new conformable fractional integrals. In the proofs, we used convexity, Hölder's inequality and mean power inequality, respectively. In other chapters, we have presented new inequalities for bounded functions, Lipschitzian Functions and functions of bounded variation. The findings of this article are reduced to previously established results in specific cases.

1. Introduction

A formal definition for convex function may be stated as follows:

Definition 1.1. *[6]Let I be convex set on $\mathbb{R}$. The function $\mathfrak{F}: I \to \mathbb{R}$ is called convex on I, if it satisfies the following inequality:*

$$\mathfrak{F}(\vartheta \upsilon + (1-\vartheta)\,\gamma) \leq \vartheta \mathfrak{F}(\upsilon) + (1-\vartheta)\mathfrak{F}(\gamma) \tag{1}$$

for all $(\upsilon, \gamma) \in I$ and $\vartheta \in [0,1]$. The mapping $\mathfrak{F}$ is a concave on I if the inequality (1) holds in reversed direction for all $\vartheta \in [0,1]$ and $\upsilon, \gamma \in I$.

Over few years, the fractional calculus has attracted the attention of many researchers due to its has wide applications in pure and applied mathematics [5, 37, 38]. Like ordinary calculus, the fractional integral and derivative have not unique representation, with the passage of time, different authors have different representations. It is well-known that inequality is an indispensable research object in mathematics, it can give explicit error bounds for some known and some new quadrature formulae, for example, the Simpson's inequality [7–9, 12, 13, 26, 30, 31], Jensen's inequality [17, 18, 21], Hermite-Hadamard's inequality [14, 27, 32–35] and integral inequalities [3, 4, 11, 22, 24, 25, 28, 29, 36, 40].

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In terms of Newton-Cotes formulas, the Milne's formula which is of open type is parallel to the Simpson's formula which is of closed type, since they are held under the same conditions. Suppose that $\mathfrak{F} : [\kappa_1, \kappa_2] \to \mathbb{R}$ is a four times continuously differentiable mapping on (κ_1, κ_2), and let $\left\|\mathfrak{F}^{(4)}\right\|_\infty = \sup_{\upsilon \in (\kappa_1, \kappa_2)} \left|\mathfrak{F}^{(4)}(\upsilon)\right| < \infty$. Then, one has the inequality [1]

$$\left|\frac{1}{3}\left[2\mathfrak{F}(\kappa_1) - \mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + 2\mathfrak{F}(\kappa_2)\right] - \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathfrak{F}(\upsilon)d\upsilon\right| \leq \frac{7(\kappa_2-\kappa_1)^4}{23040}\left\|\mathfrak{F}^{(4)}\right\|_\infty. \tag{2}$$

In this paper we will obtain fractional version of left hand side of (2) and we will consider several new bounds by using several mapping classes.

Fractional analysis is an area that is constantly developing and trying to renew itself to produce solutions to the changing world and problems. Many fractional derivative and integral operators have been defined since the start of fractional analysis. Some of these operators, each of whom has an important place in problem solving in applied mathematics and analysis: Riemann-Liouville, conformable fractional integral operators, Caputo, Hadamard, Erdelyi-Kober, Marchaud and Riesz are just a few to name. In fractional calculus, the fractional derivatives are defined via fractional integrals. Among others, an important and useful fractional integral operator is called Riemann-Liouville fractional integrals that can be defined as the following.

Definition 1.2. *Let $\mathfrak{F} \in L_1[\kappa_1, \kappa_2]$. The Riemann-Liouville fractional integrals $\mathfrak{I}^\alpha_{\kappa_1+}\mathfrak{F}$ and $\mathfrak{I}^\alpha_{\kappa_2-}\mathfrak{F}$ of order $\alpha > 0$ are defined by*

$$\mathfrak{I}^\alpha_{\kappa_1+}\mathfrak{F}(\upsilon) = \frac{1}{\Gamma(\alpha)}\int_{\kappa_1}^{\upsilon} (\upsilon-\vartheta)^{\alpha-1}\mathfrak{F}(\vartheta)d\vartheta, \quad \upsilon > \kappa_1 \tag{3}$$

and

$$\mathfrak{I}^\alpha_{\kappa_2-}\mathfrak{F}(\upsilon) = \frac{1}{\Gamma(\alpha)}\int_{\upsilon}^{\kappa_2} (\vartheta-\upsilon)^{\alpha-1}\mathfrak{F}(\vartheta)d\vartheta, \quad \upsilon < \kappa_2, \tag{4}$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $\mathfrak{I}^0_{\kappa_1+}\mathfrak{F}(\upsilon) = \mathfrak{I}^0_{\kappa_2-}\mathfrak{F}(\upsilon) = \mathfrak{F}(\upsilon)$.

For more information about Riemann-Liouville fractional integrals, please refer to [10, 19, 23].

We recall Beta function (see, e.g., [39, Section 1.1])

$$B(\alpha, \beta) = \begin{cases} \int_0^1 \vartheta^{\alpha-1}(1-\vartheta)^{\beta-1}\,d\vartheta & (\Re(\alpha) > 0;\ \Re(\beta) > 0) \\ \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)} & \left(\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-\right). \end{cases} \tag{5}$$

and the incomplete gamma function, defined for real numbers $a > 0$ and $x \geq 0$ by

$$\Gamma(a, x) = \int_x^\infty e^{-\vartheta}\vartheta^{a-1}d\vartheta.$$

Jarad et. al. [15] has defined a new fractional integral operator. Also, they gave some properties and relations between the some other fractional integral operators, as Riemann-Liouville fractional integral, Hadamard fractional integrals, generalized fractional integral operators, with this operator.

Let $\beta \in \mathbb{C}$, $Re(\beta) > 0$, then the left and right sided fractional conformable integral operators has defined respectively, as follows;

$$^{\beta}\mathfrak{I}^\alpha_{\kappa_1^+}\mathfrak{F}(x) = \frac{1}{\Gamma(\beta)}\int_{\kappa_1}^{x}\left(\frac{(x-\kappa_1)^\alpha - (\vartheta-\kappa_1)^\alpha}{\alpha}\right)^{\beta-1}\frac{\mathfrak{F}(\vartheta)}{(\vartheta-\kappa_1)^{1-\alpha}}d\vartheta \tag{6}$$

$$ {}^{\beta}\mathfrak{J}_{\kappa_2^-}^{\alpha}\mathfrak{F}(x) = \frac{1}{\Gamma(\beta)}\int_x^{\kappa_2}\left(\frac{(\kappa_2-x)^{\alpha}-(\kappa_2-\vartheta)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathfrak{F}(\vartheta)}{(\kappa_2-\vartheta)^{1-\alpha}}d\vartheta. \tag{7} $$

The fractional integral in (6) coincides with the Riemann-Liouville fractional integral (3) when $\kappa_1 = 0$ and $\alpha = 1$. It also coincides with the Hadamard fractional integral [20] once $\kappa_1 = 0$ and $\alpha \to 0$ with the Katugampola fractional integral [16], when $\kappa_1 = 0$. Similarly, Notice that, $(Q\mathfrak{F})(t) = f(\kappa_1+\kappa_2-t)$ then we have ${}^{\beta}\mathfrak{J}_{\kappa_1^+}^{\alpha}\mathfrak{F}(x) = Q({}^{\beta}\mathfrak{J}_{\kappa_2^+}^{\alpha})\mathfrak{F}(x)$. Moreover (7) coincides with the Riemann-Liouville fractional integral (4), when $\kappa_2 = 0$ and $\alpha = 1$. It also coincides with the Hadamard fractional integral [20] once $\kappa_2 = 0$ and $\alpha \to 0$ with the Katugampola fractional integral [16], when $\kappa_2 = 0$. Further, getting more knowledge, see the paper given in [15].

With the help of ongoing work and the articles cited above, we will prove several Milne-type inequalities for the case of differentiable convex functions, including new conformable fractional integrals. The entire study consists of six chapters, including introduction and preliminary information. In Chapter 2, an identity will be established for convex functions differentiable with respect to new conformable fractional integrals. Using this identity, Milne-type inequalities will be given for convex functions with the help of new conformable fractional integrals. Then, in Chapter 3, Chapter 4 and Chapter 5, Milne-type inequality for new conformable fractional integrals containing bounded functions, Milne-type inequality for new conformable fractional integrals containing Lipschitzian functions, and Milne-type inequality for new conformable fractional integrals involving functions of bounded variation are presented, respectively. Finally, summary and concluding notes are given in Chapter 6.

2. Milne Type Inequalities for Differential Convex Functions

In this part, we present a few of the inequalities of the Milne type for differentiable convex mappings.

Lemma 2.1. *Let $\mathfrak{F} : [\kappa_1,\kappa_2] \to \mathbb{R}$ be an differentiable mapping (κ_1,κ_2) such that $\mathfrak{F}' \in L_1([\kappa_1,\kappa_2])$. Then, the following equality holds:*

$$ \begin{aligned} &\frac{1}{3\alpha^{\beta}}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{J}_{\kappa_1^+}^{\alpha}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}^{\beta}\mathfrak{J}_{\kappa_2^-}^{\alpha}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right] \\ =\;& \frac{\kappa_2-\kappa_1}{4}\int_0^1\left[\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}+\frac{1}{3\alpha^{\beta}}\right]\left[\mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1+\left(\frac{1+\vartheta}{2}\right)\kappa_2\right)-\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1+\left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right]d\vartheta \end{aligned} $$

where $\alpha,\beta > 0$, $B(x,y)$ and Γ are Euler Gamma functions, respectively.

Proof. By utilizing integration by parts, we have

$$ \begin{aligned} I_1 &= \int_0^1\left[\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}+\frac{1}{3\alpha^{\beta}}\right]\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1+\left(\frac{1-\vartheta}{2}\right)\kappa_2\right)d\vartheta \qquad (8) \\ &= -\frac{2}{\kappa_2-\kappa_1}\left[\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}+\frac{1}{3\alpha^{\beta}}\right]\mathfrak{F}\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1+\left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\Bigg|_0^1 \\ &\quad+\frac{2\beta}{\kappa_2-\kappa_1}\int_0^1\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta-1}(1-\vartheta)^{\alpha-1}\mathfrak{F}\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1+\left(\frac{1-\vartheta}{2}\right)\kappa_2\right)d\vartheta \end{aligned} $$

$$\begin{aligned}
&= \frac{2}{3\alpha^{\beta}(\kappa_2-\kappa_1)}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{8}{3\alpha^{\beta}(\kappa_2-\kappa_1)}\mathfrak{F}(\kappa_1)\\
&\quad+\left(\frac{2}{\kappa_2-\kappa_1}\right)^2\beta\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{1-\left(\frac{2}{\kappa_2-\kappa_1}\right)^{\alpha}(x-\kappa_1)^{\alpha}}{\alpha}\right)^{\beta-1}\left(\frac{2}{\kappa_2-\kappa_1}\right)^{\alpha-1}(v-\kappa_1)^{\alpha-1}\mathfrak{F}(v)dv\\
&= \frac{2}{3\alpha^{\beta}(\kappa_2-\kappa_1)}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{8}{3\alpha^{\beta}(\kappa_2-\kappa_1)}\mathfrak{F}(\kappa_1)\\
&\quad+\left(\frac{2}{\kappa_2-\kappa_1}\right)^{\alpha\beta+1}\beta\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\frac{\left(\frac{\kappa_2-\kappa_1}{2}\right)^{\alpha}-(v-\kappa_1)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\mathfrak{F}(v)}{(v-\kappa_1)^{1-\alpha}}dv\\
&= \frac{2}{3\alpha^{\beta}(\kappa_2-\kappa_1)}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\frac{8}{3\alpha^{\beta}(\kappa_2-\kappa_1)}\mathfrak{F}(\kappa_1)+\left(\frac{2}{\kappa_2-\kappa_1}\right)^{\alpha\beta+1}\Gamma(\beta+1)\,{}^{\beta}\mathfrak{I}_{\kappa_1^+}^{\alpha}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
I_2 &= \int_0^1\left[\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}+\frac{1}{3\alpha^{\beta}}\right]\mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1+\left(\frac{1+\vartheta}{2}\right)\kappa_2\right)d\vartheta \qquad (9)\\
&= -\frac{2}{3\alpha^{\beta}(\kappa_2-\kappa_1)}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\frac{8}{3\alpha^{\beta}(\kappa_2-\kappa_1)}\mathfrak{F}(\kappa_2)-\left(\frac{2}{\kappa_2-\kappa_1}\right)^{\alpha\beta+1}\Gamma(\beta+1)\,{}^{\beta}\mathfrak{I}_{\kappa_2^-}^{\alpha}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right).
\end{aligned}$$

From the equalities (8) and (9), the following result is obtained:

$$\begin{aligned}
\frac{\kappa_2-\kappa_1}{4}[I_2-I_1] &= \frac{1}{3\alpha^{\beta}}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]\\
&\quad-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{I}_{\kappa_1^+}^{\alpha}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}^{\beta}\mathfrak{I}_{\kappa_2^-}^{\alpha}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right].
\end{aligned}$$

The proof of Lemma 2.1 is completed. □

Theorem 2.2. *Assume that the assumptions of Lemma 2.1 hold. Let the function $|\mathfrak{F}'|$ be a convex function on $[\kappa_1, \kappa_2]$. Then, we get the following inequality*

$$\begin{aligned}
&\left|\frac{1}{3\alpha^{\beta}}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{I}_{\kappa_1^+}^{\alpha}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}^{\beta}\mathfrak{I}_{\kappa_2^-}^{\alpha}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
&\leq \frac{\kappa_2-\kappa_1}{12}\left(\frac{3B\left(\beta+1,\frac{1}{\alpha}\right)+\alpha}{\alpha^{\beta+1}}\right)\left(|\mathfrak{F}'(\kappa_1)|+|\mathfrak{F}'(\kappa_2)|\right) \qquad (10)
\end{aligned}$$

where $\alpha, \beta > 0$, $B(x, y)$ and Γ are Euler Beta and Gamma functions, respectively.

Proof. By taking absolute value in Lemma 2.1 and utilizing the convexity of $|\mathfrak{F}'|$, we get

$$\begin{aligned}
&\left|\frac{1}{3\alpha^{\beta}}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{J}^{\alpha}_{\kappa_1^+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}^{\beta}\mathfrak{J}^{\alpha}_{\kappa_2^-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
\leq\;& \frac{\kappa_2-\kappa_1}{4}\int_0^1\left|\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}+\frac{1}{3\alpha^{\beta}}\right|\left[\left|\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1+\left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right|+\left|\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_2+\left(\frac{1-\vartheta}{2}\right)\kappa_1\right)\right|\right]d\vartheta\\
\leq\;& \frac{\kappa_2-\kappa_1}{4}\int_0^1\left[\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}+\frac{1}{3\alpha^{\beta}}\right]\left[\frac{1+\vartheta}{2}|\mathfrak{F}'(\kappa_1)|+\frac{1-\vartheta}{2}|\mathfrak{F}'(\kappa_2)|+\frac{1+\vartheta}{2}|\mathfrak{F}'(\kappa_2)|+\frac{1-\vartheta}{2}|\mathfrak{F}'(\kappa_1)|\right]d\vartheta\\
=\;& \frac{\kappa_2-\kappa_1}{4}\left(\frac{B\left(\beta+1,\frac{1}{\alpha}\right)}{\alpha^{\beta+1}}+\frac{1}{3\alpha^{\beta}}\right)(|\mathfrak{F}'(\kappa_1)|+|\mathfrak{F}'(\kappa_2)|)\\
=\;& \frac{\kappa_2-\kappa_1}{12}\left(\frac{3B\left(\beta+1,\frac{1}{\alpha}\right)+\alpha}{\alpha^{\beta+1}}\right)(|\mathfrak{F}'(\kappa_1)|+|\mathfrak{F}'(\kappa_2)|),
\end{aligned}$$

where it is easily seen that

$$\int_0^1\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}\left(\frac{1+\theta}{2}\right)d\theta=\frac{B\left(\beta+1,\frac{1}{\alpha}\right)}{\alpha^{\beta+1}}-\frac{B\left(\beta+1,\frac{2}{\alpha}\right)}{2\alpha^{\beta+1}}$$

and

$$\int_0^1\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}\left(\frac{1-\theta}{2}\right)d\theta=\frac{B\left(\beta+1,\frac{2}{\alpha}\right)}{2\alpha^{\beta+1}}.$$

So we get the desired result. □

Example 2.3. *Let consider the function* $\mathfrak{F}:[1,3]\to\mathbb{R}$, $\mathfrak{F}(t)=\frac{t^3}{3}$. *It is clear that* $|\mathfrak{F}'|$ *is convex on* $[1,3]$ *Then we have*

$$2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)=16$$

By (6), we have

$$\begin{aligned}
{}^{\beta}\mathfrak{J}^{\alpha}_{\kappa_1^+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) &= {}^{\beta}\mathfrak{J}^{\alpha}_{1^+}\mathfrak{F}(2)=\frac{1}{\Gamma(\beta)}\int_1^2\left(\frac{1-(t-1)^{\alpha}}{\alpha}\right)^{\beta-1}(t-1)^{\alpha-1}\frac{t^3}{3}dt\\
&= \frac{\alpha}{3\alpha^{\beta}\Gamma(\beta)}\int_1^2(1-(t-1)^{\alpha})^{\beta-1}\left[(t-1)^{\alpha+2}+3(t-1)^{\alpha+1}+3(t-1)^{\alpha}+(t-1)^{\alpha-1}\right]dt\\
&= \frac{1}{3\alpha^{\beta}\Gamma(\beta)}\int_0^1 u^{\beta-1}\left[(1-u)^{\frac{3}{\alpha}}+3(1-u)^{\frac{2}{\alpha}}+3(1-u)^{\frac{1}{\alpha}}+1\right]du\\
&= \frac{1}{3\alpha^{\beta}\Gamma(\beta)}\left[\mathcal{B}\left(\beta,\frac{3}{\alpha}+1\right)+3\mathcal{B}\left(\beta,\frac{2}{\alpha}+1\right)+3\mathcal{B}\left(\beta,\frac{3}{\alpha}+1\right)+\frac{1}{\beta}\right]
\end{aligned}$$

and similarly by (7), we have

$${}^{\beta}\mathfrak{J}^{\alpha}_{\kappa_2^-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)={}^{\beta}\mathfrak{J}^{\alpha}_{3}\mathfrak{F}(2)=\frac{1}{3\alpha^{\beta}\Gamma(\beta)}\left[\frac{27}{\beta}-27\mathcal{B}\left(\beta,\frac{1}{\alpha}+1\right)+9\mathcal{B}\left(\beta,\frac{2}{\alpha}+1\right)-\mathcal{B}\left(\beta,\frac{1}{\alpha}+1\right)\right].$$

Thus, the left term of the inequality (10) can be calculated as

$$\begin{aligned}
&\left|\frac{1}{3\alpha^{\beta}}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{I}^{\alpha}_{\kappa_1^+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}^{\beta}\mathfrak{I}^{\alpha}_{\kappa_2^-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
=&\left|\frac{1}{3\alpha^{\beta}}16-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)}{3\alpha^{\beta}\Gamma(\beta)}\left[12\mathcal{B}\left(\beta,\frac{2}{\alpha}+1\right)-24\mathcal{B}\left(\beta,\frac{1}{\alpha}+1\right)+\frac{28}{\beta}\right]\right|\\
=&\left|\frac{16}{3\alpha^{\beta}}-\frac{2\beta}{3\alpha^{\beta}}\left[3\mathcal{B}\left(\beta,\frac{2}{\alpha}+1\right)-6\mathcal{B}\left(\beta,\frac{1}{\alpha}+1\right)+\frac{7}{\beta}\right]\right|.
\end{aligned}$$

On the other hand, we have the the right term on the inequality (10) as

$$\begin{aligned}
&\frac{\kappa_2-\kappa_1}{12}\left(\frac{3B\left(\beta+1,\frac{1}{\alpha}\right)+\alpha}{\alpha^{\beta+1}}\right)\left(|\mathfrak{F}'(\kappa_1)|+|\mathfrak{F}'(\kappa_2)|\right)\\
=&\frac{2}{12}\left(\frac{3B\left(\beta+1,\frac{1}{\alpha}\right)+\alpha}{\alpha^{\beta+1}}\right)(1+9)\\
=&\frac{5}{3\alpha^{\beta}}\left(\frac{1}{\alpha}3B\left(\beta+1,\frac{1}{\alpha}\right)+1\right).
\end{aligned}$$

Consequently, we have the following inequality from (10)

$$\left|\frac{16}{3}-\frac{2\beta}{3}\left[3\mathcal{B}\left(\beta,\frac{2}{\alpha}+1\right)-6\mathcal{B}\left(\beta,\frac{1}{\alpha}+1\right)+\frac{7}{\beta}\right]\right|\leq\frac{5}{3}\left(\frac{1}{\alpha}3B\left(\beta+1,\frac{1}{\alpha}\right)+1\right). \tag{11}$$

One can see the validity of the inequality (11) in Figure 1.

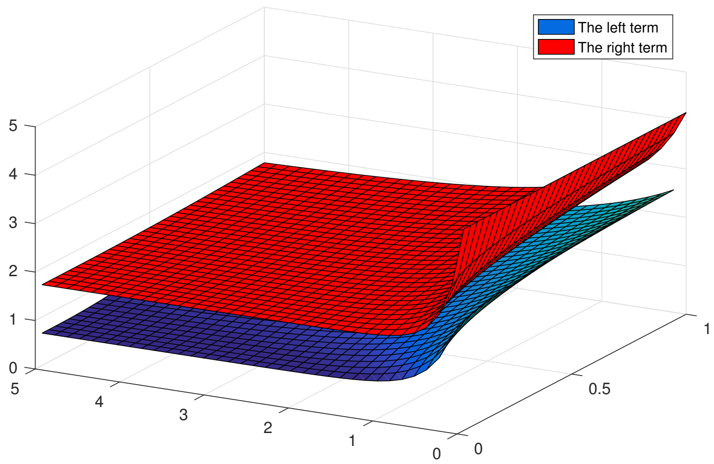

Figure 1: *An example to Theorem 2.2, depending on $\alpha \in (0,1]$ and $\beta \in (0,2]$, computed and plotted by MATLAB.*

Theorem 2.4. *Suppose that the assumptions of Lemma 2.1 hold. Suppose also that the mapping $|\mathfrak{F}'|^q$, $q>1$ is convex on $[\kappa_1,\kappa_2]$. Then, the following inequality*

$$\begin{aligned}
&\left|\frac{1}{3\alpha^{\beta}}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]-\frac{2^{\alpha\beta-1}\Gamma(\alpha+1)}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{J}^{\alpha}_{\kappa_1+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}^{\beta}\mathfrak{J}^{\alpha}_{\kappa_2^-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
\le\;& \frac{\kappa_2-\kappa_1}{4}\left(\frac{B\left(\beta p+1,\frac{1}{\alpha}\right)}{\alpha^{\beta p+1}}+\frac{1}{3^p\alpha^{\beta p}}\right)^{\frac{1}{p}}\left[\left(\frac{3|\mathfrak{F}'(\kappa_1)|^q+|\mathfrak{F}'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|\mathfrak{F}'(\kappa_1)|^q+3|\mathfrak{F}'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}\right]\\
\le\;& \frac{\kappa_2-\kappa_1}{4^{\frac{1}{q}}}\left(\frac{B\left(\beta p+1,\frac{1}{\alpha}\right)}{\alpha^{\beta p+1}}+\frac{1}{3^p\alpha^{\beta p}}\right)^{\frac{1}{p}}\left(|\mathfrak{F}'(\kappa_1)|+|\mathfrak{F}'(\kappa_2)|\right), \qquad (12)
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$, $\alpha,\beta>0$, $B(x,y)$ and Γ are Euler Beta and Gamma functions, respectively.

Proof. If the absolute value of Lemma 2.1 is taken, we get

$$\begin{aligned}
&\left|\frac{1}{3\alpha^{\beta}}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{J}^{\alpha}_{\kappa_1^+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}^{\beta}\mathfrak{J}^{\alpha}_{\kappa_2^-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
\le\;& \frac{\kappa_2-\kappa_1}{4}\left[\int_0^1\left[\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}+\frac{1}{3\alpha^{\beta}}\right]\left|\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1+\left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right|d\vartheta\right.\\
&\left.+\int_0^1\left[\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}+\frac{1}{3\alpha^{\beta}}\right]\left|\mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1+\left(\frac{1+\vartheta}{2}\right)\kappa_2\right)\right|d\vartheta\right]. \qquad (13)
\end{aligned}$$

With help of Hölder inequality in the inequality (13) and by utilizing convexity of $|\mathfrak{F}'|^q$, we get

$$\begin{aligned}
&\int_0^1\left[\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}+\frac{1}{3\alpha^{\beta}}\right]\left|\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1+\left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right|d\vartheta \qquad (14)\\
\le\;& \left(\int_0^1\left[\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}+\frac{1}{3\alpha^{\beta}}\right]^p d\vartheta\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1+\left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right|^q d\vartheta\right)^{\frac{1}{q}}\\
\le\;& \left(\int_0^1\left[\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}+\frac{1}{3\alpha^{\beta}}\right]^p d\vartheta\right)^{\frac{1}{p}}\left[\int_0^1\left(\frac{1+\vartheta}{2}|\mathfrak{F}'(\kappa_1)|^q+\frac{1-\vartheta}{2}|\mathfrak{F}'(\kappa_2)|^q\right)d\vartheta\right]^{\frac{1}{q}}\\
=\;& \left(\frac{B\left(\beta p+1,\frac{1}{\alpha}\right)}{\alpha^{\beta p+1}}+\frac{1}{3^p\alpha^{\beta p}}\right)^{\frac{1}{p}}\left(\frac{3|\mathfrak{F}'(\kappa_1)|^q+|\mathfrak{F}'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}.
\end{aligned}$$

Similarly, we have the inequality

$$\begin{aligned}
&\int_0^1\left[\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}+\frac{1}{3\alpha^{\beta}}\right]\left|\mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1+\left(\frac{1+\vartheta}{2}\right)\kappa_2\right)\right|d\vartheta \qquad (15)\\
\le\;& \left(\frac{B\left(\beta p+1,\frac{1}{\alpha}\right)}{\alpha^{\beta p+1}}+\frac{1}{3^p\alpha^{\beta p}}\right)^{\frac{1}{p}}\left(\frac{|\mathfrak{F}'(\kappa_1)|^q+3|\mathfrak{F}'(\kappa_2)|^q}{4}\right)^{\frac{1}{q}}.
\end{aligned}$$

By substituting (14) and (15) in (13), we have

$$\begin{aligned}&\left|\frac{1}{3\alpha^{\beta}}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{I}^{\alpha}_{\kappa_1^+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}^{\beta}\mathfrak{I}^{\alpha}_{\kappa_2^-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\ \leq\;& \frac{\kappa_2-\kappa_1}{4}\left(\frac{B\left(\beta p+1,\frac{1}{\alpha}\right)}{\alpha^{\beta p+1}}+\frac{1}{3^p\alpha^{\beta p}}\right)^{\frac{1}{p}}\left[\left(\frac{3\left|\mathfrak{F}'(\kappa_1)\right|^q+\left|\mathfrak{F}'(\kappa_2)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathfrak{F}'(\kappa_1)\right|^q+3\left|\mathfrak{F}'(\kappa_2)\right|^q}{4}\right)^{\frac{1}{q}}\right].\end{aligned}$$

The first inequality of the (12) is completed. For the proof of second inequality, let $a_1 = 3\left|\mathfrak{F}'(\kappa_1)\right|^q$, $b_1 = \left|\mathfrak{F}'(\kappa_2)\right|^q$, $a_2 = \left|\mathfrak{F}'(\kappa_1)\right|^q$ and $b_2 = 3\left|\mathfrak{F}'(\kappa_2)\right|^q$. Using the facts that,

$$\sum_{k=1}^{n}(a_k+b_k)^s \leq \sum_{k=1}^{n}a_k^s+\sum_{k=1}^{n}b_k^s,\ 0\leq s<1$$

and $1+3^{\frac{1}{q}}\leq 4$, then the required result can be established directly. The proof of Theorem 2.4 is finished. □

Example 2.5. *Let us consider the same function given in Example 2.3 with $p = q = 2$. Then, the left-hand side of* (12) *reduces to*

$$\begin{aligned}&\left|\frac{1}{3\alpha^{\beta}}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{I}^{\alpha}_{\kappa_1^+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}^{\beta}\mathfrak{I}^{\alpha}_{\kappa_2^-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\ =\;& \left|\frac{16}{3\alpha^{\beta}}-\frac{2\beta}{3\alpha^{\beta}}\left[3\mathcal{B}\left(\beta,\frac{2}{\alpha}+1\right)-6\mathcal{B}\left(\beta,\frac{1}{\alpha}+1\right)+\frac{7}{\beta}\right]\right|.\end{aligned}$$

On the other hand, we have the the mid term and right term on the inequality (12) as

$$\begin{aligned}&\frac{\kappa_2-\kappa_1}{4}\left(\frac{B\left(\beta p+1,\frac{1}{\alpha}\right)}{\alpha^{\beta p+1}}+\frac{1}{3^p\alpha^{\beta p}}\right)^{\frac{1}{p}}\left[\left(\frac{3\left|\mathfrak{F}'(\kappa_1)\right|^q+\left|\mathfrak{F}'(\kappa_2)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathfrak{F}'(\kappa_1)\right|^q+3\left|\mathfrak{F}'(\kappa_2)\right|^q}{4}\right)^{\frac{1}{q}}\right]\\ =\;& \frac{2}{4}\left(\frac{B\left(\beta p+1,\frac{1}{\alpha}\right)}{\alpha^{2\beta+1}}+\frac{1}{3^2\alpha^{2\beta}}\right)^{\frac{1}{2}}\left[\left(\frac{3+9}{4}\right)^{\frac{1}{2}}+\left(\frac{1+27}{4}\right)^{\frac{1}{2}}\right]\\ =\;& \frac{\sqrt{3}+\sqrt{7}}{3\alpha^{\beta}}\left(\frac{9}{\alpha}B\left(2\beta+1,\frac{1}{\alpha}\right)+1\right)^{\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}&\frac{\kappa_2-\kappa_1}{4^{\frac{1}{q}}}\left(\frac{B\left(\beta p+1,\frac{1}{\alpha}\right)}{\alpha^{\beta p+1}}+\frac{1}{3^p\alpha^{\beta p}}\right)^{\frac{1}{p}}\left(\left|\mathfrak{F}'(\kappa_1)\right|+\left|\mathfrak{F}'(\kappa_2)\right|\right)\\ =\;& \frac{2}{4^{\frac{1}{2}}}\left(\frac{B\left(2\beta+1,\frac{1}{\alpha}\right)}{\alpha^{2\beta+1}}+\frac{1}{3^2\alpha^{2\beta}}\right)^{\frac{1}{2}}(1+9)\\ =\;& \frac{10}{3\alpha^{\beta}}\left(\frac{1}{\alpha}9B\left(2\beta+1,\frac{1}{\alpha}\right)+1\right)^{\frac{1}{2}}.\end{aligned}$$

respectively. Consequently, we have the following inequality from (12)

$$\begin{aligned}&\left|\frac{16}{3}-\frac{2\beta}{3}\left[3\mathcal{B}\left(\beta,\frac{2}{\alpha}+1\right)-6\mathcal{B}\left(\beta,\frac{1}{\alpha}+1\right)+\frac{7}{\beta}\right]\right| \qquad (16)\\ \leq\ & \frac{\sqrt{3}+\sqrt{7}}{3\alpha^{\beta}}\left(\frac{9}{\alpha}B\left(2\beta+1,\frac{1}{\alpha}\right)+1\right)^{\frac{1}{2}}\\ \leq\ & \frac{10}{3\alpha^{\beta}}\left(\frac{1}{\alpha}9B\left(2\beta+1,\frac{1}{\alpha}\right)+1\right)^{\frac{1}{2}}.\end{aligned}$$

One can see the validity of the inequality (16) in Figure 2.

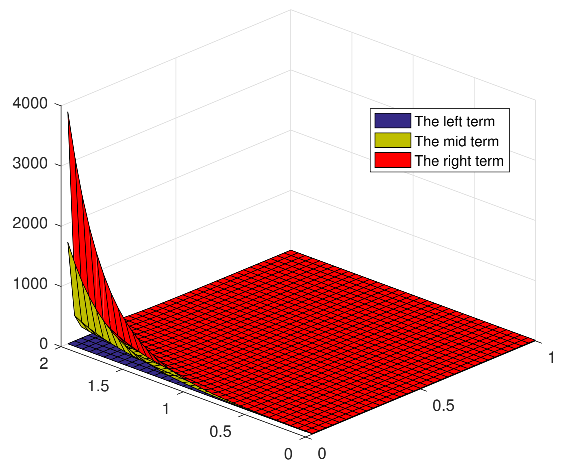

Figure 2: *An example to Theorem 2.4, depending on $\alpha \in (0,1]$ and $\beta \in (0,2]$, computed and plotted by MATLAB.*

Theorem 2.6. *Note that all the assumptions of Lemma 2.1 hold. If the mapping $|\mathfrak{F}'|^q$, $q \geq 1$ is convex on $[\kappa_1,\kappa_2]$, then we get the following inequality*

$$\begin{aligned}&\left|\frac{1}{3\alpha^{\beta}}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{I}^{\alpha}_{\kappa_1^+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}^{\beta}\mathfrak{I}^{\alpha}_{\kappa_2^-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \qquad (17)\\ \leq\ & \frac{\kappa_2-\kappa_1}{4\alpha^{\beta}}\left(\frac{1}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)+\frac{1}{3}\right)^{1-\frac{1}{q}}\\ &\times\left(\left[\left(\frac{1}{4}+\frac{1}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)-\frac{1}{2\alpha}B\left(\beta+1,\frac{2}{\alpha}\right)\right)|\mathfrak{F}'(\kappa_1)|^q+\left(\frac{1}{12}+\frac{1}{2\alpha}B\left(\beta+1,\frac{2}{\alpha}\right)\right)|\mathfrak{F}'(\kappa_2)|^q\right]^{\frac{1}{q}}\right.\\ &\left.+\left[\left(\frac{1}{12}+\frac{1}{2\alpha}B\left(\beta+1,\frac{2}{\alpha}\right)\right)|\mathfrak{F}'(\kappa_1)|^q+\left(\frac{1}{4}+\frac{1}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)-\frac{1}{2\alpha}B\left(\beta+1,\frac{2}{\alpha}\right)\right)|\mathfrak{F}'(\kappa_2)|^q\right]^{\frac{1}{q}}\right)\end{aligned}$$

where $\alpha,\beta > 0$, $B(x,y)$ and Γ are Euler Beta and Gamma functions, respectively.

Proof. With help of the power-mean inequality in (13) and considering the convexity of $|\mathfrak{F}'|^q$, we get

$$\int_0^1 \left[\left(\frac{1-(1-\vartheta)^\alpha}{\alpha}\right)^\beta + \frac{1}{3\alpha^\beta}\right]\left|\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1 + \left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right| d\vartheta \tag{18}$$

$$\leq \left(\int_0^1 \left[\left(\frac{1-(1-\vartheta)^\alpha}{\alpha}\right)^\beta + \frac{1}{3\alpha^\beta}\right] d\vartheta\right)^{1-\frac{1}{q}} \left(\int_0^1 \left[\left(\frac{1-(1-\vartheta)^\alpha}{\alpha}\right)^\beta + \frac{1}{3\alpha^\beta}\right]\left|\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1 + \left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right|^q d\vartheta\right)^{\frac{1}{q}}$$

$$\leq \left(\frac{B\left(\beta+1,\frac{1}{\alpha}\right)}{\alpha^{\beta+1}} + \frac{1}{3\alpha^\beta}\right)^{1-\frac{1}{q}} \left[\int_0^1 \left[\left(\frac{1-(1-\vartheta)^\alpha}{\alpha}\right)^\beta + \frac{1}{3\alpha^\beta}\right]\left(\frac{1+\vartheta}{2}\left|\mathfrak{F}'(\kappa_1)\right|^q + \frac{1-\vartheta}{2}\left|\mathfrak{F}'(\kappa_2)\right|^q\right)^{\frac{1}{q}}\right]$$

$$= \left(\frac{B\left(\beta+1,\frac{1}{\alpha}\right)}{\alpha^{\beta+1}} + \frac{1}{3\alpha^\beta}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\frac{1}{4\alpha^\beta} + \frac{B\left(\beta+1,\frac{1}{\alpha}\right)}{\alpha^{\beta+1}} - \frac{B\left(\beta+1,\frac{2}{\alpha}\right)}{2\alpha^{\beta+1}}\right)\left|\mathfrak{F}'(\kappa_1)\right|^q + \left(\frac{1}{12\alpha^\beta} + \frac{B\left(\beta+1,\frac{2}{\alpha}\right)}{2\alpha^{\beta+1}}\right)\left|\mathfrak{F}'(\kappa_2)\right|^q\right]^{\frac{1}{q}}.$$

By similar method used in (18), we have

$$\int_0^1 \left[\left(\frac{1-(1-\vartheta)^\alpha}{\alpha}\right)^\beta + \frac{1}{3\alpha^\beta}\right]\left|\mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1 + \left(\frac{1+\vartheta}{2}\right)\kappa_2\right)\right| d\vartheta \tag{19}$$

$$\leq \left(\frac{B\left(\beta+1,\frac{1}{\alpha}\right)}{\alpha^{\beta+1}} + \frac{1}{3\alpha^\beta}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\frac{1}{12\alpha^\beta} + \frac{B\left(\beta+1,\frac{2}{\alpha}\right)}{2\alpha^{\beta+1}}\right)\left|\mathfrak{F}'(\kappa_1)\right|^q + \left(\frac{1}{4\alpha^\beta} + \frac{B\left(\beta+1,\frac{1}{\alpha}\right)}{\alpha^{\beta+1}} - \frac{B\left(\beta+1,\frac{2}{\alpha}\right)}{2\alpha^{\beta+1}}\right)\left|\mathfrak{F}'(\kappa_2)\right|^q\right]^{\frac{1}{q}}.$$

Substituting (18) and (19) in (13), then we get

$$\left|\frac{1}{3\alpha^\beta}\left[2\mathfrak{F}(\kappa_1) - \mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + 2\mathfrak{F}(\kappa_2)\right] - \frac{2^{\alpha\beta-1}\Gamma(\beta+1)}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[{}^\beta\mathfrak{I}^\alpha_{\kappa_1^+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + {}^\beta\mathfrak{I}^\alpha_{\kappa_2^-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|$$

$$\leq \frac{\kappa_2-\kappa_1}{4}\left(\frac{B\left(\beta+1,\frac{1}{\alpha}\right)}{\alpha^{\beta+1}} + \frac{1}{3\alpha^\beta}\right)^{1-\frac{1}{q}}$$

$$\times\left(\left[\left(\frac{1}{4\alpha^\beta} + \frac{B\left(\beta+1,\frac{1}{\alpha}\right)}{\alpha^{\beta+1}} - \frac{B\left(\beta+1,\frac{2}{\alpha}\right)}{2\alpha^{\beta+1}}\right)\left|\mathfrak{F}'(\kappa_1)\right|^q + \left(\frac{1}{12\alpha^\beta} + \frac{B\left(\beta+1,\frac{2}{\alpha}\right)}{2\alpha^{\beta+1}}\right)\left|\mathfrak{F}'(\kappa_2)\right|^q\right]^{\frac{1}{q}}\right.$$

$$\left.+\left[\left(\frac{1}{12\alpha^\beta} + \frac{B\left(\beta+1,\frac{2}{\alpha}\right)}{2\alpha^{\beta+1}}\right)\left|\mathfrak{F}'(\kappa_1)\right|^q + \left(\frac{1}{4\alpha^\beta} + \frac{B\left(\beta+1,\frac{1}{\alpha}\right)}{\alpha^{\beta+1}} - \frac{B\left(\beta+1,\frac{2}{\alpha}\right)}{2\alpha^{\beta+1}}\right)\left|\mathfrak{F}'(\kappa_2)\right|^q\right]^{\frac{1}{q}}\right).$$

This completes the proof. □

Example 2.7. *Let us consider the same function given in Example 2.3 with $q = 2$ in. Then, by simple calculations and by the inequality (17), we have the inequality*

$$\begin{aligned}
&\left|\frac{16}{3} - \frac{2\beta}{3}\left[3\mathcal{B}\left(\beta, \frac{2}{\alpha}+1\right) - 6\mathcal{B}\left(\beta, \frac{1}{\alpha}+1\right) + \frac{7}{\beta}\right]\right| \\
\leq\ & \frac{1}{4\alpha^{\beta}}\left(\frac{1}{\alpha}B\left(\beta+1, \frac{1}{\alpha}\right) + \frac{1}{3}\right)^{\frac{1}{2}} \\
&\times\left(\left[\frac{1}{4} + \frac{1}{\alpha}B\left(\beta+1, \frac{1}{\alpha}\right) - \frac{1}{2\alpha}B\left(\beta+1, \frac{2}{\alpha}\right) + 81\left(\frac{1}{12} + \frac{1}{2\alpha}B\left(\beta+1, \frac{2}{\alpha}\right)\right)\right]^{\frac{1}{2}}\right. \\
&\left. + \left[\frac{1}{12} + \frac{1}{2\alpha}B\left(\beta+1, \frac{2}{\alpha}\right) + 81\left(\frac{1}{4} + \frac{1}{\alpha}B\left(\beta+1, \frac{1}{\alpha}\right) - \frac{1}{2\alpha}B\left(\beta+1, \frac{2}{\alpha}\right)\right)\right]^{\frac{1}{2}}\right). \qquad (20)
\end{aligned}$$

One can see the validity of the inequality (20) in Figure 3.

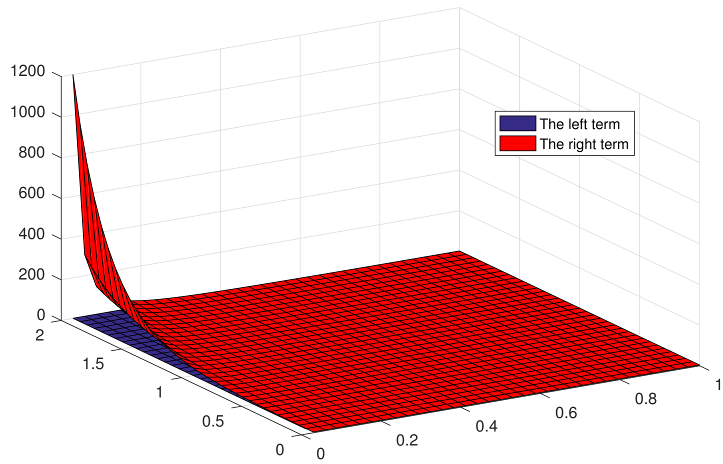

Figure 3: *An example to Theorem 2.6, depending on $\alpha \in (0,1]$ and $\beta \in (0,2]$, computed and plotted by MATLAB.*

3. Milne Type Inequality for Bounded Functions

Theorem 3.1. *Assume that the conditions of Lemma 2.1 hold. If there exist $m, M \in \mathbb{R}$ such that $m \leq \mathfrak{F}'(\vartheta) \leq M$ for $\vartheta \in [\kappa_1, \kappa_2]$, then we establish*

$$\begin{aligned}
&\left|\frac{1}{3\alpha^{\beta}}\left[2\mathfrak{F}(\kappa_1) - \mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + 2\mathfrak{F}(\kappa_2)\right] - \frac{2^{\alpha\beta-1}\Gamma(\beta+1)}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{J}^{\alpha}_{\kappa_1^+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) + {}^{\beta}\mathfrak{J}^{\alpha}_{\kappa_2^-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \qquad (21) \\
\leq\ & \frac{\kappa_2-\kappa_1}{12\alpha^{\beta}}\left(\frac{3}{\alpha}B\left(\beta+1, \frac{1}{\alpha}\right) + 1\right)(M-m)
\end{aligned}$$

where $\alpha, \beta > 0$, $B(x, y)$ and Γ are Euler Beta and Gamma functions, respectively.

Proof. With the help of Lemma 2.1, we get

$$\begin{aligned}
&\left|\frac{1}{3\alpha^{\beta}}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{I}^{\alpha}_{\kappa_1^+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}^{\beta}\mathfrak{I}^{\alpha}_{\kappa_2^-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
=\;&\frac{\kappa_2-\kappa_1}{4}\int_0^1\left[\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}+\frac{1}{3\alpha^{\beta}}\right]\left[\mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1+\left(\frac{1+\vartheta}{2}\right)\kappa_2\right)-\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1+\left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right]d\vartheta\\
=\;&\frac{\kappa_2-\kappa_1}{4}\int_0^1\left[\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}+\frac{1}{3\alpha^{\beta}}\right]\left[\mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1+\left(\frac{1+\vartheta}{2}\right)\kappa_2\right)-\frac{m+M}{2}\right]d\vartheta\\
&+\int_0^1\left[\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}+\frac{1}{3\alpha^{\beta}}\right]\left[\frac{m+M}{2}-\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1+\left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right]d\vartheta. \qquad (22)
\end{aligned}$$

Through the absolute value of (22), we have

$$\begin{aligned}
&\left|\frac{1}{3\alpha^{\beta}}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{I}^{\alpha}_{\kappa_1^+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}^{\beta}\mathfrak{I}^{\alpha}_{\kappa_2^-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
=\;&\frac{\kappa_2-\kappa_1}{4}\int_0^1\left[\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}+\frac{1}{3\alpha^{\beta}}\right]\left|\mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1+\left(\frac{1+\vartheta}{2}\right)\kappa_2\right)-\frac{m+M}{2}\right|d\vartheta\\
&+\int_0^1\left[\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}+\frac{1}{3\alpha^{\beta}}\right]\left|\frac{m+M}{2}-\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1+\left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right|d\vartheta.
\end{aligned}$$

From $m\le\mathfrak{F}'(\vartheta)\le M$ for $\vartheta\in[\kappa_1,\kappa_2]$, we get

$$\left|\mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1+\left(\frac{1+\vartheta}{2}\right)\kappa_2\right)-\frac{m+M}{2}\right|\le\frac{M-m}{2} \qquad (23)$$

and

$$\left|\frac{m+M}{2}-\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1+\left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right|\le\frac{M-m}{2}. \qquad (24)$$

Using (23) and (24), we have

$$\begin{aligned}
&\left|\frac{1}{3\alpha^{\beta}}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{I}^{\alpha}_{\kappa_1^+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}^{\beta}\mathfrak{I}^{\alpha}_{\kappa_2^-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
\le\;&\frac{\kappa_2-\kappa_1}{4}(M-m)\int_0^1\left[\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}+\frac{1}{3\alpha^{\beta}}\right]d\vartheta\\
=\;&\frac{\kappa_2-\kappa_1}{12}\left(\frac{3B\left(\beta+1,\frac{1}{\alpha}\right)+\alpha}{\alpha^{\beta+1}}\right)(M-m).
\end{aligned}$$

The proof of the theorem is finished. □

Example 3.2. *Let us consider the same function given in Example 2.3. Then we have* $m=\frac{1}{3}$ *and* $M=9$ *and by the inequality (21), we have*

$$\left|\frac{16}{3}-\frac{2\beta}{3}\left[3\mathcal{B}\left(\beta,\frac{2}{\alpha}+1\right)-6\mathcal{B}\left(\beta,\frac{1}{\alpha}+1\right)+\frac{7}{\beta}\right]\right|\le\frac{13}{9\alpha^{\beta}}\left(\frac{3}{\alpha}B\left(\beta+1,\frac{1}{\alpha}\right)+1\right). \qquad (25)$$

One can see the validity of the inequality (25) in Figure 4.

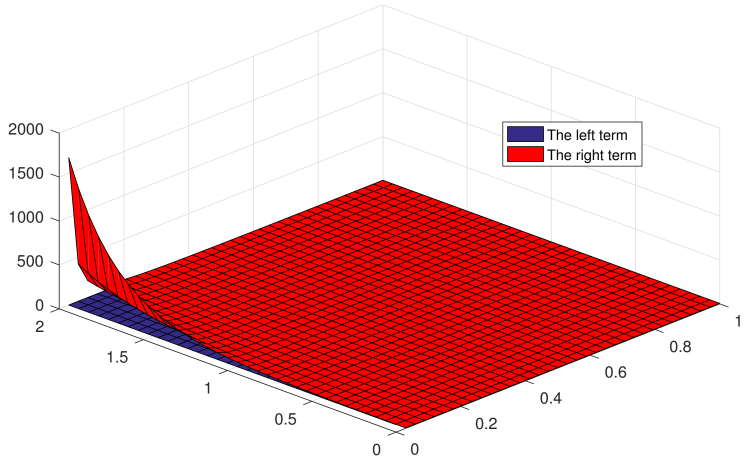

Figure 4: *An example to Theorem 3.1, depending on $\alpha \in (0,1]$ and $\beta \in (0,2]$, computed and plotted by MATLAB.*

4. Milne Type Inequality for Lipschitzian Functions

In this part, we present some fractional Milne type inequalities for Lipschitzian functions.

Theorem 4.1. *Suppose that the assumptions of Lemma 2.1 hold. If $\mathfrak{F}'$ is a L-Lipschitzian function on $[\kappa_1, \kappa_2]$, then we get the following inequality*

$$\left|\frac{1}{3\alpha^{\beta}}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{J}^{\alpha}_{\kappa_1^+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}^{\beta}\mathfrak{J}^{\alpha}_{\kappa_2^-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|$$
$$=\frac{(\kappa_2-\kappa_1)^2}{24}\left(\frac{B\left(\beta+1,\frac{1}{\alpha}\right)}{\alpha^{\beta+1}}-\frac{B\left(\beta+1,\frac{2}{\alpha}\right)}{\alpha^{\beta+1}}+\frac{1}{6\alpha^{\beta}}\right)L$$

where $\alpha, \beta > 0$, $B(x,y)$ and Γ are Euler Beta and Gamma functions, respectively.

Proof. With help of Lemma 2.1, since $\mathfrak{F}'$ is L-Lipschitzian function, we get

$$\begin{aligned}
&\left|\frac{1}{3\alpha^{\beta}}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{J}^{\alpha}_{\kappa_1^+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}^{\beta}\mathfrak{J}^{\alpha}_{\kappa_2^-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\
=&\left|\frac{\kappa_2-\kappa_1}{4}\int_0^1\left[\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}+\frac{1}{3\alpha^{\beta}}\right]\left[\mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1+\left(\frac{1+\vartheta}{2}\right)\kappa_2\right)-\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1+\left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right]d\vartheta\right|\\
\leq&\frac{\kappa_2-\kappa_1}{4}\int_0^1\left[\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}+\frac{1}{3\alpha^{\beta}}\right]\left|\mathfrak{F}'\left(\left(\frac{1-\vartheta}{2}\right)\kappa_1+\left(\frac{1+\vartheta}{2}\right)\kappa_2\right)-\mathfrak{F}'\left(\left(\frac{1+\vartheta}{2}\right)\kappa_1+\left(\frac{1-\vartheta}{2}\right)\kappa_2\right)\right|d\vartheta\\
\leq&\frac{\kappa_2-\kappa_1}{4}\int_0^1\left[\left(\frac{1-(1-\vartheta)^{\alpha}}{\alpha}\right)^{\beta}+\frac{1}{3\alpha^{\beta}}\right]L\vartheta(\kappa_2-\kappa_1)\,d\vartheta\\
=&\frac{(\kappa_2-\kappa_1)^2}{4}L\left(\frac{B\left(\beta+1,\frac{1}{\alpha}\right)}{\alpha^{\beta+1}}-\frac{B\left(\beta+1,\frac{2}{\alpha}\right)}{\alpha^{\beta+1}}+\frac{1}{6\alpha^{\beta}}\right).
\end{aligned}$$

The proof of this theorem is completed. □

5. Milne Type Inequality for Functions of Bounded Variation

In this part, we show Milne type inequality via conformable fractional integrals of bounded variation.

Theorem 5.1. *Let $\mathfrak{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ be a mapping of bounded variation on $[\kappa_1,\kappa_2]$. Then we get*

$$\begin{aligned}&\left|\frac{1}{3}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{J}^{\alpha}_{\kappa_1^+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}^{\beta}\mathfrak{J}^{\alpha}_{\kappa_2^-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right|\\ \leq\ &\frac{2}{3}\bigvee_{\kappa_1}^{\kappa_2}(\mathfrak{F})\end{aligned}$$

where $\alpha,\beta>0$, $B(x,y)$ and Γ are Euler Beta and Gamma functions and $\bigvee_{c}^{d}(\mathfrak{F})$ demonstrates the total variation of $\mathfrak{F}$ on $[c,d]$.

Proof. Define the function $K_{\alpha}^{\beta}(\upsilon)$ by

$$K_{\alpha}^{\beta}(\upsilon)=\begin{cases}-\left(\left(\frac{\kappa_2-\kappa_1}{2}\right)^{\alpha}-(\upsilon-\kappa_1)^{\alpha}\right)^{\beta}-\frac{(\kappa_2-\kappa_1)^{\alpha\beta}}{3\cdot 2^{\alpha\beta}}, & \kappa_1\leq\upsilon\leq\frac{\kappa_1+\kappa_2}{2}\\[2ex] \left(\left(\frac{\kappa_2-\kappa_1}{2}\right)^{\alpha}-(\kappa_2-\upsilon)^{\alpha}\right)^{\beta}+\frac{(\kappa_2-\kappa_1)^{\alpha\beta}}{3\cdot 2^{\alpha\beta}}, & \frac{\kappa_1+\kappa_2}{2}<\upsilon\leq\kappa_2.\end{cases}$$

Then we have

$$\begin{aligned}&\int_{\kappa_1}^{\kappa_2}K_{\alpha}^{\beta}(\upsilon)d\mathfrak{F}(\upsilon)\\ =\ &-\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\left(\left(\frac{\kappa_2-\kappa_1}{2}\right)^{\alpha}-(\upsilon-\kappa_1)^{\alpha}\right)^{\beta}+\frac{(\kappa_2-\kappa_1)^{\alpha\beta}}{3\cdot 2^{\alpha\beta}}\right)d\mathfrak{F}(\upsilon)\\ &+\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\left(\left(\left(\frac{\kappa_2-\kappa_1}{2}\right)^{\alpha}-(\kappa_2-\upsilon)^{\alpha}\right)^{\beta}+\frac{(\kappa_2-\kappa_1)^{\alpha\beta}}{3\cdot 2^{\alpha\beta}}\right)d\mathfrak{F}(\upsilon).\end{aligned}$$

By utilizing integrating by parts, we get

$$\begin{aligned}&\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\left(\left(\frac{\kappa_2-\kappa_1}{2}\right)^{\alpha}-(\upsilon-\kappa_1)^{\alpha}\right)^{\beta}+\frac{(\kappa_2-\kappa_1)^{\alpha\beta}}{3\cdot 2^{\alpha\beta}}\right)d\mathfrak{F}(\upsilon) \qquad (26)\\ =\ &-\left(\left(\left(\frac{\kappa_2-\kappa_1}{2}\right)^{\alpha}-(\upsilon-\kappa_1)^{\alpha}\right)^{\beta}+\frac{(\kappa_2-\kappa_1)^{\alpha\beta}}{3\cdot 2^{\alpha\beta}}\right)\mathfrak{F}(\upsilon)\Bigg|_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\\ &-\alpha\beta\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\left(\frac{\kappa_2-\kappa_1}{2}\right)^{\alpha}-(\upsilon-\kappa_1)^{\alpha}\right)^{\beta-1}(\upsilon-\kappa_1)^{\alpha-1}\mathfrak{F}(\upsilon)d\upsilon\\ =\ &-\frac{(\kappa_2-\kappa_1)^{\alpha\beta}}{3\cdot 2^{\alpha\beta}}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+\frac{(\kappa_2-\kappa_1)^{\alpha}}{3\cdot 2^{\alpha\beta-2}}\mathfrak{F}(\kappa_1)-\alpha^{\beta}\Gamma(\beta+1)\ {}^{\beta}\mathfrak{J}^{\alpha}_{\kappa_1^+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\end{aligned}$$

and similarly

$$\begin{aligned}
&\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\left(\left(\left(\frac{\kappa_2-\kappa_1}{2}\right)^{\alpha}-(\kappa_2-\upsilon)^{\alpha}\right)^{\beta}+\frac{(\kappa_2-\kappa_1)^{\alpha\beta}}{3\cdot 2^{\alpha\beta}}\right)d\mathfrak{F}(\upsilon) \\
=\;& \frac{(\kappa_2-\kappa_1)^{\alpha}}{3\cdot 2^{\alpha\beta-2}}\mathfrak{F}(\kappa_2)-\frac{(\kappa_2-\kappa_1)^{\alpha}}{3\cdot 2^{\alpha\beta}}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)-\alpha^{\beta}\Gamma(\beta+1)\;{}^{\beta}\mathfrak{I}^{\alpha}_{\kappa_2^-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right).
\end{aligned} \tag{27}$$

By (26) and (27), we have

$$\begin{aligned}
&\int_{\kappa_1}^{\kappa_2}K_{\alpha}^{\beta}(\upsilon)d\mathfrak{F}(\upsilon) \\
=\;& \frac{(\kappa_2-\kappa_1)^{\alpha\beta}}{2^{\alpha\beta-1}}\left\{\frac{1}{3\alpha^{\beta}}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]\right. \\
&\left.-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{I}^{\alpha}_{\kappa_1^+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}^{\beta}\mathfrak{I}^{\alpha}_{\kappa_2^-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right\}.
\end{aligned}$$

That is,

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{I}^{\alpha}_{\kappa_1^+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}^{\beta}\mathfrak{I}^{\alpha}_{\kappa_2^-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\
=\;& \frac{2^{\alpha\beta-1}}{(\kappa_2-\kappa_1)^{\alpha\beta}}\int_{\kappa_1}^{\kappa_2}K_{\alpha}(\upsilon)d\mathfrak{F}(\upsilon).
\end{aligned}$$

It is well known that if $g,\mathfrak{F}:[\kappa_1,\kappa_2]\to\mathbb{R}$ are such that g is continuous on $[\kappa_1,\kappa_2]$ and $\mathfrak{F}$ is of bounded variation on $[\kappa_1,\kappa_2]$, then $\int_{\kappa_1}^{\kappa_2}g(\vartheta)d\mathfrak{F}(\vartheta)$ exist and

$$\left|\int_{\kappa_1}^{\kappa_2}g(\vartheta)d\mathfrak{F}(\vartheta)\right|\le\sup_{\vartheta\in[\kappa_1,\kappa_2]}\left|g(\vartheta)\right|\bigvee_{\kappa_1}^{\kappa_2}(\mathfrak{F}). \tag{28}$$

Otherwise, utilizing (28), we have

$$\begin{aligned}
&\left|\frac{1}{3}\left[2\mathfrak{F}(\kappa_1)-\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+2\mathfrak{F}(\kappa_2)\right]-\frac{2^{\alpha\beta-1}\alpha^{\beta}\Gamma(\beta+1)}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[{}^{\beta}\mathfrak{I}^{\alpha}_{\kappa_1^+}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)+{}^{\beta}\mathfrak{I}^{\alpha}_{\kappa_2^-}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\right]\right| \\
=\;& \frac{2^{\alpha\beta-1}}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left|\int_{\kappa_1}^{\kappa_2}K_{\alpha}(\upsilon)d\mathfrak{F}(\upsilon)\right| \\
\le\;& \frac{2^{\alpha\beta-1}}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[\left|\int_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}\left(\left(\left(\frac{\kappa_2-\kappa_1}{2}\right)^{\alpha}-(\upsilon-\kappa_1)^{\alpha}\right)^{\beta}+\frac{(\kappa_2-\kappa_1)^{\alpha\beta}}{3\cdot 2^{\alpha\beta}}\right)d\mathfrak{F}(\upsilon)\right|\right. \\
+\;& \left.\left|\int_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}\left(\left(\left(\frac{\kappa_2-\kappa_1}{2}\right)^{\alpha}-(\kappa_2-\upsilon)^{\alpha}\right)^{\beta}+\frac{(\kappa_2-\kappa_1)^{\alpha\beta}}{3\cdot 2^{\alpha\beta}}\right)d\mathfrak{F}(\upsilon)\right|\right]
\end{aligned}$$

$$
\begin{aligned}
&\le \frac{2^{\alpha\beta-1}}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[\sup_{\upsilon\in\left[\kappa_1,\frac{\kappa_1+\kappa_2}{2}\right]}\left|\left(\left(\frac{\kappa_2-\kappa_1}{2}\right)^{\alpha}-(\upsilon-\kappa_1)^{\alpha}\right)^{\beta}+\frac{(\kappa_2-\kappa_1)^{\alpha\beta}}{3\cdot 2^{\alpha\beta}}\right|\bigvee_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(\mathfrak{F})\right.\\
&+\left.\sup_{\upsilon\in\left[\frac{\kappa_1+\kappa_2}{2},\kappa_2\right]}\left|\left(\left(\frac{\kappa_2-\kappa_1}{2}\right)^{\alpha}-(\kappa_2-\upsilon)^{\alpha}\right)^{\beta}+\frac{(\kappa_2-\kappa_1)^{\alpha\beta}}{3\cdot 2^{\alpha\beta}}\right|\bigvee_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}(\mathfrak{F})\right]\\
&= \frac{2^{\alpha\beta-1}}{(\kappa_2-\kappa_1)^{\alpha\beta}}\left[\frac{(\kappa_2-\kappa_1)^{\alpha\beta}}{3\cdot 2^{\alpha\beta-2}}\bigvee_{\kappa_1}^{\frac{\kappa_1+\kappa_2}{2}}(\mathfrak{F})+\frac{(\kappa_2-\kappa_1)^{\alpha\beta}}{3\cdot 2^{\alpha\beta-2}}\bigvee_{\frac{\kappa_1+\kappa_2}{2}}^{\kappa_2}(\mathfrak{F})\right]\\
&= \frac{2}{3}\bigvee_{\kappa_1}^{\kappa_2}(\mathfrak{F}).
\end{aligned}
$$

This finishes the proof. □

6. Conclusion

In this paper, we gave new Milne-type inequalities for convex functions. In order to prove these inequalities, we used the new conformable fractional integral operators. Our results are the generalizations of the Milne-type inequalities that ones given via Riemann-Liouville fractional integrals in [2].

In researches in the field of inequality theory, the properties of the function class, the structure of the operator, and the various methods used as proof methods sensitively affect the effectiveness of the findings and the optimality of the bounds. Convex, bounded and Lipschitzian function types and the use of the new conformable fractional integral operator can be listed as the main motivations of the study. The purpose of using different function types here is to investigate the optimal upper bounds, but the use of the new conformable fractional integral operator, whose structure is very similar to the classical derivative, ensures that the findings are compatible with the results in the literature. With these aspects, the study aims to contribute to the fields of fractional analysis and integral inequalities.

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