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Universal functions for the Sharkovsky classes of maps

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Abstract. We exhibit a single interval map (called a universal map) that admits all those orbit patterns which are available in the first Sharkovsky class. An interval map is said to be in the first Sharkovsky class if every periodic point of it is a fixed point. This provides a way to find universal maps in the class of contractions on intervals. We also characterize all such universal maps in the first Sharkovsky class.

1. Introduction and Preliminaries

The function $\sin\frac{1}{x}$ (topologist's sine curve) and its sisters $x\sin\frac{1}{x}$ and $x^2\sin\frac{1}{x}$ are encountered frequently in Real Analysis to provide counterexamples such as:

- a discontinuous function with a connected graph,
- a connected planar set that is not path-connected,
- a discontinuity of the second kind,
- a continuous function that is not of bounded variation,
- a non-rectifiable curve,
- a differentiable function that is not continuously differentiable,
- a continuous function that is not uniformly continuous,

  and so on.

In this paper, we study the function $x\sin\frac{1}{x}$ from the view of topological dynamics. While doing so, we find one glaring contrast. In Real Analysis, it had a negative role of being peculiar. But in Topological Dynamics, it plays a positive role in "synthesizing", i.e., putting all things together.

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If $\mathcal{F}$ is a class of dynamical systems, an element $f \in \mathcal{F}$ is said to be universal for $\mathcal{F}$ if f admits all orbit patterns that are available for maps in $\mathcal{F}$. Some families $\mathcal{F}$ admit universal elements, and others do not. For instance, if $\mathcal{F}$ is the class of t-simple systems on $\mathbb{R}$, there is no universal element in it (see [6]). As a note, in topological dynamics, a similar idea of universality with respect to ω-limit sets appeared earlier. For details, see [4], [8].

Two real sequences $(a_n)_{n=0}^{\infty}$ and $(b_n)_{n=0}^{\infty}$ are said to be of the same order pattern if $a_m < a_n \iff b_m < b_n$ holds for all $m, n \in \mathbb{N}_0$. We denote the order pattern of a sequence (a_n) by $P_{(a_n)}$. The order pattern of a sequence $(f^n(x))_{n=0}^{\infty}$ in a real dynamical system $(\mathbb{R}, f)$ or (I, f) is called an orbit pattern. Here I is a closed interval in the real line, and any continuous map from I to I is called an interval map. We say that an interval map f admits an order pattern $P_{(a_n)}$ if $\exists\, x \in I$ such that $(f^n(x))_{n=0}^{\infty}$ and $(a_n)_{n=0}^{\infty}$ are having the same order pattern.

We denote by $\mathcal{F}_1$ (known as maps of first Sharkovsky type) the set of all interval maps that do not admit a 2-cycle. Similarly, $\mathcal{F}_n$ is the set of all interval maps that do not admit a 2^n-cycle. We say that an orbit pattern $P_{(x_n)}$ does not force a 2-cycle if there exists an interval map f admitting an order pattern $P_{(x_n)}$ such that it does not admit a 2-cycle (in other words, $f \in \mathcal{F}_1$). The next proposition gives an equivalent description (without proof) for maps in $\mathcal{F}_1$.

Proposition 1.1. *The following are equivalent for an interval map f:*
1. *f does not admit a 2-cycle.*
2. *f is an anti-symmetric relation.*
3. *Every periodic point of f is a fixed point of f.*
4. *If x is between y and $f(y)$, then y is not between x and $f(x)$, unless they are equal.*

An element x is called a wall in its trajectory if all its future terms are on the same side of it. In a sequence (a_n), we note that every term is a wall, if for all $n \in \mathbb{N}$,
i) $a_{n+1} > a_n$ implies $a_m > a_n$ for all $m > n$,
ii) $a_{n+1} < a_n$ implies $a_m < a_n$ for all $m > n$,
iii) $a_n = a_{n+1}$ implies $a_n = a_m$ for all $m > n$.
The following result throws more light on the kind of sequences that we are studying in this paper.

Proposition 1.2. *The following are equivalent for a (convergent) sequence (a_n) in I, with distinct terms*

1. Every term is a wall; that is, all future terms lie on one side of it.

2. It is the union of an increasing sequence followed in its right by a decreasing sequence, where one of these two subsequences may be finite.

3. Its terms go nearer and nearer to its limit p, when they are on the same side of p; i.e., $|a_m - p| < |a_n - p|$ for all those $m > n$ such that a_m and a_n are on the same side of p.

4. It (as an orbit pattern) does not force a 2-cycle.

The equivalence of (1) and (4) has been proved in [3], [9]. Other implications among the above can be proved, but we omit the proof. We also mention (without proof) a well-known result, which we are going to use in successive sections.

Theorem 1.3. *([3]) If I and J are closed intervals such that $f(I)$ contains J, then there is a closed subinterval K of I such that $f(K) = J$.*

In section 2, we list all such orbit patterns which are available in $\mathcal{F}_1$ maps. It is a formidable task because there are uncountably many of them. Still, we succeed by choosing a suitable index set (see Theorem 2.2).

In section 3, we take one particular example, namely the function $rx \sin \frac{1}{x}$ on $[-1, 1]$ for some $0 < r < 1$. We prove that this example provides a universal function in the first Sharkovsky class. Moreover, Theorem 3.1 provides us a universal function in the class of contraction maps on interval also.

In section 4, we improve this result by characterizing the universal maps in $\mathcal{F}_1$. Actually, there are uncountably many conjugacy types, but these maps together have a neat description (see Theorem 4.1). The function $x \sin \frac{1}{x}$ is one among them. Moreover, section 4 concludes with some more equivalent ways of this description.

We summarize the main theorems (Theorem 2.2, Theorem 3.1, and Theorem 4.1) proved here in the following equivalent ways:

Theorem 1. *The set $\mathbb{P}$ of all orbit patterns available in $\mathcal{F}_1$ can be naturally indexed by the set $J = \{L, R\}^{\mathbb{N}} \cup \{L, R\}^*$.*

This is reminiscent of the theory of continued fractions, where every real number has a tag from $\mathbb{N}^{\mathbb{N}} \cup \mathbb{N}^*$.

Theorem 2. *The function $f(x) = rx \sin \frac{1}{x}$ for some $0 < r < 1$ is an $\mathcal{F}_1$-map on $I = [-1, 1]$ with the following universal property: If g is any $\mathcal{F}_1$-map on $[-1, 1]$, for all $p \in I$, $\exists q \in I$ such that $(g^n(p))_{n=0}^{\infty}$ and $(f^n(q))_{n=0}^{\infty}$ are of the same order pattern.*

Theorem 3. *The following are equivalent for $f \in \mathcal{F}_1$:*

a) f is universal in $\mathcal{F}_1$, i.e., all orbit patterns that are available for any g in $\mathcal{F}_1$ are available for this f.

b) f admits a fixed point p for which arbitrarily near p, on both sides of p, the values taken by f swing both above p and below p.

2. Enlisting orbit patterns in $\mathcal{F}_1$

Let (x_n) be such that $x_n = f(x_{n-1})$ where $n \geq 2$, for some $f \in \mathcal{F}_1$. Then $x_n \to p$ for some p and $f(p) = p$. It is important to note that if some $x_k = p$, then $x_i = p$ for $i \geq k$.

Label every term of (x_n) whenever $x_n \neq p$ with L or R according as it moves to its left or right. When k is the least natural number such that $x_k = p$, then label up to $k-1$-th term of (x_n) with L or R according as it moves to its left or right. Moreover, it is important to note the following result.

Proposition 2.1. *Let (x_n) be an orbit of an $\mathcal{F}_1$-map such that $x_n \to p$, then*
x_i is labeled as L if and only if $x_i > p$ and
x_i is labeled as R if and only if $x_i < p$.

Proof. This follows from Proposition 1.2. Indeed, if x_i is labeled as L then $x_{i+1} < x_i$ and hence $x_n < x_i$ for all $n > i$. Therefore $p \leq x_i$. But $x_i \neq p$. Therefore $x_i > p$.

Conversely, if possible, $x_i > p$ and x_i is labeled as R. Then by our preceding argument, we get $x_n > x_i$ and $p \geq x_i$, which is a contradiction.

Similarly, for the second also. □

We choose the index set J as the union of $\{L, R\}^{\mathbb{N}}$ and $\{L, R\}^*$. Here $\{L, R\}^{\mathbb{N}}$ denotes the set of all sequences over $\{L, R\}$ and $\{L, R\}^*$ denotes the set of all words over $\{L, R\}$. Let $\mathbb{P}$ denote the set of all orbit patterns available in $\mathcal{F}_1$.

In general, the sequence over $\{L, R\}$ need not determine the orbit pattern. In other words, two different patterns may have the same L-R-sequence. Here is an example, $((-1)^n \frac{1}{n})$ and $((-1)^n n)$ are having the same L-R-sequence (namely $\overline{RL}$) but with different orbit patterns. This can be better understood as follows: The L-R-sequence contains only a part of the information that an order pattern provides. The order pattern provides the information: For each pair (m, n) of positive integers, which is smaller between the two numbers x_m and x_n? The L-R-sequence provides a part of this information, namely when $n = m + 1$.

We do not expect this partial information to determine the full information. But it happens when we confine our discussion to the orbit patterns available in $\mathcal{F}_1$. To the question: given $m < n$, is $f^m(x) < f^n(x)$ or not?, the answer gets determined as follows: if the $(m+1)$-th term in the L-R-sequence is R then $f^m(x) < f^n(x)$; if it is L, then $f^n(x) < f^m(x)$. The next theorem shows that J is a natural index set for $\mathbb{P}$. Moreover, J can be called the set of orbit pattern tags for $\mathcal{F}_1$.

Theorem 2.2. *There exists a natural bijection $\phi : \mathbb{P} \to J$.*

Proof. Let us define $\phi : \mathbb{P} \to J$ by taking $\phi(P_{(x_n)})$ as that element of J whose i-th term is the labeled of x_i as defined earlier, for all i. The function ϕ is well-defined. Indeed if (a_n) and (b_n) are representatives of two orbits that have the same order pattern, then ϕ takes them to the same sequence or word over $\{L, R\}$. This is because the n-th term of $\phi(P_{(a_n)})$ is L iff $a_n > a_{n+1}$. This happens iff $b_n > b_{n+1}$. This happens if and only if the n-th term of $\phi(P_{(b_n)})$ is also L. It is important to note that the empty word corresponds to the constant orbit, i.e., the orbit of a fixed point.

Let $w = w_1w_2w_3... \in J$. Take

$$x_1 = \begin{cases} 1 & \text{if } w_1 = L \\ -1 & \text{if } w_1 = R \\ 0 & \text{if } w_1 \text{ does not exist.} \end{cases}$$

Suppose x_1, ..., x_k have been defined. To define x_{k+1}:

$$x_{k+1} = \begin{cases} \frac{1}{k+1} & \text{if } w_{k+1} = L \\ -\frac{1}{k+1} & \text{if } w_{k+1} = R \\ 0 & \text{if } w_{k+1} \text{ is not there.} \end{cases}$$

The set $\{x_k : k \in \mathbb{N}\}$ constructed above is a discrete set. If $0 \in \{x_k : k \in \mathbb{N}\}$, it is finite. If $0 \notin \{x_k : k \in \mathbb{N}\}$ then $\{x_k : k \in \mathbb{N}\} \subset \{\pm\frac{1}{n} : n \in \mathbb{N}\}$. For every element (except at most two at the boundary), there is a next element and a previous element. For every pair of adjacent elements say x_m and x_n we join the point (x_n, x_{n+1}) and (x_m, x_{m+1}) by a line segment in the plane. When this is done for all such pairs and $(0, 0)$ is also taken, the graph of a function is ready; let the function be called f.

By the construction of the map f, it is clear that f is continuous. Observe that $|f(x)| < |x|$ on $[\inf \{x_k\}, \sup \{x_k\}]$ except 0, where $f(0) = 0$. If $p \neq 0$, then $|f^2(p)| < |p|$ i.e., p can not be a periodic point of period 2. Hence $f \in \mathcal{F}_1$ where $\phi(P_{(x_n)}) = w$. Hence ϕ is a surjection.

Let $\alpha \in J$. Then $\phi(P_{(x_n)}) = \alpha$ where (x_n) is a representative of an orbit in $\mathcal{F}_1$. For $m < n$ in $\mathbb{N}$, we can decide if $x_m < x_n$ or $x_m = x_n$ or $x_m > x_n$ using the wall condition. Let α_m be the m-th term of α. If α_m is L, then $x_m > x_n$. If α_m is R, then $x_m < x_n$. If α_m is empty, then $x_m = x_n$. Thus the orbit pattern of (x_n) is decided by α. This proves: ϕ is an injection. □

Remark 2.3. *Enlisting all orbit patterns available for $\mathcal{F}_1$-maps serves two other purposes (apart from the fact that it is natural on its own):*

1. It paves the way for the proof of the later theorems of this paper.

2. It can be used to choose one representative from each orbit pattern, from the set $\{\pm\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ as we have done in the proof of Theorem 2.2.

3. Motivating example of a universal function

The dynamics of a system is said to be completely understood if we can describe all its orbit patterns. An excellent monograph about combinatorial orbit patterns for one dimensional maps is [1]. In elementary textbooks like [5], examples have been provided where such a complete understanding is possible. In all those examples, only finitely many, or at most countably many order patterns are available. It will indeed be nice if the same is achieved for a more complicated example. This is what we do in this section. We are able to describe all the order patterns available for the interval map $rx \sin \frac{1}{x}$ on $[-1, 1]$ for $0 < r \leq 1$.

Theorem 3.1. *Let $0 < r < 1$. Then the function $f(x) := rx \sin \frac{1}{x}$ on $[-1, 1]$ is universal. In other words, for any $\alpha \in J$, $\exists x_\alpha \in [-1, 1]$ whose orbit pattern tag is $\phi(\alpha)$.*

Proof. It is easy to see that $f(x) = rx\sin\frac{1}{x}$ for $0 < r < 1$ is a first Sharkovsky type map. Note that 0 is the only fixed point. We will prove this theorem in three steps.

Step-I: Let $\delta > 0$. Then there exist four closed intervals L_1, L_2, L_3, L_4 such that:

L_1 and L_2 are inside the open interval $(0, \delta)$,

L_3 and L_4 are inside the open interval $(-\delta, 0)$,

$f(L_1)$ and $f(L_3)$ are of the form $[0, \eta]$,

$f(L_2)$ and $f(L_4)$ are of the form $[-\eta, 0]$ for some $\eta > 0$. Indeed for a given $\delta > 0$ choose $a \in (0, \delta)$ such that $f(a) = 0$ and f is increasing at a. Choose $b \in (a, \delta)$ such that f is increasing on (a, b). Take $L_1 = [a, b]$. Then $f(L_1) = [0, f(b)]$ as we want. Similarly, L_2, L_3, and L_4 are chosen with corresponding modifications. Observe that L_1, L_2, L_3 and L_4 do not contain 0.

Step-II: Let $s = (s_n)$ be in $\{L, R\}^{\mathbb{N}}$. Define

$$I_1 = \begin{cases} [\frac{1}{\pi} - \delta_1, \frac{1}{\pi} + \delta_1] & \text{if } s_1 = L \\ [-\frac{1}{\pi} - \delta_1, -\frac{1}{\pi} + \delta_1] & \text{if } s_1 = R \end{cases}$$

for a small enough $\delta_1 > 0$ such that f is monotone on I_1. Observe that $f(I_1)$ is an interval containing 0 as an interior point.

By Step-I, we choose a closed interval $J_1 \subset f(I_1)$ not containing 0 such that $f(J_1)$ is equal either $[0, \delta_2]$ or $[-\delta_2, 0]$ for some $\delta_2 > 0$. In fact, if $s_2 = L$, we choose $J_1 \subset (0, \delta_1) \cap f(I_1)$, if $s_2 = R$, we choose $J_1 \subset (-\delta_1, 0) \cap f(I_1)$. By Theorem 1.3, $\exists I_2 \subset I_1$ such that $f(I_2) = J_1$.

Suppose we have constructed $I_1 \supset I_2 \supset ... \supset I_k$ such that $f^j(I_j)$ is equal to $[0, \delta_j]$ or $[-\delta_j, 0]$ for some $\delta_j > 0$ where $2 \leq j \leq k$. To construct I_{k+1}, first, we take a closed interval $J_k \subset f^k(I_k)$ not containing 0 such that $f(J_k)$ is equal to either $[0, \delta_{k+1}]$ or $[-\delta_{k+1}, 0]$ for some $\delta_{k+1} > 0$. In fact, if $s_{k+1} = L$, we choose $J_k \subset (0, \delta_k) \cap f^k(I_k)$, if $s_{k+1} = R$, we choose $J_k \subset (-\delta_k, 0) \cap f^k(I_k)$. By Theorem 1.3, $\exists I_{k+1} \subset I_k$ such that $f^k(I_{k+1}) = J_k$.

Thus we have recursively constructed a nest of intervals $I_1 \supset I_2 \supset ... \supset I_n \supset ....$ Using NIT (Nested Interval Theorem), find an element x common to all these I_n's. Observe that for all $x \in I_1$, the orbit pattern tag of x starts with s_1. For all points $x \in I_2$, the orbit pattern tag of x starts with $s_1 s_2$. More generally for all $n \in \mathbb{N}$, $\forall x \in I_n$, the orbit pattern tag starts with $s_1 s_2 ... s_n$. Here the orbit pattern tag of x means $\phi(\alpha)$ where α is the orbit pattern of $(f^n(x))$. Therefore the orbit pattern tag of x is nothing but the given sequence $s = (s_n)$.

Step-III: Let w be a word of length n over $\{L, R\}$. We will show that there exists an element x_1 in $[-1, 1]$ such that $\phi(P_{(x_n)}) = w$ where $x_n = f(x_{n-1})$ for $n \geq 2$. If w is a word of length 1, then choose $x_1 = \frac{1}{\pi}$ for $w = L$ or $x_1 = -\frac{1}{\pi}$ for $w = R$. Assume $n \geq 2$. Construct $J_1, J_2, ..., J_{n-1}$ as in the previous. Note that $f(J_1) \supset J_2$, $f(J_2) \supset J_3$ ..., $f(J_{n-2}) \supset J_{n-1}$. As $f(J_m)$ always contain 0 for every m, choose $y_n \in J_{n-1}$ such that $f(y_n) = 0$. Choose $f(y_{j-1}) = y_j$ and $y_j \in J_{j-1}$ for $2 \leq j \leq n$. Again since $J_1 \subset f(I_1)$, then $\exists y_1 \in I_1$ such that $f(y_1) = y_2$. This y_1 has its orbit pattern represented by the given word w. □

Remark 3.2. *One can prove that $x\sin\frac{1}{x}$ is also universal in $\mathcal{F}_1$ by a careful choice of δ_n's and J_n's such that no J_n contains a fixed point. Moreover, $xsin\frac{1}{x}$ and $rx\sin\frac{1}{x}$ have the same collection of orbit patterns. One can use Theorem 4.1 to prove this (see Remark 4.2).*

Let us denote by C the set of all contraction maps on I. We next prove an interesting fact that C and $\mathcal{F}_1$ admit the same collection of orbit patterns. Moreover, we have the following consequence.

Corollary 3.3. *There exists a universal function in the class C.*

Proof. In this proof, without loss of generality, we assume the domain $I = [-1, 1]$. Let $f \in C$. If possible, let f admit a 2-cycle $\{p, q\}$ so that $f(p) = q$, $f(q) = p$. Then $|f(p) - f(q)| = |p - q|$, which contradicts the fact that f is contraction. Hence $f \in \mathcal{F}_1$. Hence {All orbit patterns available in C} $\subset$ {All orbit patterns available for $\mathcal{F}_1$}. Take $f(x) = \frac{x^3}{5}\sin\frac{1}{x}$ on $[-1, 1]$. Since $|f'(x)| \leq \frac{4}{5}$, f is contraction. A similar approach to Theorem 3.1 can be used to prove that it contains all the orbit patterns available in $\mathcal{F}_1$ on I. Therefore $\frac{x^3}{5}\sin\frac{1}{x}$ is universal for C. □

Remark 3.4. *For any $0 < r < 1$, $rx \sin \frac{1}{x}$ is not a contraction, so we are directly unable to use it to prove Corollary 3.3.*

Remark 3.5. *Our motivating example in this section serves two other purposes:*
1. In some sense $rx \sin \frac{1}{x}$ for some $0 < r < 1$ is the simplest example in $\mathcal{F}_1$ that is universal for $\mathcal{F}_1$.
2. The proof there gives a glimpse of the more complicated proof of the next section.

4. Characterization of Universal functions

Why should we give two proofs, one for the particular case of $rx \sin \frac{1}{x}$ for some $0 < r < 1$ and again for the general case? Can we not omit the proof in Section 3? We have included both because: The first proof gives the basic idea in the simplest case. It is easier. It needs refinement in the general proof. There are two main difficulties in the general proof (that were not encountered in the earlier proof). First, the set of points that go to the fixed point can have several limit points; that is, infinite fluctuations can happen at several points. Second, the graph of the function may cross the diagonal often, spoiling our easier method. This necessitates more care on the choice of J_n's. The endpoint of J_n, which is in the pre-image of the fixed point, is chosen carefully so that it is isolated from the required side. (In the notation of the proof, there is no point strictly between b_n and a_n that goes to p.) Next, for all points in J_n, the motion under f is unilateral. This is guaranteed by keeping J_n inside $\mathcal{S}$ (defined in the proof). These requirements are optional in the more straightforward case; thus, the particular proof also deserves to be grasped separately.

Theorem 4.1. *A function $f \in \mathcal{F}_1$ is universal if and only if $\exists$ a sequence (a_n) in I converging to a fixed point p of f such that*
$a_n > p$ if n is even,
$a_n < p$ if n is odd,
$f(a_n) > p$ if n or $n-1$ is a multiple of four and
$f(a_n) < p$ if $n-2$ or $n-3$ is a multiple of four.

Proof. First, we will prove the reverse implication, i.e., the conditions stated in the theorem are sufficient for a function to be universal. This part of the proof has two parts, namely the preparatory part and the recursive definition part.

In the preparatory part, first, we define b_n and study some essential properties of b_n (as Fact-I and Fact-II), which are helpful in our proof.

Let us define b_n by

$$b_n := \begin{cases} \sup \{x : f(x) = p \text{ and } x < a_n\} & \text{where } a_n > p \\ \inf \{x : f(x) = p \text{ and } x > a_n\} & \text{where } a_n < p \end{cases}$$

Fact-I: No other points between b_n and a_n go to p. Here $f(b_n) = p$ but no b_n is equal to p. And both $a_n \to p$, $b_n \to p$. Therefore there is an interval with b_n as an endpoint whose image is an interval with p as an endpoint. Moreover for any $\delta > 0$, $\exists$ $n \in \mathbb{N}$ and $0 < \epsilon < \delta$ such that
$b_{4n} \in [p, p+\delta]$ and $f(L_0^n) \supset [p, p+\eta]$ where $L_0^n = [b_{4n}, b_{4n} + \epsilon]$;
$b_{4n+1} \in [p-\delta, p]$ and $f(L_1^n) \supset [p, p+\eta]$ where $L_1^n = [b_{4n+1} - \epsilon, b_{4n+1}]$;
$b_{4n+2} \in [p, p+\delta]$ and $f(L_2^n) \supset [p-\eta, p]$ where $L_2^n = [b_{4n+2}, b_{4n+2} + \epsilon]$;
$b_{4n+3} \in [p-\delta, p]$ and $f(L_3^n) \supset [p-\eta, p]$ where $L_3^n = [b_{4n+3} - \epsilon, b_{4n+3}]$ for some $\eta > 0$.
Fact-II: Define $\mathcal{S} := \{x \in I : |f(x) - p| < |x - p|\}$. Observe that $b_n \in \mathcal{S}$, $\forall n \in \mathbb{N}$. Moreover, each b_n is an interior point of $\mathcal{S}$. Therefore, for suitable $\eta > 0$, we can choose $L_0^n, L_1^n, L_2^n, L_3^n$ contained in $\mathcal{S}$ such that Fact-I holds.

Let $s = (s_n)$ be any sequence over $\{L, R\}$. We will show that $\exists$ some element x is in the domain of f such that the orbit pattern tag of x is s.

In the recursive definition part, as a base step, first observe that $a_1 < p$ and $f(a_1) > p$, $a_2 > p$ and $f(a_2) < p$, $a_3 < p$ and $f(a_3) < p$, $a_4 > p$ and $f(a_4) > p$.

If $s_1 = L$ we will search for an element on the right side of p, and if $s_1 = R$ we will search for an element on the left side of p (because of Proposition 2.1). We choose I_1 such that

$$I_1 = \begin{cases} [b_2, b_2 + \delta_1] \text{ or } [b_4, b_4 + \delta_1] & \text{if } s_1 = L \\ [b_1 - \delta_1, b_1] \text{ or } [b_3 - \delta_1, b_3] & \text{if } s_1 = R \end{cases}$$

for a small enough $\delta_1 > 0$ so that $f(I_1)$ is an interval with p as an endpoint and $|f(x) - p| < |x - p|$ for all $x \in I_1$. Note that whether we will choose b_2 or b_4 or b_1 or b_3 will be decided by s_2.

Choose n_1 large enough such that $|a_n - p| < \delta_1$ for all $n \geq n_1$. Then choose $m_1 > n_1$ such that

m_1 is a multiple of 4 if $s_2 = L = s_3$,

m_1 is of the form $4k + 1$ if $s_2 = R$ and $s_3 = L$,

m_1 is of the form $4k + 2$ if $s_2 = L$ and $s_3 = R$,

m_1 is of the form $4k + 3$ if $s_2 = R = s_3$.

By Fact-I and Fact-II, we can choose an interval $J_1 \subset f(I_1)$ with b_{m_1} as an endpoint such that $f(J_1)$ is equal to $[p - \delta_2, p]$ or $[p, p + \delta_2]$ for some $\delta_2 > 0$ and $|f(x) - p| < |x - p|$ for $x \in J_1$. In fact, $J_1 \subset (p, p + \delta_1) \cap f(I_1)$ if $s_2 = L$ or $J_1 \subset (p - \delta_1, p) \cap f(I_1)$ if $s_2 = R$. By Theorem 1.3, $\exists I_2 \subset I_1$ such that $f(I_2) = J_1$.

Suppose we have constructed $I_1 \supset I_2 \supset ... \supset I_k$ such that $f^j(I_j)$ is equal to $[p, p + \delta_j]$ or $[p - \delta_j, p]$ for some $\delta_j > 0$ where $2 \leq j \leq k$.

To construct I_{k+1}: We make a succession of choices as under:

First, choose $n_k \in \mathbb{N}$ such that $|a_n - p| < \delta_k$ for all $n \geq n_k$.

Next, look at s_{k+1} (which may be L or R), accordingly choose $m_k > n_k$ so that b_{m_k} is as wanted (After choosing n_k, we choose m_k. Its parity or rather its conjugacy class modulo 4, is determined by s_{k+1} and s_{k+2}.) and choose $J_k \subset f^k(I_k)$ with b_{m_k} as an endpoint such that $f(J_k)$ is equal to $[p, p + \delta_{k+1}]$ or $[p - \delta_{k+1}, p]$ for some $\delta_{k+1} > 0$ where $|f(x) - p| < |x - p|$ for all $x \in J_k$. In fact, if $s_{k+1} = L$, we choose $J_k \subset (p, p + \delta_k) \cap f^k(I_k)$, if $s_{k+1} = R$, we choose $J_k \subset (p - \delta_k, p) \cap f^k(I_k)$.

Lastly, take $I_{k+1} \subset I_k$ such that $f^k(I_{k+1}) = J_k$ (by Theorem 1.3).

Thus we have recursively constructed a nest of intervals $I_1 \supset I_2 \supset ... \supset I_n \supset ....$ Using NIT, we find that $\cap_{n=1}^{\infty} I_n \neq \emptyset$. By analogous method (used in Theorem 3.1), the orbit pattern tag of $x \in \cap_{n=1}^{\infty} I_n$ is nothing but the given sequence $s = (s_n)$.

Let w be a word of length n over $\{L, R\}$. We will show that there exists an element x_1 such that $\phi(P_{(x_n)}) = w$ where $x_n = f(x_{n-1})$ for $n \geq 2$. As above, construct an interval I_w such that every element in it has its orbit tag starting with w. Lastly since $f(I_w) = J_{n-1}$, there is x in I_w whose image in J_n goes to p. The orbit pattern tag of this x is w.

Conversely, since $f \in \mathcal{F}_1$ is universal, there exists x in I with the orbit pattern $\overline{RRLL}$. Let p be the limit point of $(f^n(x))$. Take $a_{4k+3} = f^{4k}(x)$, $a_{4k+1} = f^{4k+1}(x)$, $a_{4k} = f^{4k+2}(x)$ and $a_{4k+2} = f^{4k+3}(x)$. The sequence (a_n) is the desired sequence. □

Remark 4.2. *To see that Theorem 3.1 is a particular case of Theorem 4.1, we take*

$$a_n = \begin{cases} \frac{1}{2n\pi + \frac{\pi}{2}} & \textit{if } n \textit{ is of the form } 4k, \\ \frac{1}{2n\pi - \frac{\pi}{2}} & \textit{if } n \textit{ is of the form } 4k+2, \\ -\frac{1}{2n\pi + \frac{\pi}{2}} & \textit{if } n \textit{ is of the form } 4k+3, \\ -\frac{1}{2n\pi - \frac{\pi}{2}} & \textit{if } n \textit{ is of the form } 4k+1. \end{cases}$$

Then $a_n \to 0$ and $a_n \sin \frac{1}{a_n}$ is > 0 if n or $n-1$ is a multiple of four and < 0 if $n-2$ or $n-3$ is a multiple of four. In particular, this proves that $x \sin \frac{1}{x}$ is also universal in $\mathcal{F}_1$.

Remark 4.3. *Our characterization of universal functions serves two other purposes also.*

1. It throws further light on the forcing relation on the orbit patterns. It is no longer a partial order (though it was partial order in some other instances [2], [6]).

2. It shows that polynomial interval maps in $\mathcal{F}_1$ admit only countably many order patterns.

Summary 4.4. *Let f in $\mathcal{F}_1$ admit a fixed point p, say. There are five more equivalent conditions for f to be universal:*

i) For every $\epsilon > 0$, there is $\delta > 0$ such that both $f([p, p+\epsilon])$ and $f([p-\epsilon, p])$ contain $[p-\delta, p+\delta]$.

ii) Arbitrarily near p, on either side of p, both values $> p$ and values $< p$ are taken.

iii) For every $r > 0$, $inf\ f([p, p+r]) < p$, $sup\ f([p, p+r]) > p$; $inf\ f([p-r, p]) < p$ and $sup\ f([p-r, p]) > p$.

iv) p is in the closure of these four sets, namely, $f^{-1}((p, 1]) \cap (p, 1]$, $f^{-1}((p, 1]) \cap [0, p)$, $f^{-1}([0, p)) \cap [0, p)$, $f^{-1}([0, p)) \cap (p, 1]$.

v) p is a limit point of (each of) some four sequences (a_n), (b_n), (c_n) and (d_n), where for every n, the four numbers a_n, c_n, $f(a_n)$ and $f(b_n)$ are $> p$ and the other four numbers b_n, d_n, $f(c_n)$ and $f(d_n)$ are $< p$.

Conclusion 4.5. *Theorem 3.2 and its consequences in [10] are helpful to find the admissible orbit patterns in $\mathcal{F}_n$ for each n. One may use the 'period doubling' method and possible suitable pasting or gluing to find a universal function for $\mathcal{F}_n$. We omit the proof since the construction is analogous but technically gigantic. Our discussion on orbit patterns leads to one of the main fundamental questions: which orbit patterns force which others? Enlisting those may be challenging for general orbit patterns. However, for some orbit patterns, it can be achievable (see [7]).*

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