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On set-valued multiadditive functional equations

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Abstract. In this work, we obtain some representations of set-valued solutions defined on an abelian group G with values in a Hausdorff topological vector space of the generalized multiadditive functional equations. We also investigate the Hyers-Ulam stability of the mentioned earlier set-valued functional equations. Furthermore, we prove the Hyers-Ulam stability of the set-valued multiadditive mappings by applying a fixed point theorem.

1. Introduction

It is well-known that among functional equation the additive (Cauchy) equation

$$\mathcal{A}(x+y) = \mathcal{A}(x) + \mathcal{A}(y) \tag{1}$$

plays a significant role in many parts of mathematics. More information about them (in particular, about their solutions and their applications can be found for instance in [18] and [32].

Throughout this paper, $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{R}$ are the sets of all positive integers, rationals and real numbers, respectively, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}, \mathbb{R}_+ := [0, \infty)$. Moreover, for the set E, we denote $\overbrace{E \times E \times \ldots \times E}^{n-\text{times}}$ by E^n. Let V be a commutative group, W be a linear space over $\mathbb{Q}$, and $n \in \mathbb{N}$ with $n \geq 2$. A mapping $f : V^n \longrightarrow W$ is called *multiadditive* if it satisfies (1) in each variable. It is shown in [10, Theorem 2] that a mapping f is multiadditive if and only if it satisfies

$$f(x_1 + x_2) = \sum_{i_1,\ldots,i_n \in \{1,2\}} f(x_{i_1 1}, \ldots, x_{i_n n}), \tag{2}$$

where $x_i = (x_{i1}, \ldots, x_{in}) \in V^n$ with $i \in \{1, 2\}$.

The stability problem of the functional equation, initiated by the celebrated Ulam's question [37] about the stability of group homomorphisms (answered by Hyers [16], Aoki [1], Th. M. Rassias [30] and Găvruţa

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[15] for additive and linear mappings) has been growing rapidly over the last decades and applied in sciences and engineering. Recall that a functional equation Γ is said to be *stable* if any function f satisfying the equation Γ approximately must be near to an exact solution of Γ. Here, we remember that the Ulam query about the stability of group homomorphisms and functional equations on Banach spaces has been studied and established for instance in papers and books [13], [14], [17], [19], [20], [26], [31], [33] and moreover references therein. On the other hand, a lot of information about the structure of multiadditive mappings and their Ulam stabilities are available in [9], [10] and [18, Sections 13.4 and 17.2].

In the last decades, the theory of set-valued functions in Banach spaces have been improved and developed by the authors. In 1965 and 1966, the pioneering papers by Aumann [2] and Debreu [11] were inspired by problems arising in control theory and mathematical economics. Next, some equations for set-valued functions and set-valued solutions of miscellaneous functional equations have been investigated by the authors which can be found for instance in [6], [22], [23], [24], [28], [34], [35] and [36].

In this paper, we introduce the generalized multiadditive functional equations and investigate some set-valued solutions of the such functional equations. Moreover, motivated by some results in [27] and [29], we prove the stability of the generalized multiadditive set-valued functional equations. Finally, we establish the Hyers-Ulam stability of the set-valued multiadditive mappings by using a fixed point method.

2. Set-valued solutions of generalized multiadditive mappings

Throughout this section, $(G, +)$ denotes a commutative group and X is a Hausdorff topological vector space over $\mathbb{R}$. Both the zero element of G and the origin of X will be denoted by 0. We also denote the collection of all nonempty subsets of X by $P^*(X)$.

Here, we have some sets which are necessary in this paper.

$$b(X) = \{E|E \in P^*(X),\ E \text{ is bounded}\};$$

$$cc(X) = \{E|E \in P^*(X),\ E \text{ is closed and convex}\};$$

$$bc(X) = \{E|E \in P^*(X),\ E \text{ is bounded and convex}\};$$

$$bcc(X) = \{E|E \in P^*(X),\ E \text{ is bounded, convex and closed}\}.$$

Let $A, B \in P^*(X)$ and $\lambda \in \mathbb{R}$. We consider the addition and scalar multiplication as follows:

$$A + B := \{a + b|\, a \in A, b \in B\}, \qquad \lambda A := \{\lambda a|\, a \in A\}.$$

It is easy to check that for $\lambda, \mu \geq 0$ and $A \in P^*(X)$ which is convex, we have

$$(\lambda + \mu)A = \lambda A + \mu A.$$

Recall that $C \in P^*(X)$ is called a (convex) *cone* in X if $C + C \subseteq C$ and for each positive real number $\lambda C \subseteq C$. It is clear that every convex cone is a convex set. If the zero vector in X belongs to C, then we say that C is a (convex) cone with zero. Next, we remind the following lemmas from [3] and [28] which will be useful in the proofs of our main results.

Lemma 2.1. *Let $C \in P^*(X)$ be a convex set such that there exists $1 \neq \lambda > 0$ with the property $\lambda C = C$. Then, C is a convex cone.*

Lemma 2.2. *Let $B \in P^*(X)$ be a bounded set such that there exists $1 \neq \lambda > 0$ with the property $\lambda B = B$. Then, $B = \{0\}$.*

Lemma 2.3. *Let $A, B \in cc(X)$ and $C \in P^*(X)$ be a bounded set. If $A + C = B + C$, then $A = B$.*

In this section, we investigate the set-valued solutions $f : G^n \longrightarrow P^*(X)$ of the following generalized multiadditive functional equation

$$f(x_1 + x_2) = \sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n} f(x_{i_1 1}, \dots, x_{i_n n}), \tag{3}$$

where $A_{i_1,\dots,i_n}$ are nonnegative numbers with $i_1, \dots, i_n \in \{1, 2\}$. In the case that each $A_{i_1,\dots,i_n}$ is 1, equations (2) and (3) coincide.

The first elementary result is for the constant mappings as follows.

Proposition 2.4. *Suppose that each $A_{i_1,\dots,i_n}$ is a positive real number.*

(i) If $C \in P^*(X)$ is a convex cone, then the constant mapping $f(x) = C$ is a solution of (3)*, for all $x \in G^n$;*
(ii) If $C \in P^*(X)$ is a convex set and $\sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n} \neq 1$, then the constant mapping $f(x) = C$ is a solution of (3)*, for all $x \in G^n$ if and only if C is a convex cone.*

Proof. (i) We have

$$\begin{aligned} f(x_1 + x_2) = C &= \left[\sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n}\right] C \\ &= \sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n} f(x_{i_1 1}, \dots, x_{i_n n}), \end{aligned}$$

for all $x_1, x_2 \in G^n$.

(ii) Assume that $f(x) = C$ is a solution of (3), for all $x \in G^n$, then

$$C = \left[\sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n}\right] C.$$

By Lemma 2.1, C is a convex cone in X. The other implication follows from part (i). □

Put $\mathbf{n} := \{1, \dots, n\}$, $n \in \mathbb{N}$. For a subset $T = \{l_1, \dots, l_k\}$ of $\mathbf{n}$ with $2 \leq l_1 < \dots < l_k \leq m$ such that $m \leq n - 1$, denote

$$\begin{aligned} &B_{i_1,\dots,i_{l_1-1},1,i_{l_1+1},\dots,i_{l_k-1},1,i_{l_k+1}\dots,i_n} \\ &= \sum_{i_1,\dots,i_{l_1-1},i_{l_1+1},\dots,i_{l_k-1},i_{l_k+1}\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_{l_1-1},1,i_{l_1+1},\dots,i_{l_k-1},1,i_{l_k+1}\dots,i_n}. \end{aligned}$$

Note that

$$B_{i_1,\dots,i_n} = \sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n} \text{ and } B_{1,\dots,1} = A_{1,\dots,1}.$$

We use these notations in the proof of the results in this paper. It is shown in the proof of Theorem 2 from [10] that for a commutative semigroup G and a linear space W, for every solution $f : G^n \longrightarrow W$ of (2), $f(x) = 0$ for any $x \in G^n$ with at least one component which is equal to zero. However, the same proof can be applied to show that for every solution $f : G^n \longrightarrow W$ of (3), $f(x) = 0$ for any $x \in G^n$ with at least one component which is equal to zero. In the next result, we prove a similar result when the image is set-valued.

Theorem 2.5. *The only solution $f : G^n \longrightarrow bc(X)$ of* (3) *is zero set provided that*

$$B_{i_1,\dots,i_{l_1-1},1,i_{l_1+1},\dots,i_{l_k-1},1,i_{l_k+1}\dots,i_n} \neq 1,$$

for all $1 \leq l_k \leq n$.

Proof. Letting $x_1 = x_2 = (0,\dots,0)$ in (3), we have

$$f(0,\dots,0) = B_{i_1,\dots,i_n} f(0,\dots,0).$$

It follows from Lemma 2.2 that $f(0,\dots,0) = \{0\}$. Fix $j \in \{1,\dots,n\}$. Letting $x_{1p} = 0$ for all $p \in \{1,\dots,n\}\backslash\{j\}$ and $x_{2p} = 0$ for $1 \le p \le n$ in (3), we get

$$f(0,\dots,0,x_{1j},0,\dots,0) = B_{i_1,\dots,i_{j-1},1,i_{j+1},\dots,i_n} f(0,\dots,0,x_{1j},0,\dots,0).$$

By assumptions and Lemma 2.2, we get $f(0,\dots,0,x_{1j},0,\dots,0) = \{0\}$. The processing above can be repeated to obtain

$$\begin{aligned}&f(0,\dots,0,x_{1_{l_1}},0,\dots,0,x_{1_{l_k}},0,\dots,0)\\ &= B_{i_1,\dots,i_{l_1-1},1,i_{l_1+1},\dots,i_{l_k-1},1,i_{l_k+1}\dots,i_n} f(0,\dots,0,x_{1_{l_1}},0,\dots,0,x_{1_{l_k}},0,\dots,0).\end{aligned}$$

where $1 \le l_k \le n$. Lastly, by putting $x_2 = (0,\dots,0)$ in (3), we get $f(x_1) = B_{1,\dots,1} f(x_1)$ for all $x_1 \in G^n$. Since $B_{1,\dots,1} \neq 1$, we find the result. □

Remark 2.6. *Note that in Theorem 2.5, if $B_{i_1,\dots,i_n} = 1$ but $f(0,\dots,0) = \{0\}$, then the result is again valid.*

Theorem 2.7. *Let $B_{i_1,\dots,i_n} = 1$ and each $A_{i_1,\dots,i_n}$ be positive real number. If $f : G^n \longrightarrow bcc(X)$ is a solution of* (3)*, then $f(x) = f(0,\dots,0)$ for all $x \in G^n$.*

Proof. We first note that

$$B_{i_1,\dots,i_{l_1-1},1,i_{l_1+1},\dots,i_{l_k-1},1,i_{l_k+1}\dots,i_n} + B^*_{i_1,\dots,i_{l_1-1},2,i_{l_1+1},\dots,i_{l_k-1},2,i_{l_k+1}\dots,i_n} = 1, \tag{4}$$

for all $1 \le l_k \le n$, where

$$B^*_{i_1,\dots,i_{l_1-1},2,i_{l_1+1},\dots,i_{l_k-1},2,i_{l_k+1}\dots,i_n} = \sum_{i_1,\dots,i_{l_1-1},i_{l_1+1},\dots,i_{l_k-1},i_{l_k+1}\dots,i_n \in\{1,2\}} A_{i_1,\dots,i_{l_1-1},2,i_{l_1+1},\dots,i_{l_k-1},2,i_{l_k+1}\dots,i_n}.$$

Fix $j \in \{1,\dots,n\}$. Letting $x_{1p} = 0$ for all $p \in \{1,\dots,n\}\backslash\{j\}$ and $x_{2p} = 0$ for $1 \le p \le n$ in (3), we get

$$\begin{aligned}&f(0,\dots,0,x_{1j},0,\dots,0) = B_{i_1,\dots,i_{j-1},1,i_{j+1},\dots,i_n} f(0,\dots,0,x_{1j},0,\dots,0)\\ &+ B^*_{i_1,\dots,i_{j-1},2,i_{j+1},\dots,i_n} f(0,\dots,0),\end{aligned}$$

and hence

$$\begin{aligned}&\left(1 - B_{i_1,\dots,i_{j-1},1,i_{j+1},\dots,i_n}\right) f(0,\dots,0,x_{1j},0,\dots,0)\\ &+ B_{i_1,\dots,i_{j-1},1,i_{j+1},\dots,i_n} f(0,\dots,0,x_{1j},0,\dots,0)\\ &= B_{i_1,\dots,i_{j-1},1,i_{j+1},\dots,i_n} f(0,\dots,0,x_{1j},0,\dots,0)\\ &+ B^*_{i_1,\dots,i_{j-1},2,i_{j+1},\dots,i_n} f(0,\dots,0).\end{aligned} \tag{5}$$

It follows from Lemma 2.3 that (5) is equivalent to

$$\begin{aligned}&\left(1 - B_{i_1,\dots,i_{j-1},1,i_{j+1},\dots,i_n}\right) f(0,\dots,0,x_{1j},0,\dots,0)\\ &= B^*_{i_1,\dots,i_{j-1},2,i_{j+1},\dots,i_n} f(0,\dots,0).\end{aligned}$$

Now, relation (4) implies that

$$f(0,\dots,0,x_{1j},0,\dots,0) = f(0,\dots,0).$$

Similar to the above, one can show that

$$\begin{aligned}&\left(1 - B_{i_1,\dots,i_{l_1-1},1,i_{l_1+1},\dots,i_{l_k-1},1,i_{l_k+1}\dots,i_n}\right) f(0,\dots,0,x_{1_{l_1}},0,\dots,0,x_{1_{l_k}},0,\dots,0)\\ &= B^*_{i_1,\dots,i_{l_1-1},2,i_{l_1+1},\dots,i_{l_k-1},2,i_{l_k+1}\dots,i_n} f(0,\dots,0),\end{aligned}$$

where $2 \le l_k \le n$. Therefore, by (4) we conclude that $f(x) = f(0,\dots,0)$ for all $x \in G^n$. □

3. A stability result for the generalized multiadditive functional equations

Here and subsequently, for a mapping $f : V^n \longrightarrow W$, we consider the difference operator $\mathcal{D}f : V^n \times V^n \longrightarrow W$ by

$$\mathcal{D}f(x_1, x_2) := f(x_1 + x_2) - \sum_{i_1,\dots,i_n \in\{1,2\}} A_{i_1,\dots,i_n} f(x_{i_1 1}, \dots, x_{i_n n}),$$

for all $x_1, x_2 \in V^n$. We need the next result, indicated in [18, Theorem 13.4.3].

Theorem 3.1. *Let $g : \mathbb{R}^{p^N} \longrightarrow \mathbb{R}$ be a continuous p-additive function. Then, there exist constants $c_{j_1\dots j_p} \in \mathbb{R}$, $j_1, \dots, j_p = 1, \dots, N$, such that*

$$g(x_1, \dots, x_p) = \sum_{j_1=1}^{N} \dots \sum_{j_p=1}^{N} c_{j_1\dots j_p} x_{1 j_1} \dots x_{p j_p},$$

for all $x_i = (x_{i1}, \dots, x_{iN})$ and $i = 1, \dots, p$.

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a solution of the inequality

$$|\mathcal{D}f(x_1, x_2)| < \delta. \tag{6}$$

Similar to [4] and [10, Theorem 3] and using Theorem 3.1, one can show that there exists a multiadditive function $g(r_1, \dots, r_n) = cr_1 \dots r_n$, $c \in \mathbb{R}$, such that $|f(r_1, \dots, r_n) - g(r_1, \dots, r_n)| < \varepsilon$ for all $(r_1, \dots, r_n) \in \mathbb{R}^n$, where $\varepsilon = \frac{1}{B_{i_1,\dots,i_n} - 1}\delta$. Hence, inequality (6) can be written in the form

$$\mathcal{D}f(x_1, x_2) \in B(0, \delta),$$

where $B(0, \delta) := (-\delta, \delta)$, and thus

$$f(x_1 + x_2) + B(0, \delta) \subseteq \sum_{i_1,\dots,i_n \in\{1,2\}} \left[A_{i_1,\dots,i_n} f(x_{i_1 1}, \dots, x_{i_n n}) + B(0, \delta)\right].$$

Putting $F(x) = f(x) + B(0, \delta)$ for all $x \in \mathbb{R}^n$, we have

$$F(x_1 + x_2) \subseteq \sum_{i_1,\dots,i_n \in\{1,2\}} A_{i_1,\dots,i_n} F(x_{i_1 1}, \dots, x_{i_n n}),$$

provided that $A_{i_1,\dots,i_n} \geq 1$ for all $i_1, \dots, i_n \in \{1, 2\}$. Moreover, $g(x) \subseteq F(x)$ for all $x \in \mathbb{R}^n$.

Let W be a real normed space. The family of all closed and convex subsets, containing 0, of W will be denoted by $cc(P_0^*(W))$. Recall that for a metric space (X, d), the diameter of a set $E \subset X$ is defined to be $\mathrm{diam}E = \sup\{d(x, y) : x, y \in E\}$.

In this section, let V be a real vector space, $K \subseteq V$ be a convex cone with zero and W be a Banach space.

Theorem 3.2. *Let $F : K^n \longrightarrow cc(P_0^*(W))$ be a set-valued mapping satisfying*

$$F(x_1 + x_2) \subseteq \sum_{i_1,\dots,i_n \in\{1,2\}} A_{i_1,\dots,i_n} F(x_{i_1 1}, \dots, x_{i_n n}), \tag{7}$$

and

$$\sup\{\mathrm{diam}F(x) : x \in K^n\} < \infty,$$

for all $x_1, x_2 \in K^n$ and $B_{i_1,\dots,i_n} \neq 1$. Then, there exists a unique multiadditive mapping $\mathcal{A} : K^n \longrightarrow W$ such that $\mathcal{A}(x) \in F(x)$ for all $x \in K^n$.

Proof. Putting $x_1 = x_2 = x$ in (7), we get

$$F(2x) \subseteq \sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n} F(x) = B_{i_1,\dots,i_n} F(x), \tag{8}$$

for all $x \in K^n$. Replacing x by $2^m x$ with $m \in \mathbb{N}$, in (8), we obtain

$$F(2^{m+1}x) \subseteq B_{i_1,\dots,i_n} F(2^m x),$$

for all $x \in K^n$ and so

$$\frac{F(2^{m+1}x)}{2^{m+1}} \subseteq \frac{B_{i_1,\dots,i_n}}{2} \times \frac{F(2^m x)}{2^m},$$

for all $x \in K^n$. Thus

$$\frac{F(2^{m+1}x)}{2^{m+1}} \subseteq \frac{F(2^m x)}{2^m},$$

for all $x \in K^n$. Let $m \in \mathbb{N}_0$. Set $F_m(x) = \frac{F(2^m x)}{2^m}$ for all $x \in K^n$. It is easily seen that $\{F_m(x)\}_{m\in\mathbb{N}_0}$ is a decreasing sequence of closed subsets of W. Moreover,

$$\mathrm{diam} F_m(x) = \frac{1}{2^m} \mathrm{diam} F(2^m x).$$

Since $\sup\{\mathrm{diam}(F(x)) : x \in K^n\}$ is finite, we find $\lim_{m\to\infty} \mathrm{diam}(F_m(x)) = 0$ for all $x \in K^n$. Applying the Cantor theorem for the sequence $\{F_m(x)\}_{m\in\mathbb{N}_0}$, we see that the intersection $\bigcap_{m\in\mathbb{N}_0} F_m(x)$ is a singleton set and we denote it by $\mathcal{A}(x)$ for all $x \in K^n$ which is in fact a map $\mathcal{A} : K^n \longrightarrow W$. In other words, $\mathcal{A}(x) \in F_0(x) = F(x)$ for all $x \in K^n$. Here, we show that $\mathcal{A}$ is a generalized multiadditive mapping. We have

$$\begin{aligned}
F_m(x_1 + x_2) &= \frac{F(2^m(x_1 + x_2))}{2^m} = \frac{F(2^m x_1 + 2^m x_2)}{2^m} \\
&\subseteq \frac{\sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n} F\left(2^m(x_{i_1 1}, \dots, x_{i_n n})\right)}{2^m} \\
&= \sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n} \frac{F\left(2^m(x_{i_1 1}, \dots, x_{i_n n})\right)}{2^m} \\
&= \sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n} F_m\left(x_{i_1 1}, \dots, x_{i_n n}\right).
\end{aligned}$$

By the above relation and the definition of $\mathcal{A}$, we arrive at

$$\mathcal{A}(x_1 + x_2) = \bigcap_{m\in\mathbb{N}_0} F_m(x_1 + x_2) \subseteq \bigcap_{m\in\mathbb{N}_0} \left(\sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n} F_m\left(x_{i_1 1}, \dots, x_{i_n n}\right) \right), \tag{9}$$

for all $x_1, x_2 \in K^n$. In addition, one can show that

$$\sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n} \mathcal{A}\left(x_{i_1 1}, \dots, x_{i_n n}\right) \in \sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n} F_m\left(x_{i_1 1}, \dots, x_{i_n n}\right). \tag{10}$$

Fix now $m \in \mathbb{N}$ and $x_1, x_2 \in K^n$. It follows from (9) that there exist $\mathfrak{A}_{i_1,\dots,i_n} \in F_m\left(x_{i_1 1}, \dots, x_{i_n n}\right)$ such that

$$\mathcal{A}(x_1 + x_2) = \sum_{i_1,\dots,i_n\in\{1,2\}} A_{i_1,\dots,i_n} \mathfrak{A}_{i_1,\dots,i_n}. \tag{11}$$

On the other hand, relation (10) implies that there exist $\mathfrak{B}_{i_1,\ldots,i_n} \in F_m\left(x_{i_1 1}, \ldots, x_{i_n n}\right)$ such that

$$\sum_{i_1,\ldots,i_n \in \{1,2\}} A_{i_1,\ldots,i_n} \mathcal{A}\left(x_{i_1 1}, \ldots, x_{i_n n}\right) = \sum_{i_1,\ldots,i_n \in \{1,2\}} A_{i_1,\ldots,i_n} \mathfrak{B}_{i_1,\ldots,i_n}. \tag{12}$$

It follows from (11) and (12) that

$$\mathcal{A}(x_1 + x_2) - \sum_{i_1,\ldots,i_n \in \{1,2\}} A_{i_1,\ldots,i_n} \mathcal{A}\left(x_{i_1 1}, \ldots, x_{i_n n}\right) = \sum_{i_1,\ldots,i_n \in \{1,2\}} A_{i_1,\ldots,i_n} \left(\mathfrak{A}_{i_1,\ldots,i_n} - \mathfrak{B}_{i_1,\ldots,i_n}\right). \tag{13}$$

Since $\mathfrak{A}_{i_1,\ldots,i_n}, \mathfrak{B}_{i_1,\ldots,i_n} \in F_m\left(x_{i_1 1}, \ldots, x_{i_n n}\right)$, we obtain

$$\begin{aligned}
&\left\|\mathcal{A}(x_1 + x_2) - \sum_{i_1,\ldots,i_n \in \{1,2\}} A_{i_1,\ldots,i_n} \mathcal{A}\left(x_{i_1 1}, \ldots, x_{i_n n}\right)\right\| \\
&\leq \sum_{i_1,\ldots,i_n \in \{1,2\}} A_{i_1,\ldots,i_n} \left\|\mathfrak{A}_{i_1,\ldots,i_n} - \mathfrak{B}_{i_1,\ldots,i_n}\right\| \\
&\leq \sum_{i_1,\ldots,i_n \in \{1,2\}} A_{i_1,\ldots,i_n} \operatorname{diam} F_m\left(x_{i_1 1}, \ldots, x_{i_n n}\right),
\end{aligned}$$

which goes to zero as m tends to infinity. Thus

$$\mathcal{A}(x_1 + x_2) = \sum_{i_1,\ldots,i_n \in \{1,2\}} A_{i_1,\ldots,i_n} \mathcal{A}\left(x_{i_1 1}, \ldots, x_{i_n n}\right).$$

This completes the proof. □

4. Stability results for the multiadditive mappings

In this section, we prove the Hyers-Ulam stability for the multiadditive mapping (2) of the set-valued. Let W be a Banach space. The dual space of W is denoted by W^*. For a given set $A \in P(W)$, the distance function $d(\cdot, A)$ and the support function $s(\cdot, A)$ are respectively defined by

$$d(x, A) = \inf\{\|x - y\| : y \in A\}, \qquad x \in W,$$
$$s(x^*, A) = \sup\{\langle x^*, y\rangle : y \in A\}, \qquad x^* \in W^*.$$

For each pair $K, K' \in b(W)$, the Hausdorff distance between K and K' defined by

$$H(K, K') = \inf\{\lambda > 0 : K \subseteq K' + \lambda B_W, \quad K' \subseteq K + \lambda B_W\},$$

where B_W is the closed unit ball in W. For any $C, C' \in cc(W)$, we denote by $C \oplus C' = \overline{C + C'}$. Some properties of the Hausdorff distance is in the next proposition.

Proposition 4.1. *Let $C, C', K, K' \in bcc(W)$ and $\lambda > 0$. Then, the following properties hold:*

(i) $H(C \oplus C', K \oplus K') \leq H(C, K) + H(C', K')$;
(ii) $H(\lambda C, \lambda K) = \lambda H(C, K)$.

Let W be a Banach space and $(bcc(W), \oplus, H)$ be endowed with the Hausdorff distance H. Then, $(bcc(W), \oplus, H)$ is a complete metric semigroup; see [7]. Moreover, Debreu [7] showed that $(bcc(W), \oplus, H)$ is isometrically embedded in a Banach space as follows.

Lemma 4.2. *Let $C(B_{W^*})$ be the Banach space of continuous real-valued functions on B_{W^*} endowed with the uniform norm $\|\cdot\|_u$. Then, the mapping $j : (bcc(W), \oplus, H) \longrightarrow C(B_{W^*})$ defined through $j(K) = s(\cdot, K)$ satisfies the following properties:*

(i) $j(A \oplus B) = j(A) + j(B)$;
(ii) $j(\lambda A) = \lambda j(A)$;
(iii) $H(A, B) = \|j(A) - j(B)\|_u$;
(iv) $j(bcc(W))$ *is closed in* $C(B_{W^*})$.

for all $A, B \in bcc(W)$ *and all* $\lambda \geq 0$.

Let $f : X \longrightarrow (bcc(W), H)$ be a set-valued function from a complete finite measure space $(X, \sum, \nu)$ into $bcc(W)$. Then, f is said to be *Debreu integrable* if the composition $j \circ f$ is Bochner integrable [8]. In this case, the Debreu integral of f in X is the unique element $(D) \int_X f d\nu \in bcc(W)$ such that $j\left((D) \int_X f d\nu\right)$ is the Bochner integral of $j \circ f$. The set of Debreu integrable functions from X to $bcc(W)$ will be denoted by $D(X, bcc(W))$. Moreover, on $D(X, bcc(W))$, we define $(f + g)(v) = f(v) \oplus g(v)$ for all $f, g \in D(X, bcc(W))$ and so we find that $((X, bcc(W)), +)$ is an abelian semigroup.

The upcoming theorem was presented in [12] which is useful to our goals for the rest of the paper.

Theorem 4.3. *Let* (Ω, d) *be a complete generalized metric space and* $\mathcal{J} : \Omega \longrightarrow \Omega$ *be a mapping with Lipschitz constant* $L < 1$. *Then, for each element* $x \in \Omega$, *either* $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$ *for all* $n \geq 0$, *or there exists a natural number* n_0 *such that*

(i) $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty$ *for all* $n \geq n_0$;
(ii) *the sequence* $\{\mathcal{J}^n x\}$ *is convergent to a unique fixed point* z *of* $\mathcal{J}$ *which is in the set*

$$\Lambda = \{y \in \Omega : d(\mathcal{J}^{n_0} x, y) < \infty\};$$

(iii) $d(y, z) \leq \frac{1}{1-L} d(y, \mathcal{J} y)$ *for all* $y \in \Lambda$.

Definition 4.4. *Let* V *be a vector space,* W *be a normed space and* $f : V^n \longrightarrow bcc(W)$. *The set-valued multiadditive functional equation is defined by*

$$f(x_1 + x_2) = \bigoplus_{i_1, \ldots, i_n \in \{1,2\}} f(x_{i_1 1}, \ldots, x_{i_n n}),$$

for $x_1, x_2 \in V^n$. *Every solution of the set-valued multiadditive functional equation is called a set-valued multiadditive mapping.*

Here, we present the main theorem of this section.

Theorem 4.5. *Let* $\beta \in \{-1, 1\}$, V *be a topological vector space and* W *be a Banach space. Let* $\varphi : V^n \times V^n \longrightarrow [0, \infty)$ *be a function such that there exists an* $L < 1$ *with*

$$\varphi(x_1, x_2) \leq \frac{L}{2^{n\beta}} \varphi\left(2^\beta x_1, 2^\beta x_2\right),$$

for all $x_1, x_2 \in V^n$. *Suppose that a mapping* $f : V^n \longrightarrow (bcc(W), H)$ *satisfying*

$$H\left(f(x_1 + x_2), \bigoplus_{i_1, \ldots, i_n \in \{1,2\}} f(x_{i_1 1}, \ldots, x_{i_n n})\right) \leq \varphi(x_1, x_2), \tag{14}$$

for all $x_1, x_2 \in V^n$. *If* r *and* M *are positive real numbers with* $r \neq n$ *and* $\operatorname{diam} f(x) \leq M \sum_{j=1}^n \|x_{1j}\|^r$ *for all* $x = x_1 \in V^n$, *then there exists a unique set-valued multiadditive mapping* $\mathcal{A} : V^n \longrightarrow (bcc(W), H)$ *such that*

$$H(f(x), \mathcal{A}(x)) \leq \frac{L^{\frac{\beta+1}{2}}}{2^n(1-L)} \varphi(x, x), \tag{15}$$

for all $x \in V^n$.

Proof. We first bring the proof for the case $\beta = 1$ and $r > n$. Putting $x_1 = x_2 = x$ in (14) and using the convexity of $f(x)$, we have

$$H\left(f(2x), 2^n f(x)\right) \leq \varphi(x, x), \tag{16}$$

and so

$$H\left(f(x), 2^n f\left(\frac{x}{2}\right)\right) \leq \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \leq \frac{L}{2^n}\varphi(x, x), \tag{17}$$

for all $x \in V^n$ (and for the rest of this proof, all the equations and inequalities are valid for all $x \in V^n$). Set $\Omega := \{f : V^n \longrightarrow bcc(W) | \, f(0, \dots, 0) = \{0\}\}$. Consider the generalized metric d on Ω as follows:

$$d(g, h) := \inf\{\lambda_{g,h} \in (0, \infty) : H(g(x), h(x)) \leq \lambda_{g,h}\varphi(x, x), \quad x \in V^n\},$$

where as usual $\inf \emptyset = \infty$. Similar to the proof of [5, Theorem 2.2] and [21, Lemma 2.1] (see also the proof of [25, Theorem 2.3]), one can show that (Ω, d) is a complete generalized metric space. We define a mapping $\mathcal{J} : \Omega \longrightarrow \Omega$ via

$$\mathcal{J} f(x) := 2^n f\left(\frac{x}{2}\right).$$

for all $x \in V^n$. We show that $\mathcal{J}$ is a strictly contractive operator with the Lipschitz constant L. To do this, take $g, h \in \Omega$ and $\lambda_{g,h} \in (0, \infty)$ with $d(g, h) = \lambda_{g,h}$. Then, $H(g(x), h(x)) \leq \lambda_{g,h}\varphi(x, x)$ and hence

$$\begin{aligned} H(\mathcal{J} g(x), \mathcal{J} h(x)) &= H\left(2^n g\left(\frac{x}{2}\right), 2^n h\left(\frac{x}{2}\right)\right) \\ &= 2^n H\left(g\left(\frac{x}{2}\right), h\left(\frac{x}{2}\right)\right) \\ &\leq 2^n \lambda_{g,h}\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \leq \lambda_{g,h} L\varphi(x, x). \end{aligned}$$

Therefore, $d(\mathcal{J} g, \mathcal{J} h) \leq \lambda_{g,h} L$. This shows that $d(\mathcal{J} g, \mathcal{J} h) \leq L d(g, h)$, as claimed. Let us next observe that from (17) it follows that

$$d(\mathcal{J} f(x), f(x)) \leq \frac{L}{2^n}.$$

We can now apply Theorem 4.3 for the space (Ω, d), the operator $\mathcal{J}$, $n_0 = 0$ and $x = f$ to deduce that the sequence $(\mathcal{J}^l f)_{l \in \mathbb{N}}$ is convergent in (Ω, d) and its limit, $\mathcal{A}$ is a unique fixed point of $\mathcal{J}$ in the set $\Lambda = \{g \in \Omega : d(g, f) < \infty\}$. In other words, $\mathcal{A}(x) = \lim_{l\to\infty} \mathcal{J}^l f(x) = \lim_{l\to\infty} 2^{nl} f\left(\frac{x}{2^l}\right)$ and

$$\mathcal{A}(x) = \frac{1}{2^n}\mathcal{A}(2x). \tag{18}$$

Moreover, there exists a $\lambda \in (0, \infty)$ satisfying $H(f(x), \mathcal{A}(x)) \leq \lambda\varphi(x, x)$. Next, note that $f \in \Lambda$ and therefore, part (iii) of Theorem 4.3 implies that

$$d(f, \mathcal{A}) \leq \frac{1}{1 - L} d(\mathcal{J} f(x), f(x)) \leq \frac{L}{2^n(1 - L)},$$

which proves (15). We now have

$$\begin{aligned} &H\left(\mathcal{A}(x_1 + x_2), \bigoplus_{i_1,\dots,i_n \in\{1,2\}} \mathcal{A}(x_{i_1 1}, \dots, x_{i_n n})\right) \\ &= \lim_{l\to\infty} 2^{nl} H\left(f\left(\frac{x_1 + x_2}{2^l}\right) \bigoplus_{i_1,\dots,i_n \in\{1,2\}} f\left(\frac{1}{2^l}\left(x_{i_1 1}, \dots, x_{i_n n}\right)\right)\right) \\ &\leq \lim_{l\to\infty} 2^{nl}\varphi\left(\frac{x_1}{2^l}, \frac{x_2}{2^l}\right) = 0, \end{aligned}$$

for all $x_1, x_2 \in V^n$. On the other hand, we have $\operatorname{diam} f(x) \leq M \sum_{j=1}^{n} \|x_{1j}\|^r$ and so

$$\operatorname{diam}\left(2^{nl} f\left(\frac{x}{2^l}\right)\right) \leq M \frac{2^{nl}}{2^{rl}} \sum_{j=1}^{n} \|x_{1j}\|^r.$$

Therefore, $\mathcal{A}(x) = 2^{nl} f\left(\frac{x}{2^l}\right)$ is a singleton set and

$$\mathcal{A}(x_1 + x_2) = \bigoplus_{i_1,\ldots,i_n \in \{1,2\}} \mathcal{A}\left(x_{i_1 1}, \ldots, x_{i_n n}\right),$$

for all $x_1, x_2 \in V^n$.

For the case $\beta = -1$ and $r < n$, from (16), we have

$$H\left(f(x), \frac{1}{2^n} f(2x)\right) \leq \frac{1}{2^n} \varphi(x, x).$$

The rest of the proof is similar to the previous part. □

Corollary 4.6. *Let p be a positive real number such that $p \neq n$, $\theta \geq 0$ be real numbers, and V be a real normed space. Suppose that $f : V^n \longrightarrow (bcc(W), H)$ is a mapping satisfying*

$$H\left(f(x_1 + x_2), \bigoplus_{i_1,\ldots,i_n \in \{1,2\}} f(x_{i_1 1}, \ldots, x_{i_n n})\right) \leq \theta \sum_{i=1}^{2} \sum_{j=1}^{n} \|x_{ij}\|^p,$$

for all $x_1, x_2 \in V^n$. If r and M are positive real numbers with $r \neq n$ and $\operatorname{diam} f(x) \leq M \sum_{j=1}^{n} \|x_{1j}\|^r$ for all $x = x_1 \in V^n$, then there exists a unique set-valued additive mapping $\mathcal{A} : V^n \longrightarrow (bcc(W), H)$ such that

$$H\left(f(x), \mathcal{A}(x)\right) \leq \frac{2\theta}{|2^n - 2^p|} \sum_{j=1}^{n} \|x_{1j}\|^p,$$

for all $x \in V^n$.

Proof. The proof follows from Theorem 4.5 by taking $L = 2^{-|n-p|}$ and $\varphi(x_1, x_2) = \theta \sum_{i=1}^{2} \sum_{j=1}^{n} \|x_{ij}\|^p$. □

Questions and Remarks

The current work provides guidelines for further research and proposals for new directions and open problems with relevant discussions. Here, we give some questions and information on the connections between the fixed point theory and the Hyers-Ulam stability.

(1) What are the set-valued solutions of a multi-quadratic mapping?
(2) How to establish the Hyers-Ulam stability of the set-valued multi-Jensen and multi-quadratic mappings by applying a fixed point theorem? What about the multi-cubic mappings?
(3) Following Senasukh and Saejung [33], one can discuss the stability of multiadditive and multi-quadratic mappings on certain groupoids.

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