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Approximation by N-dimensional max-product and max-min kind discrete operators with applications

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Abstract. In this study, we construct max-product and max-min kind pseudo-linear discrete operators. First, we obtain some approximation results for bounded and uniformly continuous functions. Then, the rate of convergence for these approximations using a suitable Lipschitz class of continuous functions is also discussed. Some applications for verifying our results are also given. Furthermore, we approximate fuzzy numbers using our operators. Finally, we apply our approximation in image processing.

1. Introduction

Discrete operators have been examined in different ways and have significant applications (see [5–7, 9, 21–26, 47]). On the other hand, it is known that nonlinear approximations have widely been studied in approximation theory since they may give better results compared to their linear counterparts (see, for instance [4, 8, 10–14]). To this end, Bede et al. (see [19]) defined new operators with weakening the condition of linearity. These operators are so-called max-product and max-min operators and provide the pseudo-linearity feature. In the literature, there are many studies on max-product operators (see [1–3, 15–18, 28–35, 37, 39, 42–45, 49]). However, there are only a few investigations on max-min operators [19, 38, 40, 41]. We should remark that max-product and max-min operators have quite beneficial applications on fuzzy logic (see, for instance, [27, 46, 50, 51]). In addition, pseudo-linear operators could perform satisfactory results in image processing [20].

In this paper, our first motivation is to construct pseudo-linear discrete operators via max-product and max-min operations. Then, we investigate the approximation properties of these operators for bounded and uniformly continuous functions with the help of N-dimensional max-product and max-min kind discrete operators. To the best of our knowledge, this is the first study of max-product and max-min kind operators using discrete kernels. Moreover, we obtain the rate of convergence for these approximations by using proper Lipschitz classes. Then, we give some applications that tackle the approximation to functions of one and two variables using different kernels. We also support our results with some graphical representations

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and some approximations to fuzzy continuous numbers, which can be represented by quasi-concave functions. At the end, we use our new operators for digital image processing.

Some notations and definitions are given below.

- In this study, $N-$ dimensional maximum (supremum) operation $\bigvee \ldots \bigvee_{(k_1,\ldots,k_N)\in\mathbb{Z}^N}$ is indicated by "$\bigvee_{\mathbf{k}\in\mathbb{Z}^N}$".
- By $\|\cdot\|$, we mean the usual supremum norm on $\mathbb{R}^N$.
- $BUC_+\left(\mathbb{R}^N\right) := \{f : \mathbb{R}^N \to [0,\infty) \mid f$ is bounded and uniformly continuous on $\mathbb{R}^N\}$.
- $UC_{[0,1]}(\mathbb{R}^N) := \{f : \mathbb{R}^N \to [0,1] \mid f$ is uniformly continuous on $\mathbb{R}^N\}$.
- $N-$ dimensional vector $(u_1,\ldots,u_N)$ is denoted by "$\mathbf{u}$" and euclidean norm of $\mathbf{u}$ will be shown by "$|\mathbf{u}|$".

Discrete operators, which we consider in this paper are introduced in [5] such that

$$T_w(f;x) = \sum_{k\in\mathbb{Z}} f\left(x - \tfrac{k}{w}\right) l_{k,w}$$

where the discrete kernel $l_{k,w}$ satisfies the approximate identities, i.e.,

- $\sup\limits_{w\in\mathbb{N}} \sum\limits_{k\in\mathbb{Z}} |l_{k,w}| = A < \infty.$
- $\forall w \in \mathbb{N},\ \sum\limits_{k\in\mathbb{Z}} l_{k,w} = 1$
- There exists a number $r > 0$ such that $\lim\limits_{w\to\infty} \sum\limits_{|k|\geq r} |l_{k,w}| = 0.$

2. Approximation By Max-Product Kind Discrete Operators

We define $N-$ dimensional max-product kind discrete operators as follows:

$$T_w^{(M)}(f;\mathbf{x}) = \bigvee_{\mathbf{k}\in\mathbb{Z}^N} \frac{f\left(\mathbf{x} - \frac{\mathbf{k}}{w}\right) l_{\mathbf{k},w}}{\bigvee\limits_{\mathbf{i}\in\mathbb{Z}^N} l_{\mathbf{i},w}} \quad \left(\mathbf{x}\in\mathbb{R}^N,\ w \in \mathbb{N}\right) \tag{1}$$

where $f : \mathbb{R}^N \to [0,\infty)$ is bounded and $l_{\mathbf{k},w} := l_{(k_1,\ldots,k_N),w}$ is a family of $N-$ dimensional discrete kernels for all $w \in \mathbb{N}$, where $\bigvee\limits_{\mathbf{i}\in\mathbb{Z}^N} l_{\mathbf{i},w} \neq 0$ for sufficiently large $w \in \mathbb{N}$. For simplicity, throughout the paper, we use the notation $L_{\mathbf{k},w} := \dfrac{l_{\mathbf{k},w}}{\bigvee\limits_{\mathbf{i}\in\mathbb{Z}^N} l_{\mathbf{i},w}}$. In order to obtain more general results, instead of $\bigvee\limits_{\mathbf{k}\in\mathbb{Z}^N} L_{\mathbf{k},w} = 1$ for all $w \in \mathbb{N}$, now we assume more general condition (l_2) given below.

In this section, our aim is to prove the following convergence result

$$\lim_{w\to\infty} \left\| T_w^{(M)}(f) - f \right\| = 0$$

for all $f \in BUC_+\left(\mathbb{R}^N\right)$. For this reason, we need the following modified conditions on the kernel of the operator (1).

(l_1) $\bigvee\limits_{\mathbf{k}\in\mathbb{Z}^N} \left|L_{\mathbf{k},w}\right| = A_w \leq A < \infty,$

(l_2) $\lim\limits_{w\to\infty} \bigvee\limits_{\mathbf{k}\in\mathbb{Z}^N} L_{\mathbf{k},w} = 1,$

(l_3) There exists a number $r_0 > 0$ such that $\lim_{w\to\infty} \bigvee_{|\mathbf{k}|\geq r_0} \left|L_{\mathbf{k},w}\right| = 0.$

Lemma 2.1. *If (l_2) holds, then there exists a $\mathbf{k}' \in \mathbb{Z}^N$ such that $L_{\mathbf{k}',w} > 0$ for sufficiently large $w \in \mathbb{N}$.*

Proof. From (l_2), for all $\varepsilon > 0$, there exists a $w_0 \in \mathbb{N}$ such that $\left|\bigvee_{\mathbf{k}\in\mathbb{Z}^N} L_{\mathbf{k},w} - 1\right| < \varepsilon$ for all $w \geq w_0$. So, we can also find a number $\bar{w}_0$ for $\varepsilon = \frac{1}{2}$ such that $\left|\bigvee_{\mathbf{k}\in\mathbb{Z}^N} L_{\mathbf{k},w} - 1\right| < \frac{1}{2}$, which implies that

$$\frac{1}{2} < \bigvee_{\mathbf{k}\in\mathbb{Z}^N} L_{\mathbf{k},w} < \frac{3}{2} \tag{2}$$

for all $w \geq \bar{w}_0$. Then it is not hard to see that there exist $\mathbf{k}' = (k_1', k_2', \ldots, k_N') \in \mathbb{Z}^N$ satisfying that $\frac{1}{2} < L_{\mathbf{k}',w} < \frac{3}{2}$ for sufficiently large $w \in \mathbb{N}$. □

Lemma 2.2. *(See [17]) If for any $a_k, b_k \in \mathbb{R}$ ($k \in \mathbb{N}$) satisfies $\bigvee_{k\in\mathbb{N}} a_k < \infty$ or $\bigvee_{k\in\mathbb{N}} b_k < \infty$, then we get*

$$\left|\bigvee_{k\in\mathbb{N}} a_k - \bigvee_{k\in\mathbb{N}} b_k\right| \leq \bigvee_{k\in\mathbb{N}} |a_k - b_k| \tag{3}$$

Here, we should state that the above lemma is also valid for all $k \in \mathbb{Z}$.

Lemma 2.3. *If (l_2) holds, then our operator $T_w^{(M)}$ provides the property of pseudo-linearity for sufficiently large $w \in \mathbb{N}$ in the sense that*

$$T_w^{(M)}(\alpha f \bigvee \beta g; \mathbf{x}) = \alpha T_w^{(M)}(f; \mathbf{x}) \bigvee \beta T_w^{(M)}(g; \mathbf{x})$$

for any $\alpha, \beta \geq 0$ and for all $f, g \in BUC_+\left(\mathbb{R}^N\right)$.

Proof. By the previous lemma, it is clear. □

The following lemma shows that $T_w^{(M)}(f; \mathbf{x})$ is well-defined for all $w \in \mathbb{N}$.

Lemma 2.4. *If f is bounded and (l_1) holds, then $\left\|T_w^{(M)}(f)\right\| < \infty$ for all $w \in \mathbb{N}$, that is, our operator is well-defined.*

Proof. Since f is bounded, then there exists a positive number M such that $\left|f(\mathbf{x})\right| \leq M$ for all $\mathbf{x} \in \mathbb{R}^N$. Then from Lemma 2.2, there holds

$$\begin{aligned}\left|T_w^{(M)}(f; \mathbf{x})\right| &= \left|\bigvee_{\mathbf{k}\in\mathbb{Z}^N} f\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right) L_{\mathbf{k},w}\right| \\ &\leq \bigvee_{\mathbf{k}\in\mathbb{Z}^N} \left|f\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right)\right| \left|L_{\mathbf{k},w}\right|.\end{aligned}$$

Considering (l_1) in the previous inequality, we obtain that

$$\left|T_w^{(M)}(f; \mathbf{x})\right| \leq \bigvee_{\mathbf{k}\in\mathbb{Z}^N} \left|f\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right)\right| \left|L_{\mathbf{k},w}\right| \leq MA$$

for all $w \in \mathbb{N}$. Finally taking supremum over $\mathbf{x}\in\mathbb{R}^N$, we conclude

$$\left\|T_w^{(M)}(f)\right\| \leq MA < \infty.$$

□

Lemma 2.5. *Assume that* (l_1) *holds. If* $f \in BUC_+\left(\mathbb{R}^N\right)$, *then* $T_w^{(M)}(f) \in BUC_+\left(\mathbb{R}^N\right)$ *for every* $w \in \mathbb{N}$.

Proof. By the previous lemma, it is clear that if f is bounded, then $T_w^{(M)}(f)$ is also bounded. Now, let $\varepsilon > 0$ be given and let $\left|\mathbf{x} - \mathbf{y}\right| < \delta$ where δ corresponds to uniform continuity of given f. Then, using Lemma 2.2 we observe

$$\left|T_w^{(M)}(f;\mathbf{x}) - T_w^{(M)}(f;\mathbf{y})\right| \leq \bigvee_{\mathbf{k}\in\mathbb{Z}^N} \left|L_{\mathbf{k},w}\right| \left|f\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right) - f\left(\mathbf{y} - \tfrac{\mathbf{k}}{w}\right)\right|.$$

Since $\left|\left(\mathbf{x} - \frac{\mathbf{k}}{w}\right) - \left(\mathbf{y} - \frac{\mathbf{k}}{w}\right)\right| = |\mathbf{x} - \mathbf{y}| < \delta$, we finally have from (l_1) that

$$\left|T_w^{(M)}(f;\mathbf{x}) - T_w^{(M)}(f;\mathbf{y})\right| < A\varepsilon$$

for all $w \in \mathbb{N}$. □

Our main approximation theorem is given below.

Theorem 2.6. *Assume that* $(l_1) - (l_3)$ *hold. Then, for all* $f \in BUC_+\left(\mathbb{R}^N\right)$ *we have*

$$\lim_{w\to\infty} \left\|T_w^{(M)}(f) - f\right\| = 0.$$

Proof. From Lemma 2.2 and Lemma 2.3, we may write that

$$\begin{aligned}\left|T_w^{(M)}(f;\mathbf{x}) - f(\mathbf{x})\right| &\leq \left|\bigvee_{\mathbf{k}\in\mathbb{Z}^N} L_{\mathbf{k},w} f\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right) - \bigvee_{\mathbf{k}\in\mathbb{Z}^N} L_{\mathbf{k},w} f(\mathbf{x})\right| \\ &\quad + \left|\bigvee_{\mathbf{k}\in\mathbb{Z}^N} L_{\mathbf{k},w} f(\mathbf{x}) - f(\mathbf{x})\right| \\ &\leq \bigvee_{\mathbf{k}\in\mathbb{Z}^N} \left|L_{\mathbf{k},w}\right| \left|f\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right) - f(\mathbf{x})\right| \\ &\quad + M\left|\bigvee_{\mathbf{k}\in\mathbb{Z}^N} L_{\mathbf{k},w} - 1\right|\end{aligned}$$

for sufficiently large $w \in \mathbb{N}$. Now taking supremum over $\mathbf{x} \in \mathbb{R}^N$, we obtain

$$\begin{aligned}\left\|T_w^{(M)}(f) - f\right\| &\leq \bigvee_{\mathbf{k}\in\mathbb{Z}^N} |L_{\mathbf{k},w}| \left\|f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot)\right\| \\ &\quad + M\left|\bigvee_{\mathbf{k}\in\mathbb{Z}^N} L_{\mathbf{k},w} - 1\right| \\ &=: C_1 + MC_2.\end{aligned}$$

Directly from (l_2), for a given $\varepsilon > 0$, we can find a number w' such that for all $w \geq w'$, $C_2 < \varepsilon$. Now, we concentrate on the continuity of f. Since f is uniformly continuous, for every $\varepsilon > 0$ we can find a $\delta > 0$ such that

$$\left|f(\mathbf{x}) - f(\mathbf{y})\right| < \varepsilon \tag{4}$$

whenever $\left|\mathbf{x}-\mathbf{y}\right| < \delta$. Then, for a fixed r_0 given in (l_3), it is easy to find a number $w_1 \in \mathbb{N}$ such that

$$\left|\frac{r_0}{w}\right| < \delta$$

for all $w \geq w_1$.

Now, if we divide C_1 as follows

$$\begin{aligned} C_1 &\leq \bigvee_{|\mathbf{k}|<r_0} \left|L_{\mathbf{k},w}\right| \left\| f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot) \right\| \\ &\quad + \bigvee_{|\mathbf{k}|\geq r_0} \left|L_{\mathbf{k},w}\right| \left\| f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot) \right\| \\ &=: C_1^1 + C_1^2 \end{aligned}$$

from (4) and (l_1)

$$C_1^1 \leq A\varepsilon$$

holds, since $\left|\mathbf{x} - \frac{\mathbf{k}}{w} - \mathbf{x}\right| = \left|\frac{\mathbf{k}}{w}\right| < \frac{r_0}{w} < \delta$.

In C_1^2, from (l_3) we easily see that

$$C_1^2 < 2\left\|f\right\|\varepsilon$$

for sufficiently large $w \in \mathbb{N}$. Hence we complete the proof. □

3. Rate of Convergence for Max-Product Operators

In this section, we investigate the rate of approximation, and therefore we need the following Lipschitz class.

For any given $\alpha > 0$, define $Lip_N(\alpha)$ as follows:

$$Lip_N(\alpha) = \left\{ f \in BUC_+\left(\mathbb{R}^N\right) : \left\| f(\cdot - \mathbf{t}) - f(\cdot) \right\| = O\left(|\mathbf{t}|^\alpha\right) \text{ as } \mathbf{t} \to \mathbf{0} \right\}$$

where $f(\mathbf{t}) = O(g(\mathbf{t}))$ as $\mathbf{t} \to \mathbf{0}$ means that, there exist $\delta, N > 0$ such that $\left|f(\mathbf{t})\right| \leq N\left|g(\mathbf{t})\right|$ for $|\mathbf{t}| < \delta$.

We also require the following assumptions for a given $\alpha > 0$:

$$\left| \bigvee_{\mathbf{k}\in\mathbb{Z}^N} L_{\mathbf{k},w} - 1 \right| = O\left(1/w^\alpha\right) \text{ as } w \to \infty, \tag{5}$$

There exists a constant $r_1 > 0$ such that

$$\bigvee_{|\mathbf{k}|\geq r_1} \left|L_{\mathbf{k},w}\right| = O\left(1/w^\alpha\right) \text{ as } w \to \infty. \tag{6}$$

We obtain the following estimation.

Theorem 3.1. *Let $\alpha > 0$ and (l_1), (5) and (6) hold. Then, for all $f \in Lip_N(\alpha)$*

$$\left\| T_w^{(M)}(f) - f \right\| = O\left(1/w^\alpha\right) \textit{ as } w \to \infty.$$

Proof. From the proof of Theorem 2.6, we observe that

$$\begin{aligned} \left\| T_w^{(M)}(f) - f \right\| &\leq \bigvee_{\mathbf{k}\in\mathbb{Z}^N} |L_{\mathbf{k},w}| \left\| f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot) \right\| + M \left| \bigvee_{\mathbf{k}\in\mathbb{Z}^N} L_{\mathbf{k},w} - 1 \right| \\ &=: D_1 + D_2 \end{aligned}$$

holds. In D, for a fixed $r_1 > 0$, we can find a number w_2 such that for all $w \geq w_2$, $\left|\mathbf{x} - \frac{\mathbf{k}}{w} - \mathbf{x}\right| = \left|\frac{\mathbf{k}}{w}\right| < \frac{r_1}{w} < \delta$. Since $f \in Lip_N(\alpha)$, then there exists a constant $K > 0$ such that

$$\left\|f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot)\right\| \leq K\left|\tfrac{\mathbf{k}}{w}\right|^{\alpha}$$

for sufficiently large $w \in \mathbb{N}$. Then, we get

$$\begin{aligned} D_1 \leq & \bigvee_{|\mathbf{k}|<r_1} \left|L_{\mathbf{k},w}\right| \left\|f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot)\right\| \\ & + \bigvee_{|\mathbf{k}|\geq r_1} \left|L_{\mathbf{k},w}\right| \left\|f\left(\cdot - \tfrac{\mathbf{k}}{w}\right) - f(\cdot)\right\| \\ \leq & K(r_1)^{\alpha} \bigvee_{|\mathbf{k}|<r_1} \frac{\left|L_{\mathbf{k},w}\right|}{w^{\alpha}} \\ & + 2\|f\| \bigvee_{|\mathbf{k}|\geq r_1} \left|L_{\mathbf{k},w}\right| \\ \leq & K(r_1)^{\alpha} \tfrac{1}{w^{\alpha}} A \\ & + 2\|f\| \bigvee_{|\mathbf{k}|\geq r_1} \left|L_{\mathbf{k},w}\right| \\ =: & D_1^1 + D_1^2 \end{aligned}$$

It is clear that

$$D_1^1 = O(1/w^{\alpha}) \text{ as } w \to \infty.$$

From (6), there holds

$$D_1^2 = O(1/w^{\alpha}) \text{ as } w \to \infty.$$

Also from (5),

$$D_2 = O(1/w^{\alpha}) \text{ as } w \to \infty.$$

Hence, the proof is completed. □

4. Approximation by Max-Min Kind Discrete Operators

In this part, we give an approximation theorem for max-min kind discrete operators.

- We will use the same conditions on $L_{\mathbf{k},w}$ that are $(l_1), (l_2), (l_3)$ with one exception, that is $L_{\mathbf{k},w} \in [0,1]$ for all $\mathbf{k} \in \mathbb{Z}^N$ and $w \in \mathbb{N}$. So that (l_1) holds for $A = 1$.

Now, we can define our operator as follows:

$$T_w^{(m)}(f;\mathbf{x}) = \bigvee_{\mathbf{k}\in\mathbb{Z}^N} L_{\mathbf{k},w} \wedge f\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right) \quad \left(\mathbf{x}\in\mathbb{R}^N,\ w \in \mathbb{N}\right) \tag{7}$$

where $f : \mathbb{R}^N \to [0,1]$.

One can easily see that our operator is well-defined.

Lemma 4.1. *(See [19]) If for any* $x, y, z \in [0,1]$, *then*

$$|x \wedge y - x \wedge z| \leq x \wedge |y - z|$$

holds.

Lemma 4.2. *If* (l_2) *holds, then our operator* $T_w^{(m)}$ *is pseudo-linear for all* $w \in \mathbb{N}$ *in the sense that*

$$T_w^{(m)}((\alpha \wedge f) \vee (\beta \wedge g); \mathbf{x}) = (\alpha \wedge T_w^{(m)}(f; \mathbf{x})) \vee (\beta \wedge T_w^{(m)}(g; \mathbf{x}))$$

for every $f, g \in UC_{[0,1]}(\mathbb{R}^N)$ *and for all* $\alpha, \beta \in [0,1]$.

Proof. It is clear. □

Lemma 4.3. *Assume that* (l_1) *holds. If* $f \in UC_{[0,1]}\left(\mathbb{R}^N\right)$, *then* $T_w^{(m)}(f) \in UC_{[0,1]}\left(\mathbb{R}^N\right)$ *for all* $w \in \mathbb{N}$.

Proof. It is clear that $T_w^{(m)}(f)$ is bounded by 1 for all $w \in \mathbb{N}$. Now, let $\varepsilon > 0$ be given and let $\left|\mathbf{x} - \mathbf{y}\right| < \delta$ where δ corresponds to uniform continuity of given f. Then, using Lemma 2.2 and Lemma 4.1 we get

$$\left|T_w^{(m)}(f; \mathbf{x}) - T_w^{(m)}(f; \mathbf{y})\right| \leq \bigvee_{\mathbf{k} \in \mathbb{Z}^N} L_{\mathbf{k},w} \wedge \left|f\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right) - f\left(\mathbf{y} - \tfrac{\mathbf{k}}{w}\right)\right|$$

and since $\left|\left(\mathbf{x} - \frac{\mathbf{k}}{w}\right) - \left(\mathbf{y} - \frac{\mathbf{k}}{w}\right)\right| = |\mathbf{x} - \mathbf{y}| < \delta$, we finally have

$$\left|T_w^{(m)}(f; \mathbf{x}) - T_w^{(m)}(f; \mathbf{y})\right| < \varepsilon$$

for all $w \in \mathbb{N}$. □

Now, we get the following approximation theorem.

Theorem 4.4. *Assume that* $(l_1) - (l_3)$ *hold. Then, for all* $f \in UC_{[0,1]}\left(\mathbb{R}^N\right)$ *we have*

$$\lim_{w \to \infty} \left\|T_w^{(m)}(f) - f\right\| = 0.$$

Proof. From Lemma 2.2, Lemma 4.1 and Lemma 4.2, we may get

$$\begin{aligned}
\left|T_w^{(m)}(f; \mathbf{x}) - f(\mathbf{x})\right| &\leq \left|\bigvee_{\mathbf{k} \in \mathbb{Z}^N} L_{\mathbf{k},w} \wedge f\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right) - \bigvee_{\mathbf{k} \in \mathbb{Z}^N} L_{\mathbf{k},w} \wedge f(\mathbf{x})\right| \\
&\quad + \left|\bigvee_{\mathbf{k} \in \mathbb{Z}^N} L_{\mathbf{k},w} \wedge f(\mathbf{x}) - f(\mathbf{x})\right| \\
&\leq \bigvee_{\mathbf{k} \in \mathbb{Z}^N} \left|L_{\mathbf{k},w} \wedge f\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right) - L_{\mathbf{k},w} \wedge f(\mathbf{x})\right| \\
&\quad + f(\mathbf{x}) \wedge \left|\bigvee_{\mathbf{k} \in \mathbb{Z}^N} L_{\mathbf{k},w} - 1\right| \\
&\leq \bigvee_{\mathbf{k} \in \mathbb{Z}^N} L_{\mathbf{k},w} \wedge \left|f\left(\mathbf{x} - \tfrac{\mathbf{k}}{w}\right) - f(\mathbf{x})\right| \\
&\quad + f(\mathbf{x}) \wedge \left|\bigvee_{\mathbf{k} \in \mathbb{Z}^N} L_{\mathbf{k},w} - 1\right|.
\end{aligned}$$

Now taking supremum over $\mathbf{x} \in \mathbb{R}^N$, there holds

$$\begin{aligned}\|T_w^{(m)}(f) - f\| &\leq \bigvee_{\mathbf{k}\in\mathbb{Z}^N} L_{\mathbf{k},w} \wedge \left\| f\left(\cdot - \frac{\mathbf{k}}{w}\right) - f(\cdot)\right\| + \left|\bigvee_{\mathbf{k}\in\mathbb{Z}^N} L_{\mathbf{k},w} - 1\right| \\ &=: E_1 + E_2\end{aligned}$$

for all $w \in \mathbb{N}$. Here, using that f is uniformly continuous, for every $\varepsilon > 0$ we can find a $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{y})| < \varepsilon$ whenever $|\mathbf{x} - \mathbf{y}| < \delta$. Then for a fixed r_0 in (l_3), it is easy to find that a number $w_3 \in \mathbb{N}$ such that

$$\left|\frac{r_0}{w}\right| < \delta$$

for all $w \geq w_3$.

Now, dividing E_1 into two parts,

$$\begin{aligned}E_1 &\leq \bigvee_{|\mathbf{k}|<r_0} \left(L_{\mathbf{k},w} \wedge \left\| f\left(\cdot - \frac{\mathbf{k}}{w}\right) - f(\cdot)\right\|\right) \\ &\quad + \bigvee_{|\mathbf{k}|\geq r_0} \left(L_{\mathbf{k},w} \wedge \left\| f\left(\cdot - \frac{\mathbf{k}}{w}\right) - f(\cdot)\right\|\right) \\ &=: E_1^1 + E_1^2\end{aligned}$$

we may easily see that

$$E_1^1 < \bigvee_{|\mathbf{k}|<r_0} (L_{\mathbf{k},w} \wedge \varepsilon)$$

for all $w \geq w_3$. Then by the pseudo-linearity, we have

$$E_1^1 < \varepsilon \wedge \bigvee_{|\mathbf{k}|<r_0} L_{\mathbf{k},w} \leq \varepsilon.$$

In E_1^2, from (l_3)

$$E_1^2 \leq \bigvee_{|\mathbf{k}|\geq r_0} (L_{\mathbf{k},w} \wedge 1) \leq \bigvee_{|\mathbf{k}|\geq r_0} L_{\mathbf{k},w} < \varepsilon$$

for all $w \in \mathbb{N}$. Finally from (l_2), $E_2 \to 0$ as $w \to \infty$, which completes the proof. □

5. Rate of Convergence for Max-min Operators

In this section we investigate the rate of approximation, and for this aim, we define the following Lipschitz class.

For any given $\alpha > 0$, define $\tilde{Lip}_N(\alpha)$ as follows:

$$\tilde{Lip}_N(\alpha) = \left\{ f \in UC_{[0,1]}\left(\mathbb{R}^N\right) : \left\| f(\cdot - \mathbf{t}) - f(\cdot)\right\| = O\left(|\mathbf{t}|^\alpha\right) \text{ as } \mathbf{t} \to \mathbf{0}\right\}.$$

Theorem 5.1. *Let $\alpha > 0$ and (l_1), (5) and (6) hold. Then, for all $f \in \tilde{Lip}_N(\alpha)$*

$$\left\| T_w^{(m)}(f) - f\right\| = O\left(1/w^\alpha\right) \text{ as } w \to \infty.$$

Proof. By the proof of Theorem 4.4, we have

$$\begin{aligned}\|T_w^{(m)}(f) - f\| \leq & \bigvee_{|\mathbf{k}|<r_1} \left(L_{\mathbf{k},w} \wedge \left\| f\left(\cdot - \frac{\mathbf{k}}{w}\right) - f(\cdot)\right\|\right) \\ & + \bigvee_{|\mathbf{k}|\geq r_1} \left(L_{\mathbf{k},w} \wedge \left\| f\left(\cdot - \frac{\mathbf{k}}{w}\right) - f(\cdot)\right\|\right) \\ & + \left|\bigvee_{\mathbf{k}\in\mathbb{Z}^N} L_{\mathbf{k},w} - 1\right| \\ =: & F_1 + F_2 + F_3.\end{aligned}$$

In F_1, for a fixed $r_1 > 0$, we can find a number w_3 such that for all $w \geq w_3$, $\left|\mathbf{x} - \frac{\mathbf{k}}{w} - \mathbf{x}\right| = \left|\frac{\mathbf{k}}{w}\right| < \frac{r_1}{w} < \delta$. Since $f \in \tilde{Lip}_N(\alpha)$, then there exists a constant $K' > 0$ such that

$$\left\| f\left(\cdot - \frac{\mathbf{k}}{w}\right) - f(\cdot)\right\| \leq K' \left|\frac{\mathbf{k}}{w}\right|^\alpha$$

for all $w \in \mathbb{N}$. By the pseudo-linearity, (5) and (6), we get

$$F_1 \leq \bigvee_{|\mathbf{k}|<r_1} \left(L_{\mathbf{k},w} \wedge K' \frac{(r_1)^\alpha}{w^\alpha}\right) \leq K' \frac{(r_1)^\alpha}{w^\alpha} = O(1/w^\alpha) \text{ as } w \to \infty,$$

$$F_2 \leq \bigvee_{|\mathbf{k}|\geq r_1} (L_{\mathbf{k},w} \wedge 1) \leq \bigvee_{|\mathbf{k}|\geq r_1} L_{\mathbf{k},w} = O(1/w^\alpha) \text{ as } w \to \infty$$

and

$$F_3 = \left|\bigvee_{\mathbf{k}\in\mathbb{Z}^N} L_{\mathbf{k},w} - 1\right| = O(1/w^\alpha) \text{ as } w \to \infty.$$

□

6. Applications

In the present section, we give some applications of the operators of types (1) and (7).

6.1. Univariate Max-Product Kind Discrete Operators

In this part, the discrete kernel is given by

$$L_{k,w} = \begin{cases} \dfrac{\sin((k-3)w)}{(k-3)w}; & \text{if } k \neq 3 \\ 1; & \text{if } k = 3. \end{cases}$$

Using this kernel, operator (1) becomes the following

$$T_w^{(M)}(f;x) = \bigvee_{k\in\mathbb{Z}} L_{k,w} f(x - \tfrac{k}{w}).$$

- We can easily write that

$$\bigvee_{k\in\mathbb{Z}} |L_{k,w}| = \bigvee_{k\in\mathbb{Z}} L_{k,w} = 1 \tag{8}$$

  which implies (l_1) and (l_2).

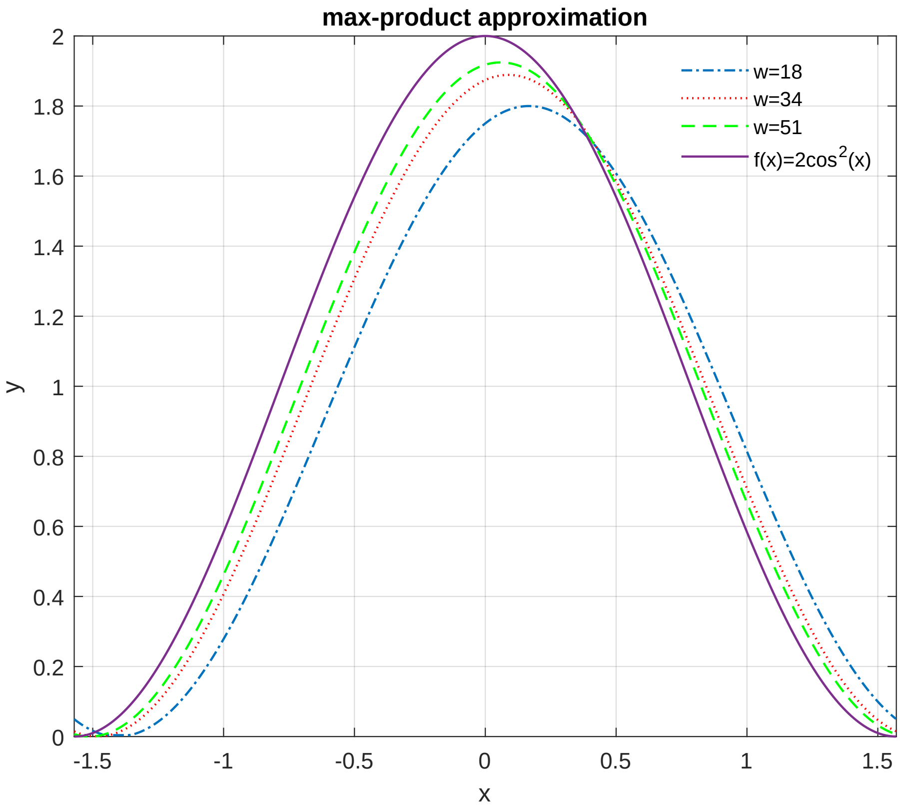

Figure 1: Approximation to $f(x) = 2\cos^2 x$ by means of the operators $T_w^{(M)}(f)$ for $w = 18, 34, 51$.

- And also, for $\alpha = 1$ it can easily be seen that

$$\left| \bigvee_{k\in\mathbb{Z}} L_{k,w} - 1 \right| = 0 = O\left(1/w\right) \text{ as } w \to \infty \tag{9}$$

  holds. Then, condition (5) is done.

- Assume that $r_0 = r_1 = 4$ and $\alpha = 1$. Then, we get

$$\bigvee_{|k|\geq 4} |L_{k,w}| \leq \frac{1}{w} = O\left(1/w\right) \text{ as } w \to \infty \tag{10}$$

  which gives (l_3) and (6).

Therefore, $L_{k,w}$ fullfils the conditions $(l_1) - (l_3)$ and (5), (6). Then for every $f \in BUC_+(\mathbb{R})$, we get

$$\lim_{w\to\infty} \left\| T_w^{(M)}(f) - f \right\| = 0.$$

This uniform approximation is displayed in Figure 1, for $f(x) = 2\cos^2 x$.

By the following expression, we see that $f \in Lip_1(1)$,

$$\begin{aligned}|f(x-t)-f(x)| &\leq 2||\cos^2(x-t)|-|\cos^2 x|| \\ &\leq 4|\cos(x-t)-\cos x| \\ &= 4|\sin(\tfrac{\pi}{2}-x+t)-\sin(\tfrac{\pi}{2}-x)| \\ &\leq 4|\tfrac{\pi}{2}-x+t-(\tfrac{\pi}{2}-x)| \\ &= 4|t|.\end{aligned}$$

Furthermore, from (9) and (10), we observe that

$$\left\|T_w^{(M)}(f)-f\right\| = O(1/w) \text{ as } w \to \infty.$$

6.2. Univariate Max-Min Kind Discrete Operators

In this application, we take the discrete kernel as the following

$$L_{k,w} = \frac{w}{(w+2)(|k-3|+1)^w}.$$

Using this kernel, operator (7) turns into the following one

$$T_w^{(m)}(f;x) = \bigvee_{k\in\mathbb{Z}} \left(\frac{w}{(w+2)(|k-3|+1)^w} \wedge f(x-\tfrac{k}{w}) \right).$$

Now , we prove that $L_{k,w}$ fullfils the conditions $(l_1)-(l_3)$ and (5), (6).

- We first obtain the condition (l_1) from the following expressions

  $$\bigvee_{k\in\mathbb{Z}} L_{k,w} = \frac{w}{w+2} = A_w \leq 1 \tag{11}$$

  and $\lim_{w\to\infty} A_w = 1$.

- In addition,

  $$\left|\bigvee_{k\in\mathbb{Z}} L_{k,w} - 1\right| = \left|\frac{w}{w+2}-1\right| = \frac{2}{w+2} = O(1/w) \text{ as } w \to \infty \tag{12}$$

  which immediately gives us (l_2) and (5) for $\alpha = 1$.

- Assume $r_0 = r_1 = 4$. Then, for $\alpha = 1$,

  $$\bigvee_{|k|\geq 4} L_{k,w} = \frac{w}{(w+2)2^w} \leq \frac{1}{2^w} \leq \frac{1}{w} = O(1/w) \text{ as } w \to \infty \tag{13}$$

  holds. So, we get (l_3) and (6).

Therefore, from Theorem 4.4 we have

$$\lim_{w\to\infty} \left\|T_w^{(m)}(f)-f\right\| = 0$$

for every $f \in UC_{[0,1]}(\mathbb{R})$. This uniform approximation is indicated in Figure 2 for the function $f(x) = |\sin x|$. Moreover, $f \in \tilde{Lip}_1(1)$. Indeed, f satisfies the following inequality

$$|f(x-t)-f(x)| = ||\sin(x-t)|-|\sin x|| \leq |\sin(x-t)-\sin x| \leq |t|.$$

On the other hand, for the rate of convergence, considering $f \in \tilde{Lip}_1(1)$, (12) and (13), we get that

$$\left\|T_w^{(m)}(f)-f\right\| = O(1/w) \text{ as } w \to \infty.$$

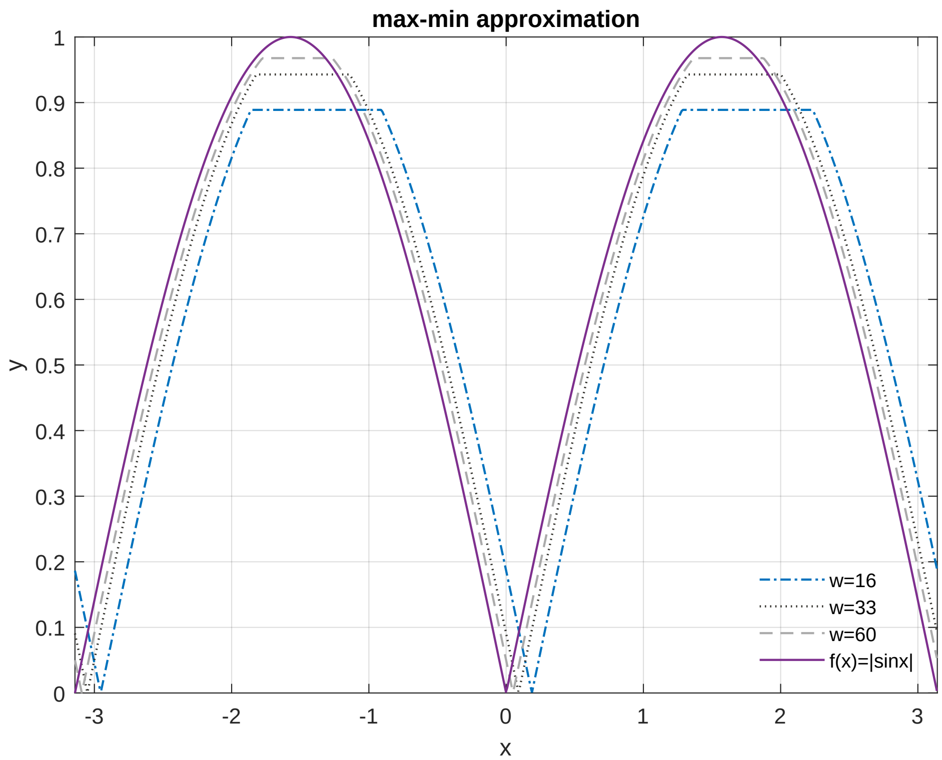

Figure 2: Approximation to $f(x) = |\sin x|$ by means of the operators $T_w^{(m)}(f)$ for $w = 16, 33, 60$.

6.3. Bivariate Max-Product Type Discrete Operators

In this part, we examine the following discrete kernel defined by

$$L_{(k_1,k_2),w} = \begin{cases} \left(\frac{w+2}{w}\right) \frac{\sin(w\sqrt{k_1^2+k_2^2})}{w\sqrt{k_1^2+k_2^2}}; & \text{if } (k_1,k_2) \neq (0,0) \\ 1; & \text{if } (k_1,k_2) = (0,0). \end{cases}$$

Then, our operator reduces to

$$T_w^{(M)}(f;x,y) = \bigvee_{(k_1,k_2)\in\mathbb{Z}^2} L_{(k_1,k_2),w} f\left(x - \tfrac{k_1}{w}, y - \tfrac{k_2}{w}\right).$$

Here, following inequality indicates that condition (l_1) is satisfied

$$\bigvee_{(k_1,k_2)\in\mathbb{Z}^2} \left|L_{(k_1,k_2),w}\right| = \bigvee_{(k_1,k_2)\in\mathbb{Z}^2} L_{(k_1,k_2),w} = \frac{w+2}{w} = A_w \leq 3. \tag{14}$$

In addition, it is stated in the following expression that conditions (l_2) and (5) are satisfied

$$\left(\bigvee_{(k_1,k_2)\in\mathbb{Z}^2} L_{(k_1,k_2),w} - 1\right) = \left(\frac{w+2}{w} - 1\right) = \frac{2}{w} = O\left(1/w\right) \text{ as } w \to \infty. \tag{15}$$

And also, assume that $r_0 = r_1 = 1$ and $\alpha = 1$, then (l_3) and (6) are provided by the followings

$$\bigvee_{\sqrt{k_1^2+k_2^2}\geq 1} \left|L_{(k_1,k_2),w}\right| \leq \frac{w+2}{w}\left(\frac{1}{w}\right) \leq \frac{3}{w} = O\left(1/w\right) \text{ as } w \to \infty. \tag{16}$$

Now, the kernel ensures the conditions $(l_1) - (l_3)$, so that following uniform approximation is obtained for every $f \in BUC_+(\mathbb{R}^2)$

$$\lim_{w\to\infty} \left\|T_w^{(M)}(f) - f\right\| = 0.$$

This approximation is plotted in Figure 3 for the function $f(x,y) = |\sin x|$.

Besides, for the function $f(x,y) = |\sin x|$, the following inequality holds

$$\begin{aligned} \left|f(x-t_1, y-t_2) - f(x,y)\right| &= \left||\sin(x-t_1)| - |\sin x|\right| \\ &\leq |\sin(x-t_1) - \sin x| \\ &\leq |t_1| \\ &= \sqrt{t_1^2} \\ &\leq \sqrt{t_1^2+t_2^2} \\ &= |\mathbf{t}|, \end{aligned}$$

which gives $f \in Lip_2(1)$. Using this together with (15) and (16), we have

$$\left\|T_w^{(M)}(f) - f\right\| = O\left(1/w\right) \text{ as } w \to \infty.$$

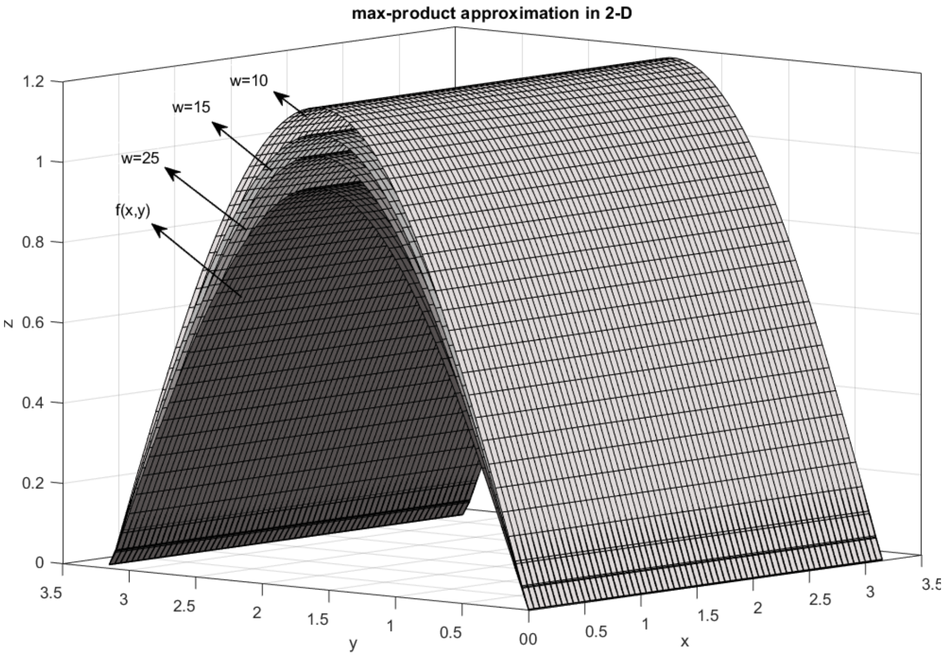

Figure 3: Approximation to $f(x,y) = |\sin x|$ by means of the operators $T_w^{(M)}(f)$ for $w = 10, 15, 25$.

6.4. Bivariate Max-Min Type Discrete Operators

Take the discrete kernel given by

$$L_{(k_1,k_2),w} = \frac{w}{(w+2)(\sqrt{(k_1-3)^2+(k_2-3)^2}+1)^w}.$$

Then, in this case, our operator is obtained as follows:

$$T_w^{(m)}(f;x,y) = \bigvee_{(k_1,k_2)\in\mathbb{Z}^2} \frac{w}{(w+2)(\sqrt{(k_1-3)^2+(k_2-3)^2}+1)^w} \wedge f\left(x-\tfrac{k_1}{w}, y-\tfrac{k_2}{w}\right).$$

Now, by the following inequality the condition (l_1) is satisfied,

$$\bigvee_{(k_1,k_2)\in\mathbb{Z}^2} L_{(k_1,k_2),w} = \frac{w}{w+2} = A_w \leq 1. \tag{17}$$

In addition, it is shown that conditions (l_2) and (5) are provided from the following expression

$$\left(\bigvee_{(k_1,k_2)\in\mathbb{Z}^2} L_{(k_1,k_2),w} - 1\right) = \frac{-2}{w+2} = O\left(1/w\right) \text{ as } w\to\infty. \tag{18}$$

And also assume that $r_0 = r_1 = 4$ and $\alpha = 1$, then the conditions (l_3) and (6) are satisfied:

$$\bigvee_{\sqrt{k_1^2+k_2^2}\geq 4} L_{(k_1,k_2),w} \leq \frac{w}{(w+2)2^w} \leq \frac{1}{2^w} \leq \frac{1}{w} = O\left(1/w\right) \text{ as } w\to\infty. \tag{19}$$

Now, all the conditions of Theorem 4.4 are fulfilled. Therefore, we get

$$\lim_{w\to\infty} \left\|T_w^{(m)}(f) - f\right\| = 0$$

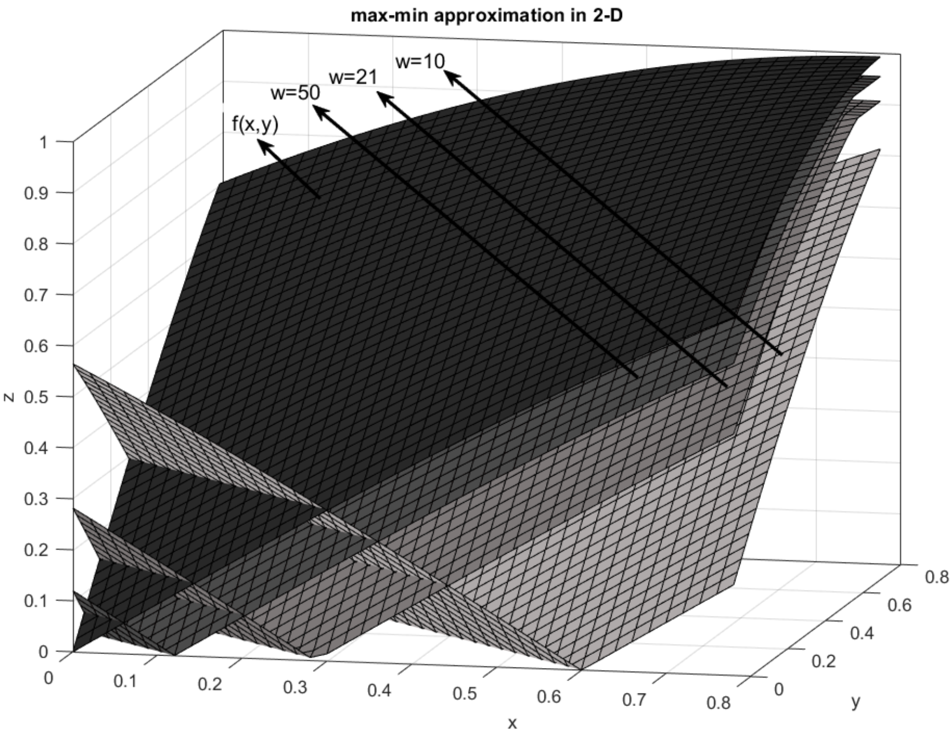

Figure 4: Approximation to $f(x,y) = |\sin(x+y)|$ by means of the operators $T_w^{(m)}(f)$ for $w = 10, 21, 50$.

for every $f \in UC_{[0,1]}(\mathbb{R}^2)$. This approximation is illustrated in Figure 4 for $f(x,y) = |\sin(x+y)|$.

It is easy to see that $f \in \tilde{Lip}_2(1)$ from the following inequality

$$\begin{aligned}\left|f(x-t_1, y-t_2) - f(x,y)\right| &= \left||\sin((x-t_1)+(y-t_2))| - |\sin(x+y)|\right| \\ &\leq |t_1 + t_2| \\ &\leq \sqrt{t_1^2} + \sqrt{t_2^2} \\ &\leq 2\sqrt{t_1^2 + t_2^2} \\ &= 2|\mathbf{t}|.\end{aligned}$$

Furthermore, considering $f \in \tilde{Lip}_2(1)$, (18) and (19), then conditions of Theorem 5.1 are provided and hence, we have

$$\left\|T_w^{(m)}(f) - f\right\| = O(1/w) \text{ as } w \to \infty.$$

6.5. Applications to Fuzzy Logic

A continuous fuzzy number could be characterized by a quasi-concave function [32, 48]. Using this argument, we approximate the triangular fuzzy number with the help of our max-product and max-min operators. For this aim, we consider the following kernel

$$L_{k,w} = \frac{1}{\left(\frac{|k|}{100} + 1\right)^w} \quad (k \in \mathbb{Z}, w \in \mathbb{N}). \tag{20}$$

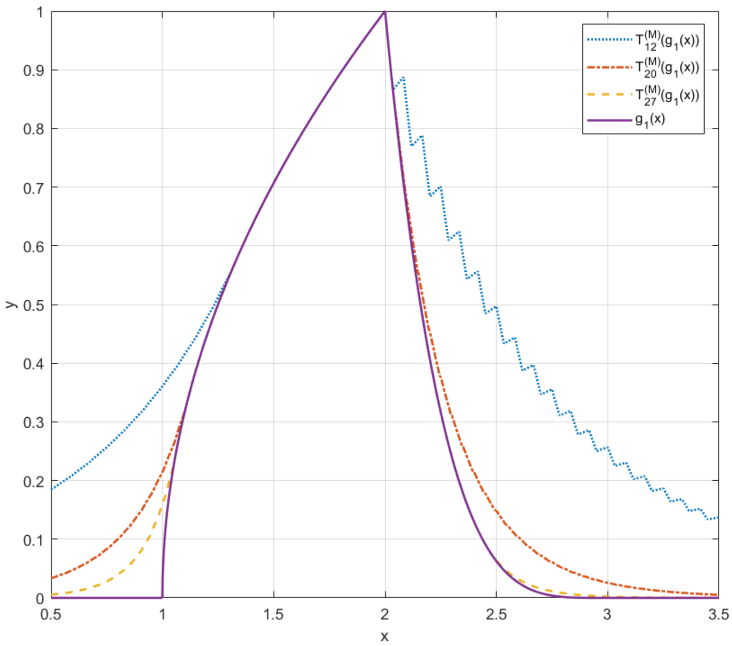

Figure 5: Approximation to g_1 by means of $T_w^{(M)}(g_1)$ for $w = 12, 20, 27$.

This kernel satisfies all the conditions of Theorem 2.6 and Theorem 4.4. We first consider the operator (1) and approximate to the given quasi-concave function, defined on the whole real line as follows:

$$g_1(x) = \begin{cases} 0; & \text{if } x < 1 \\ \sqrt{x-1}; & \text{if } 1 \le x < 2 \\ (x-3)^4; & \text{if } 2 \le x \le 3 \\ 0; & \text{if } x > 3. \end{cases}$$

Since g_1 is bounded and uniformly continuous, it is possible to approximate to g_1 with the help of our max-product operator. This approximation is shown in Figure 5 for the values $w = 12, 20, 27$.

Furthermore, in this graph we can easily see that our operator does not preserve the monotonicity for $w = 12$. Indeed, $g_1(x) = (x-3)^4$ is a decreasing function on $[2, 3]$. Then taking the points $x_1 = 2.206$ and $x_2 = 2.244$, we orderly get $g_1(2.206) = 0.3974$ and $g_1(2.244) = 0.3266$. However, for the same points, we obtain that $T_w^{(M)}(g_1; 2.206) = 0.6857$ and $T_w^{(M)}(g_1; 2.244) = 0.6992$, which shows that $T_w^{(M)}(g_1)$ is not decreasing when g_1 is decreasing on $[2, 3]$.

Now, for the second approximation, we consider the max-min operator in (7) and we take into account the following quasi-concave function, given by:

$$g_2(x) = \begin{cases} 0; & \text{if } x < 0 \\ 4x^2; & \text{if } 0 \le x < \frac{1}{2} \\ 4(x-1)^2; & \text{if } \frac{1}{2} \le x \le 1 \\ 0; & \text{if } x > 1. \end{cases}$$

This approximation is indicated in Figure 6 for $w = 11, 20, 50$.

We can also see that our operators do not maintain the monotonicity for a given function with this figure. Here, $g_2(x) = 4(x-1)^2$ is a decreasing function on $[\frac{1}{2}, 1]$. Indeed, taking $w = 11$ and choosing the

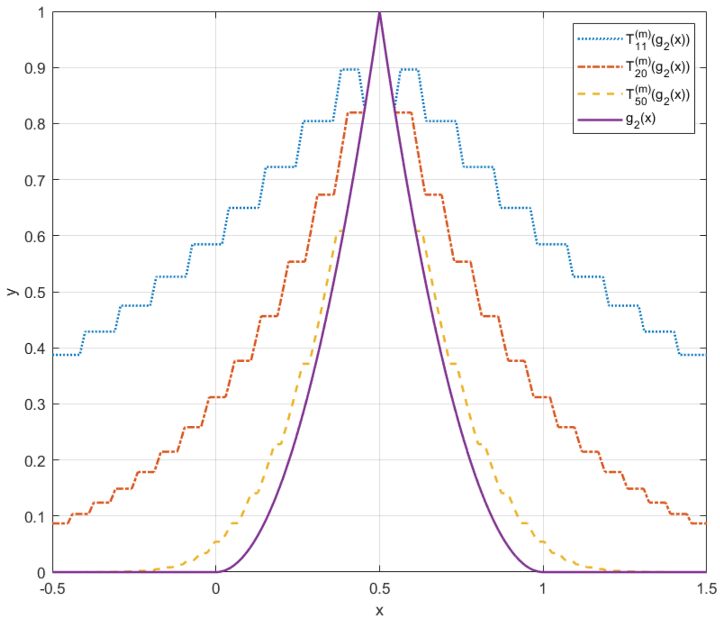

Figure 6: Approximation to g_2 by means of $T_w^{(m)}(g_2)$ for $w = 11, 20, 50$.

points $x_1 = 0.550$ and $x_2 = 0.578$, we have $g_2(0.550) = 0.81$ and $g_2(0.578) = 0.71$ respectively. However, we get that $T_w^{(m)}(g_2; 0.550) = 0.843$ and $T_w^{(M)}(g_2; 0.578) = 0.896$ which means $T_w^{(m)}(g_2)$ is not decreasing when g_2 is decreasing on $[\frac{1}{2}, 1]$.

6.6. Applications to Digital Image Processing

A digital image can be considered as a discrete signal and we can represent it by a two dimensional matrix. Now, we examine a square ($n \times n$ pixel) grayscale image, that is, $M = \left(m_{ij}\right)$, where m_{ij} is the grayscale level of the image for $i, j = 1, \ldots, n$. As given in [36], we can model any digital grayscale image as a step function (which is clearly in $L^1\left(\mathbb{R}^2\right)$) with a compact support, i.e., $I = I(x, y)$ such that

$$I(x, y) = \sum_{i=1}^{n} \sum_{j=1}^{n} m_{ij}.\mathbf{1}_{ij}(x, y) \quad (x, y) \in \mathbb{R}^2,$$

where

$$\mathbf{1}_{ij}(x, y) = \begin{cases} 1; & \text{if } (x, y) \in (i-1, i] \times (j-1, j] \text{ for all } i, j = 1, \ldots, n \\ 0; & \text{otherwise.} \end{cases}$$

On the other hand, it is well-known that compactly supported continuous functions are dense in L^1, and this enables us to approximate to the function I by $T_w^{(M)}(I)$ and $T_w^{(m)}(I)$. Here, we should remark that, in max-min operator, we must normalize the pixel values of the image from $[0, 255]$ to $[0, 1]$ using ”im2double” code on matlab. Using these approximations, if we increase the sampling rate, we get a new image, whose resolution is greater than the original one.

Now, we consider 128×128 pixel image of “cameraman”. We use the following discrete *sinc* kernels $L_{(k_1,k_2),w} := L_{k_1,w} L_{k_2,w}$, where

$$L_{k,w} := \begin{cases} \dfrac{\sin(kw)}{kw}; & \text{if } k \neq 0 \\ 1; & \text{if } k = 0 \end{cases}$$

Figure 7: Reconstructed "cameraman" by $T_{11}^{(M)}(I)$ and original "Lena" with 128×128 pixel respectively. PSNR=73.3730

Figure 8: Reconstructed "cameraman" by $T_{11}^{(m)}(I)$ and original "Lena" with 128×128 pixel respectively. PSNR=41.3676

and $|L_{(k_1,k_2),w}|$ for max-product and max-min operators respectively. By using Matlab program, we get the following figures (FIGURE 7,8,9,10).

In Figures 9,10, we evaluated *PSNR* (Peak signal to noise ratio) values on matlab, which is a quality metric in image processing, defined by

$$PSNR = 10 \log_{10} \frac{(2^r - 1)^2}{MSE}$$

where $2^r - 1$ is the number of gray-scale levels of the original image and *MSE* (Mean Square Error) is defined by

$$MSE = \frac{1}{(N_1 N_2)^2} \sum_{i=0}^{N_1-1} \sum_{j=0}^{N_2-1} (image\,(i,j) - L\,(i,j))^2$$

where $image\,(i,j)$ and $L\,(i,j)$ is the gray-scale level of the pixel (i,j) of the original image and its approximation for a $N_1 \times N_2$ pixel valued image.

6.7. Further Results

- If f is negative, bounded and uniformly continuous, then

$$\lim_{w\to\infty} \left\| -T_w^{(M)}(-f) - f \right\| = 0$$

  holds.

- If $f \geq 1$ is bounded and uniformly continuous, then

$$\lim_{w\to\infty} \left\| \left(T_w^{(m)}\left(\frac{1}{f}\right) \right)^{-1} - f \right\| = 0$$

  holds.

- Considering these features, it is also possible to extend Theorem 2.6 and Theorem 4.4 for all bounded and uniformly continuous functions, depending on their range.

Figure 9: "cameraman", obtained by $T_{11}^{(M)}(I)$ with increased resolution (256×256 pixel) and original "Lena" (256×256 pixel) respectively. PSNR=25.2593

Figure 10: "cameraman", obtained by $T_{11}^{(m)}(I)$ with increased resolution (256 × 256 pixel) and original "Lena" (256 × 256 pixel) respectively. PSNR=25.2312

7. Concluding Remarks

In this study, we construct max-product and max-min kind discrete operators with discrete kernels. Then we approximate the quasi-concave functions in applications. We note that our operators with the kernel (20) do not preserve the quasi-concavity for some w. However, for sufficiently large w, it might be possible but this, of course, needs to be proved. For now, it is an open problem whether our operators with discrete kernel satisfying $(l_1) - (l_3)$ preserves the quasi-concavity or not.

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