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Multiplicity of solutions for an elliptic problem involving GJMS operator

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Abstract. Given a compact Riemannian manifold (M, g) of dimension $n \geq 3$ without boundary, using the variational methods, we study the existence of solutions for the elliptic equation

$$P_g^k u = f|u|^{N-2}u + \lambda h|u|^{q-2}u, \tag{1}$$

where P_g^k is the GJMS operator of order $2k < n$, $h, f \in C^\infty(M)$, $1 < q < 2$, $\lambda > 0$ and N is the critical Sobolev exponent for the space $H_k^2(M)$. We apply Ljusternik-Schnirelmann theory on C^1-manifolds to prove that under some conditions, the equation (1) admits infinitely many solutions. At the end, we give two applications, one for Paneitz-Branson operator and the second is for the GJMS operator when $k = 3$.

1. Introduction and motivation

Let (M, g) be a compact and connected Riemannian manifold of dimension $n \geq 3$ without boundary. The GJMS operators are a family of conformally covariant differential operators introduced for the first time by Graham-Jenne-Mason-Sparling in their celebrated paper [10]. For any positive integer $k < n/2$, there exists a GJMS operator $P_g^k : C^\infty(M) \to C^\infty(M)$ with the following properties.

- The operator P_g^k can be written as

$$P_g^k = \Delta_g^k + \sum_{l=0}^{k-1} (-1)^l \nabla^{j_l \dots j_1} \left(A_l(g)_{i_1 \dots i_l} \nabla^{i_1 \dots i_l} \right),$$

  where, for $l \in \{0, 1, \dots, k-1\}$, $A_l(g)$ is a symmetric T_{2l}^0-tensor fields and $\Delta_g = -\operatorname{div}_g(\nabla_g)$ (See [12]).

- The operator P_g^k is formally self-adjoint with respect to the L^2-scalar product.

- For all smooth diffeomorphisms $\phi : C^\infty(M) \to C^\infty(M)$, we have $\phi^\star P_g^k = P_{\phi^\star g}^k$.

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- If $\tilde{g}$ is a conformal metric to g, given by

$$\tilde{g} = u^{\frac{N-2}{k}} g,$$

where $u \in C^\infty(M)$, $u > 0$ and $N = \frac{2n}{n-2k}$, then $\forall \phi \in C^\infty(M)$,

$$P_g^k(\phi u) = u^{N-1} P_{\tilde{g}}^k \phi.$$

In particular, by taking $\phi \equiv 1$ in the equation above, we get

$$P_g^k u = \frac{n-2k}{2} Q_{\tilde{g}}^k u^{N-1},$$

where

$$Q_{\tilde{g}}^k = \frac{2}{n-2k} P_{\tilde{g}}^k(1).$$

The quantity Q_g^k is called the Q-curvature associated to P_g^k. The problem of prescribing the Q-curvature within the conformal class of g is equivalent to the problem of finding a positive solution to the equation

$$P_g^k u = f|u|^{N-2} u,$$

for a given function $f \in C^\infty(M)$. The fact that N is the critical Sobolev exponent for the space $H_k^2(M)$ makes this problem particularly hard. The most prominent difficulty appears when $k > 1$, i.e. the fact that the absolute value of a function in $H_k^2(M)$ does not necessarily belong to $H_k^2(M)$, as well as the absence of a maximum principle for the operator P_k^g. Therefore, the positivity of eventual solutions is difficult to prove.

However, if the scalar curvature of (M, g) is constant and positive, then P_g^k can be factorised as follows

$$P_g^k = \prod_{l=1}^{k} (\Delta_g + c_l S_g),$$

where S_g denotes the scalar curvature of (M, g) and

$$c_l = \frac{(n+2l-2)(n-2l)}{4n(n-1)}.$$

As a result of that, by applying the strong maximum principle k times, we get that

- If $u \in C^{2k}(M)$ satisfies $P_g^k u \geq 0$ then either $u > 0$ or $u \equiv 0$.

The problem of prescribing the Q-curvature is particularly interesting in the case where $k = 1$, the operator P_g^1 being none other than the conformal Laplacian and the Q-curvature being the scalar curvature of the manifold multiplied by a constant. This problem of prescribing the scalar curvature has been extensively studied in the literature (see for example [3]). In particular Aubin proved that it is always possible to prescribe a constant scalar curvature on (M, g).

An other interesting case that has been extensively studied in the literature is when $k = 2$. The operator P_g^2 is known as the Paneitz-Branson operator and is given by

$$P_g^2 u = \Delta_g^2 u - \mathrm{div}_g \left[A^\# du \right] + \frac{n-4}{2} Q_g^2 u,$$

where the symbol ♯ stands for the musical isomorphism, A is the Schouten tensor :

$$A = \frac{(n-2)^2 + 4}{2(n-1)(n-2)} S_g g - \frac{4}{n-2} Ric_g,$$

and

$$Q_g^2 = \frac{1}{2(n-1)}\Delta_g S_g + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}S_g^2 - \frac{2}{(n-2)^2}\left|Ric_g\right|_g^2.$$

In [9], Esposito and Robert proved the following result:

Theorem 1.1. *Let (M,g) be a compact Riemannian n-manifold, $n \geq 5$, f, h be two functions in $C^\eta(M)$, $0<\eta<1$, $q \in \left(1, 2^\sharp - 1\right)$. We assume that P_g^2 is coercive, that f is positive and that there exists $v_0 \in H_2^2(M)\backslash\{0\}$ such that*

$$\sup_{t\geq 0} E(tv_0) < \frac{2}{nK_0^{\frac{n}{4}}\left(\sup_M f\right)^{\frac{n-4}{4}}},$$

where

$$E(u) := \frac{1}{2}\int_M uP_g^k u\, dv_g - \frac{1}{N}\int_M f|u|^N dv_g - \frac{1}{q+1}\int_M h|u|^{q+1}dv_g,$$

then the equation

$$P_g^2 u = f|u|^{2^\sharp-2}u + h|u|^{q-1}u$$

possesses a non-trivial solution $u \in C^{2k}(M)$.

In [4], by using the method of Nehari manifolds Benalili and Tahri proved the following result:

Theorem 1.2. *Let (M,g) be a compact n-dimensional Riemannian manifold, $n \geq 6$. Let $a, b, f \in C^\infty(M)$ with f positive and $x_o \in M$ such that $f(x_o) = \max_{x\in M} f(x)$. Let $0<\sigma<2$ and $0<\mu<4$. $1<q<2$ and $\lambda \in \mathbb{R}$. Denote by L the operator defined on $H_2^2(M)$ by $u \to L(u) = \Delta^2 u - \nabla^i\left(a\rho^{-\mu}\nabla_i u\right) + \rho^{-\alpha}bu$. Suppose that the operator P_g is coercive and*

$$\begin{cases} \frac{\Delta_g f(x_o)}{2(n-2)f(x_o)} + \frac{S_g(x_o)}{6(n-1)} < 0 \text{ and } S_g(x_o) > 0 & \text{in case } n > 6 \\ S_g(x_\circ) > 0 & \text{in case } n = 6 \end{cases}.$$

Then there exists $\lambda_* > 0$ such that if $\lambda \in (0, \lambda_*)$, the equation

$$Lu = \lambda|u|^{q-2}u + f(x)|u|^{N-2}u$$

possesses at least two distinct non trivial solutions in the distribution sense.

Motivated by their works, we study in this paper the existence of solutions to the following equation

$$P_g^k u = f|u|^{N-2}u + \lambda h|u|^{q-2}u,$$

where P_g^k is assumed to be coercive, $f, h \in C^\infty(M)$ are positive, $q \in (1,2)$ and $\lambda > 0$.

Under the assumptions above, we prove the following theorem:

Theorem 1.3. *There exists $\lambda_\star > 0$ such that if $\lambda \in (0, \lambda_\star)$ then equation (1) admits infinitely many pairs of weak solutions in $H_k^2(M)$ with negative energies. Moreover if $\exists v \in H_k^2(M)\backslash\{0\}$ such that*

$$\sup_{t>0} J_0(tv) < \frac{k}{n\left(\max_{x\in M} f(x)\right)^{\frac{n}{2k}-1} K_0^{\frac{n}{2k}}},$$

where

$$J_0(u) := \frac{1}{2}\int_M uP_g^k u\, dv_g - \frac{1}{N}\int_M f|u|^N dv_g.$$

Then equation (1) admits an other pair of non-trivial weak solutions in $H_k^2(M)$ with positive energies.

This paper is organized as follows : In Sect. 2, we introduce some notations and definitions as well as some results that will be useful later. In Sect. 3, we prove some useful properties of the Nehari manifold corresponding to equation (1). In Sect. 4, we prove the existence of Infinitely many pairs of weak solutions to equation (1), which have negative energies. In Sect. 5, we prove that under some condition, equation (1) admits another pair of weak solutions with positive energies, completing this way the proof of Theorem 1.3. In Sec. 6, we apply Theorem 1.3 to the Paneitz-Branson operator using some well chosen test functions. In the last section, we give an other application of our theorem for the GJMS operator of sixth order.

2. Notations and preliminaries

The Sobolev space $H_k^2(M)$ is defined as the completion of the space $C_0^\infty(M)$ with respect to the norm :

$$\|\cdot\|_{H_k^2(M)} : u \longmapsto \left(\sum_{l=0}^{k} \|\nabla_g^l u\|_2^2 \right)^{\frac{1}{2}}.$$

We denote by K_0 the sharp constant for the Euclidian Sobolev inequality $\|u\|_N^2 \leq K_0 \|\nabla_g^k u\|_2^2$. From [14], we know that

$$\frac{1}{K_0} = \pi^k \left(\frac{\Gamma(n/2)}{\Gamma(n)} \right)^{2k/n} \prod_{i=-k}^{k-1} (n+2i).$$

In [11], Mazumdar proved that for any $\epsilon > 0$, there exists a constant $B_\epsilon > 0$ such that for all $u \in H_k^2(M)$ one has:

$$\left(\int_M |u|^N dv_g \right)^{\frac{2}{N}} \leq (K_0 + \epsilon) \int_M \left(\Delta_g^{k/2} u \right)^2 dv_g + B_\epsilon \|u\|^2_{H^2_{k-1}(M)}. \tag{2}$$

Throughout this paper P_g^k is assumed to be coercive i.e. there exists $\Lambda > 0$ such that

$$\int_M u P_g^k u \, dv_g \geq \Lambda \|u\|^2_{H_k^2(M)}, \quad \forall u \in H_k^2(M). \tag{3}$$

As a consequence of that,

$$\|\cdot\|_{P_g^k} : u \longmapsto \left(\int_M u P_g^k u \, dv_g \right)^{1/2}$$

is a norm on $H_k^2(M)$ equivalent to $\|\cdot\|_{H_k^2(M)}$ (See [12]).

Also, it is easy to verify that for any sequence $(u_m)_{m \in \mathbb{N}}$ that converges weakly to u in $H_k^2(M)$, we have

$$\|u\|_{P_g^k} \leq \liminf_{m \to \infty} \|u_m\|_{P_g^k}.$$

Applicable examples of manifolds for which the operator P_g^k is coercive, are given in the following proposition.

Proposition 2.1. *Let k be a positive integer and (M, g) be a compact Riemannian manifold of dimension $n > 2k$. If S_g is constant and positive then the operator P_g^k is coercive.*

Proof. As mentioned in Sect. 1, if S_g is constant and positive, then

$$P_g^k = \prod_{l=1}^{k} (\Delta_g + c_l S_g).$$

By expanding the expression of P_g^k, we get

$$P_g^k = \sum_{l=0}^{k} a_l \Delta_g^{k-l},$$

where

$$a_l = \sum_{1 \leq i_1 < i_2 < \ldots < i_l \leq k} \left(\prod_{j=1}^{l} c_{i_j} S_g^{i_j} \right), \quad \forall l \in \{0, 2, \ldots, k\}.$$

Notice that $a_l > 0$, $\forall l \in \{0, 2, \dots, k\}$. Thus, $\forall u \in H_k^2(M)$, we have

$$\int_M u P_g^k u \, dv_g = \sum_{l=0}^{k} a_l \|\nabla_g^l u\|_2^2 \geq \underbrace{\min_{0 \leq l \leq k} a_l}_{>0} \|u\|_{H_k^2(M)}^2.$$

In other terms, P_g^k is coercive. □

We recall the notion of Palais-Smale condition :

Definition 2.2. *Let H be a Hilbert space, $\mathcal{M}$ be a C^1-submanifold of H and let $J \in C^1(\mathcal{M}, \mathbb{R})$. We say that J satisfies the Palais-Smale condition at level c if :*

Any sequence $(u_m)_{m \in \mathbb{N}}$ in $\mathcal{M}$, such that $J(u_m) \to c$ and $dJ(u_m) \to 0$, admits a strongly convergent subsequence.

We recall, the definition of the Krasnoselski genus (in [2]).

Definition 2.3. *Let H be a Hilbert space and let A be a subset of H symmetric with respect to the origin i.e. $\forall u \in A$, $-u \in A$. We define the (Krasnoselski) genus of A as*

$$\gamma(A) = \min\{m \in \mathbb{N} \mid \text{there exists an odd continuous map } \phi : H \longrightarrow \mathbb{R}^n \backslash \{0\}\}.$$

By convention, we set $\gamma(\emptyset) = 0$ and we set $\gamma(A) = +\infty$ if $\forall n \in \mathbb{N}$, there exists no odd continuous map from H to $\mathbb{R}^n \backslash \{0\}$.

As in [2], we define

$$\gamma_k(A) = \sup\{\gamma(K) \mid K \subset A, \ -K = K \text{ and } K \text{ is compact}\}.$$

In particular, if S is the unit sphere inside an infinite dimensional Hilbert space H, then $\gamma_k(S) = \gamma(S) = +\infty$. The following lemma (in [2]) will be useful.

Lemma 2.4. *Let H be a Hilbert space and let A be a subset of H symmetric with respect to the origin. If $\eta : A \to H$ is odd and continuous, then $\gamma(\eta(A)) \geq \gamma(A)$.*

We will use the following well known theorem (in [2, 13]) to prove the existence of infinitely many weak solutions to equation (1).

Theorem 2.5. *Let H be a infinite dimensional Hilbert space and let $\mathcal{M}$ be a closed C^1-submanifold of H. Let $J \in C^1(\mathcal{M}, \mathbb{R})$ be an even functional bounded from below. If the following conditions are satisfied :*

1. *$\mathcal{M}$ is symmetric with respect to the origin of H.*
2. *$\forall c < \sup_{u \in \mathcal{M}} J(u)$, the functional J satisfies the Palais-Smale condition at level c.*

then J admits at least $\gamma_k(M)$ distinct pairs of critical points in $\mathcal{M}$.

Let's now introduce the variational setting for this problem. The energy functional corresponding to equation (1) is given by

$$J_\lambda(u) := \frac{1}{2} \int_M u P_g^k u \, dv_g - \frac{1}{N} \int_M f|u|^N dv_g - \frac{\lambda}{q} \int_M h|u|^q dv_g, \quad \forall u \in H_k^2(M).$$

We introduce the Nehari manifold for this problem

$$\mathcal{N}_\lambda = \{u \in H_k^2(M) \backslash \{0\} \mid \Phi_\lambda(u) = 0\},$$

where

$$\Phi_\lambda(u) = J_\lambda'(u)u = \int_M u P_g^k u \, dv_g - \int_M f|u|^N dv_g - \lambda \int_M h|u|^q dv_g$$

We also introduce the two subsets :

$$\mathcal{N}_\lambda^+ = \{u \in \mathcal{N}_\lambda \mid \Phi'_\lambda(u)u > 0\},$$

$$\mathcal{N}_\lambda^- = \{u \in \mathcal{N}_\lambda \mid \Phi'_\lambda(u)u < 0\}$$

which are two C^1-submanifolds of $H_k^2(M)$ of codimension 1, and also the subset

$$\mathcal{N}_\lambda^0 = \{u \in \mathcal{N}_\lambda \mid \Phi'_\lambda(u)u = 0\}.$$

Lemma 2.6. *Let $J_\lambda\Big|_{\mathcal{N}_\lambda^\pm}$ denote the restriction of J_λ to $\mathcal{N}_\lambda^\pm$. Any critical point of $J_\lambda\Big|_{\mathcal{N}_\lambda^\pm}$, is a critical point of J_λ in $H_k^2(M)$ and a weak solution of (1).*

Proof. Let $u \in \mathcal{N}_\lambda^\pm$ be a critical point of $J_\lambda\Big|_{\mathcal{N}_\lambda^\pm}$ i.e there exists $\mu \in \mathbb{R}$ such that

$$J'_\lambda(u)v = \mu\Phi'_\lambda(u)v, \quad \forall v \in H_k^2(M).$$

By taking $v = u$, we get

$$\mu\Phi'_\lambda(u)u = J'_\lambda(u)u = \Phi_\lambda(u) = 0,$$

and, as $\Phi'_\lambda(u)u \neq 0$, we conclude that $\mu = 0$ and we have $J'_\lambda(u) = 0$.

In other terms, u is a critical point of J_λ in $H_k^2(M)$ and a weak solution of (1). □

3. Properties of the Nehari manifold

Lemma 3.1. *The following statements are true.*

- *If $u \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0$, then $\|u\|_{P_g^k} \leq \rho_\lambda$.*
- *If $u \in \mathcal{N}_\lambda^- \cup \mathcal{N}_\lambda^0$, then $\|u\|_{P_g^k} \geq \rho$.*

where

$$\rho_\lambda = \left(\frac{(N-q)\lambda \max_{x\in M} h(x) V(M)^{1-\frac{q}{N}} \max(K_0+\epsilon, B_\epsilon)^{q/2}\Lambda^{-q/2}}{N-2}\right)^{\frac{1}{2-q}},$$

and

$$\rho = \left(\frac{(2-q)}{(N-q)\max_{x\in M} f(x)\Lambda^{-N/2}\max(K_0+\epsilon, B_\epsilon)^{N/2}}\right)^{\frac{1}{N-2}}.$$

Proof. Let $u \in \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^0$, we have

$$0 \leq \Phi'_\lambda(u)u = 2\|u\|_{P_g^k}^2 - N\int_M f|u|^N dv_g - q\lambda\int_M h|u|^q dv_g.$$

Using the fact that

$$\Phi_\lambda(u) = \|u\|_{P_g^k}^2 - \int_M f|u|^N dv_g - \lambda\int_M h|u|^q dv_g = 0,$$

we get

$$0 \leq \Phi'_\lambda(u)u = (2-N)\|u\|_{P_g^k}^2 + (N-q)\lambda\int_M h|u|^q dv_g.$$

Thus,

$$(N-2)\|u\|^2_{P^k_g} \le (N-q)\lambda \int_M h|u|^q dv_g.$$

Using Hölder and Sobolev's inequalities and (3), we get

$$(N-2)\|u\|^2_{P^k_g} \le (N-q)\lambda \max_{x\in M} h(x) V(M)^{1-\frac{q}{N}} \max(K_0+\epsilon, B_\epsilon)^{q/2}\Lambda^{-q/2}\|u\|^q_{P^k_g}.$$

Hence, $\|u\|_{P^k_g} \le \rho_\lambda$, which proves the first statement.

Let $u \in \mathcal{N}^-_\lambda \cup \mathcal{N}^0_\lambda$, we have

$$0 \ge \Phi'_\lambda(u)u = (2-q)\|u\|^2_{P^k_g} - (N-q)\int_M f|u|^N dv_g,$$

By the Sobolev inequality and (3), we get

$$0 \ge (2-q)\|u\|^2_{P^k_g} - (N-q)\max_{x\in M} f(x) \max(K_0+\epsilon, B_\epsilon)^{N/2}\Lambda^{-N/2}\|u\|^N.$$

Hence, $\|u\|_{P^k_g} \ge \rho$. The second statement is now proved.
□

Lemma 3.2. *There exists $\lambda_\circ > 0$ such that if $\lambda \in (0,\lambda_\circ)$, then*

1. $\rho_\lambda < \rho$.
2. $\mathcal{N}^0_\lambda = \emptyset$
3. $\forall u \in H^2_k(M)$, *there exists $t^+ > 0$ such that $t^+u \in \mathcal{N}^+_\lambda$.*
4. $\forall u \in H^2_k(M)$, *there exists $t^- > 0$ such that $t^-u \in \mathcal{N}^-_\lambda$.*

Proof. As $\rho_\lambda \to 0$ when $\lambda \to 0$, by taking $\lambda_\circ > 0$ small enough, we get

$$\rho_\lambda < \rho, \quad \forall \lambda \in (0,\lambda_\circ).$$

Let $\lambda \in (0,\lambda_\circ)$. From the previous lemma, we know that if $u \in \mathcal{N}^0_\lambda$, then $\|u\|_{P^k_g} \le \rho_\lambda$ but also $\|u\|_{P^k_g} \ge \rho$. As $\rho_\lambda < \rho$, we conclude that $\mathcal{N}^0_\lambda = \emptyset$. The first and second statements are now proved.

Let $u \in H^2_k(M)\backslash\{0\}$, consider the map

$$\begin{aligned} F_u : \mathbb{R}_+ &\longrightarrow \mathbb{R} \\ t &\longmapsto \Phi_\lambda(tu). \end{aligned}$$

We have

$$F_u(t) = \|u\|^2_{P^k_g} t^2 - \left(\int_M f|u|^N dv_g\right) t^N - \lambda\left(\int_M h|u|^q dv_g\right) t^q.$$

Naturally, we get

$$F'_u(t) = \frac{\Phi'_\lambda(tu)(tu)}{t}$$

.

Using Hölder and Sobolev's inequalities and (3), we get

$$\int_M f|u|^N dv_g \le \max_{x\in M} f(x)\Lambda^{-N/2}\max(K_0+\epsilon, B_\epsilon)^{N/2}\|u\|^N,$$

and

$$\int_M h|u|^q dv_g \leq \max_{x\in M} h(x) V(M)^{1-\frac{q}{N}} \max(K_0+\epsilon, B_\epsilon)^{q/2}\Lambda^{-q/2}\|u\|^q_{P^k_g}.$$

Let's assume, without loss of generality, that $\|u\|_{P^k_g} = 1$, we get

$$F_u(t) \geq t^2 - \alpha t^N - \lambda\beta t^q,$$

where

$$\alpha = \max_{x\in M} f(x)\Lambda^{-N/2}\max(K_0+\epsilon, B_\epsilon)^{N/2} \quad \text{and} \quad \beta = \max_{x\in M} h(x)V(M)^{1-\frac{q}{N}}\max(K_0+\epsilon, B_\epsilon)^{q/2}\Lambda^{-q/2}.$$

Put

$$\lambda_\circ = \frac{1}{\beta}t_\circ^{2-q} - \frac{\alpha}{\beta}t_\circ^{N-q},$$

where $t_0 > 0$ is taken small enough, this way we have $\mathcal{N}^0_\lambda = \emptyset$ and $F_u(t_\circ) > 0$.

As $F_u(t) < 0$ for $t > 0$ small enough, we infer that there exists $t^+ \in (0, t_\circ)$ such that :

- $F_u(t^+) = 0$.
- $F'_u(t^+) \geq 0$.

In other terms, $t^+u \in \mathcal{N}^+_\lambda \cup \mathcal{N}^0_\lambda$. As $\mathcal{N}^0_\lambda = \emptyset$, we conclude that $t^+u \in \mathcal{N}^+_\lambda$.

Also, as $\lim_{t\to+\infty} F_u(t) = -\infty$, we infer that there exists $t^- \in (t_\circ, +\infty)$ such that :

- $F_u(t^-) = 0$.
- $F'_u(t^-) \leq 0$.

In other terms, $t^-u \in \mathcal{N}^-_\lambda \cup \mathcal{N}^0_\lambda$. As $\mathcal{N}^0_\lambda = \emptyset$, we conclude that $t^-u \in \mathcal{N}^-_\lambda$.

The proof is now complete.

□

Lemma 3.3. *Let $\lambda \in (0, \lambda_\circ)$, there exists an odd C^1-diffeomorphism $\psi^+ : S \to \mathcal{N}^+_\lambda$, where*

$$S = \{u \in H^2_k(M) \mid \|u\|_{H^2_k(M)} = 1\}.$$

Proof. Consider the map

$$\begin{aligned} F : H^2_k(M) \times \mathbb{R}_+ &\longrightarrow \mathbb{R} \\ (u,t) &\longmapsto \Phi_\lambda(tu). \end{aligned}$$

Naturally, we have

$$\frac{dF}{dt}(u,t) = \frac{\Phi'_\lambda(tu)(tu)}{t}.$$

Let $u \in S$. We claim that there exists a unique $t^+ > 0$ such that $t^+u \in \mathcal{N}^+_\lambda$. The existence of t^+ is already given by Lemma 3.2. Let's prove its uniqueness.

Suppose that there exist t_1 and t_2 with $t_2 > t_1 > 0$ and such that

- $t_1u \in \mathcal{N}^+_\lambda$ i.e. $F(u,t_1) = 0$ and $\frac{dF}{dt}(u,t_1) > 0$.
- $t_2u \in \mathcal{N}^+_\lambda$ i.e. $F(u,t_2) = 0$ and $\frac{dF}{dt}(u,t_2) > 0$.

Therefore, there exists $t_3 \in (t_1, t_2)$ such that :

- $F(u, t_3) = 0$
- $\frac{dF}{dt}(u, t_3) \leq 0$.

In other terms, $t_3 u \in \mathcal{N}_\lambda^- \cup \mathcal{N}_\lambda^0$. Thus, we get $\|t_2 u\|_{P_g^k} > \|t_3 u\|_{P_g^k} \geq \rho$, which is absurd given that $\|t_2 u\|_{P_g^k} \leq \rho_\lambda < \rho$.

We conclude that $\forall u \in S$, there exists a unique $t^+(u) > 0$ such that $t^+(u)u \in \mathcal{N}_\lambda^+$

This gives us a map

$$\begin{aligned} \psi^+ : & S \longrightarrow \mathcal{N}_\lambda^+ \\ & u \longmapsto t^+(u)u \end{aligned}$$

Obviously, ψ^+ is odd.

As F is C^1 and $\frac{dF}{dt}(u, t^+(u)) > 0$, $\forall u \in S$, we know from the implicit function theorem that $u \mapsto t^+(u)$ is a C^1-map. Hence, $\psi^+ \in C^1(S, \mathcal{N}_\lambda^+)$.

Also, ψ^+ possesses an obvious inverse which is the projection onto the sphere : $u \mapsto \frac{u}{\|u\|_{H_k^2(M)}}$.

□

4. Solutions on $\mathcal{N}_\lambda^+$

Lemma 4.1. *Let $\lambda \in (0, \lambda_\circ)$, then $J_\lambda(\mathcal{N}_\lambda^+)$ is a subset of $\mathbb{R}_-^\star$ and bounded below.*

Proof. Let $u \in \mathcal{N}_\lambda^+$, as $\Phi_\lambda(u) = 0$, we have

$$J_\lambda(u) = \frac{N-2}{2N}\|u\|_{P_g^k}^2 - \lambda\frac{N-q}{qN}\int_M h|u|^q dv_g.$$

Also, as $\Phi'_\lambda(u)u = (2-N)\|u\|_{P_g^k}^2 + (N-q)\lambda\int_M h|u|^q dv_g > 0$, we get

$$\lambda\int_M h|u|^q dv_g > \frac{N-2}{N-q}\|u\|_{P_g^k}^2.$$

We infer that

$$J_\lambda(u) < \underbrace{\left(\frac{1}{2N} - \frac{1}{qN}\right)}_{<0}(N-2)\|u\|_{P_g^k}^2 < 0.$$

We know from Lemma 3.1 that $\|u\|_{P_g^k} \leq \rho_\lambda$. Using Hölder and Sobolev's inequalities and (3), we find that

$$\lambda\int_M h|u|^q dv_g \leq \lambda \max_{x\in M} h(x)V(M)^{1-\frac{q}{N}} \max(K_0 + \epsilon, B_\epsilon)^{q/2}\Lambda^{-q/2}\rho_\lambda^q.$$

This implies that

$$J_\lambda(u) > -\lambda\frac{N-q}{qN}\max_{x\in M} h(x)V(M)^{1-\frac{q}{N}} \max(K_0 + \epsilon, B_\epsilon)^{q/2}\Lambda^{-q/2}\rho_\lambda^q,$$

which completes the proof. □

Lemma 4.2. *There exists $\lambda_1 \in (0, \lambda_\circ)$ such that $\forall\lambda \in (0, \lambda_1)$ the following statement is true.*

- *For any sequence $(u_m)_{m\in\mathbb{N}}$ in $\mathcal{N}_\lambda^+$ such that $J_\lambda(u_m) \to c$, where $c < 0$, we have*

$$\liminf_{m\to\infty} \Phi'_\lambda(u_m)u_m > 0$$

Proof. Let $(u_m)_{m\in\mathbb{N}}$ in $\mathcal{N}_\lambda^+$ such that $J_\lambda(u_m) \to c < 0$.

As J_λ is continuous over $H_k^2(M)$, we infer that there exists $\epsilon > 0$ such that $\forall m \in \mathbb{N}$, $\|u_m\|_{P_g^k} > \epsilon$ (otherwise 0 would be an accumulation point for $(J_\lambda(u_m))_{m\in\mathbb{N}}$).

As $\Phi(u_m) = 0$, we have

$$\Phi'_\lambda(u_m)u_m = (2-q)\|u_m\|^2 - (N-q)\int_M f|u|^N dv_g.$$

Using the Sobolev inequality and (3) we get

$$\Phi'_\lambda(u_m)u_m \geq (2-q)\|u_m\|^2 - (N-q)\max_{x\in M} f(x)\Lambda^{-N/2}\max(K_0+\epsilon, B_\epsilon)^{N/2}\|u_m\|^N,$$

and, as $\|u_m\| \leq \rho_\lambda$, we obtain

$$\Phi'_\lambda(u_m)u_m \geq \|u_m\|^2\left((2-q) - (N-q)\max_{x\in M} f(x)\Lambda^{-N/2}\max(K_0+\epsilon, B_\epsilon)^{N/2}\rho_\lambda^{N-2}\right).$$

Let

$$B = \left(\frac{(2-q)}{(N-q)\Lambda^{-N/2}\max(K_0+\epsilon, B_\epsilon)^{N/2}\max_{x\in M} f(x)}\right)^{\frac{1}{N-2}}.$$

By taking $\lambda \in (0, \lambda_1)$, where

$$\lambda_1 = \min\left(\lambda_\circ, \ \frac{B^{2-q}(N-2)}{\max_{x\in M} h(x)V(M)^{1-\frac{q}{N}}\max(K_0+\epsilon, B_\epsilon)^{q/2}\Lambda^{-q/2}(N-q)}\right),$$

we get

$$\rho_\lambda = \left(\frac{(N-q)\lambda\max_{x\in M} h(x)V(M)^{1-\frac{q}{N}}\max(K_0+\epsilon, B_\epsilon)^{q/2}\Lambda^{-q/2}}{N-2}\right)^{\frac{1}{2-q}} < B,$$

and thus,

$$\Phi'_\lambda(u_m)u_m \geq \epsilon^2\underbrace{\left((2-q) - (N-q)\max_{x\in M} f(x)\Lambda^{-N/2}\max(K_0+\epsilon, B_\epsilon)^{N/2}\rho_\lambda^{N-2}\right)}_{>0} > 0.$$

The proof is now complete. □

Proposition 4.3. *There exists $\lambda_\star^+ \in (0, \lambda_1)$ such that $\forall \lambda \in (0, \lambda_\star^+)$, the restriction of J_λ to $\mathcal{N}_\lambda^+$, satisfies the Palais-Smale condition at level c, $\forall c < \sup_{u\in\mathcal{N}_\lambda^+} J_\lambda(u)$*

Proof. Suppose that $\lambda \in (0, \lambda_1)$. We know from Lemma 4.1 that $\sup_{u\in\mathcal{N}_\lambda^+} J_\lambda(u) \leq 0$. Let $(u_m)_{m\in\mathbb{N}}$ be a Palais-Smale sequence at level $c < 0$ in $\mathcal{N}_\lambda^+$ i.e.

1. $J_\lambda(u_m) \to c$.
2. $J'_\lambda(u_m) - \mu_m\Phi'(u_m) \to 0$, where

$$\mu_m = \frac{\langle J'_\lambda(u_m), \Phi'_\lambda(u_m)\rangle}{\|\Phi'_\lambda(u_m)\|^2}.$$

Given that the sequence $(u_m)_{m\in\mathbb{N}}$ is bounded (from Lemma 3.1) and the space $H_k^2(M)$ is reflexive, we may assume $(u_m)_{m\in\mathbb{N}}$ to be weakly convergent to u in $H_k^2(M)$. Thus,$(u_m)_{m\in\mathbb{N}}$ converges strongly to u in $L^p(M)$ for $p < N$.

As $(u_m)_{m\in\mathbb{N}}$ converges weakly to u and $J'_\lambda(u_m) - \mu_m\Phi'_\lambda(u_m) \to 0$ strongly in $(H_k^2(M))^\star$, we have

$$\lim_{m\to\infty} \mu_m\phi'_\lambda(u_m)u_m = \lim_{m\to\infty}\left(\underbrace{J'_\lambda(u_m)u_m}_{=0} - \left[J'_\lambda(u_m) - \mu_m\phi'_\lambda(u_m)\right]u_m\right) = 0,$$

and we know from Lemma 4.2 that $\liminf_{m\to\infty}\Phi'_\lambda(u_m)u_m > 0$. We conclude that $\mu_m \to 0$.

We can now prove that $\Phi_\lambda(u) = 0$, we have

$$\Phi_\lambda(u) = J'_\lambda(u)u = \left(\lim_{m\to\infty}\int_M uP_g^k u_m\, dv_g - \int_M f|u_m|^{N-2}u_m u dv_g - \lambda\int_M h|u_m|^{q-2}u_m u dv_g\right),$$

and as

$$\lim_{m\to\infty} J'_\lambda(u_m)u = \lim_{m\to\infty}\left(J'_\lambda(u_m)u - \underbrace{J'_\lambda(u_m)u_m}_{=0} - \underbrace{\mu_m\Phi'_\lambda(u_m)u_m}_{=o(1)} + \mu_m \underbrace{\Phi'_\lambda(u_m)u}_{=\Phi'_\lambda(u)u+o(1)}\right),$$

we get

$$\Phi_\lambda(u) = \lim_{m\to\infty} J'_\lambda(u_m)u = \lim_{m\to\infty}(J'_\lambda(u_m) - \mu_m\Phi'_\lambda(u_m))(u - u_m) = 0.$$

Moreover, as $\Phi_\lambda(u) = 0$, we have

$$\Phi'_\lambda(u)u = (2-N)\|u\|^2_{P_g^k} + (N-q)\lambda\int_M h|u|^q dv_g.$$

Thus,

$$\begin{aligned}\Phi'_\lambda(u)u &\geq \limsup_{m\to+\infty}\left((2-N)\|u_m\|^2 + (N-q)\lambda\int_M h|u_m|^q dv_g\right)\\ &\geq \limsup_{m\to+\infty}\Phi'_\lambda(u_m)u_m > 0.\end{aligned}$$

We conclude that $u \in \mathcal{N}_\lambda^+$.

As $u_m - u \rightharpoonup 0$ in $H_k^2(M)$, we have

$$\lim_{m\to+\infty}\left(J'_\lambda(u_m) - J'_\lambda(u)\right)(u_m - u) = \lim_{m\to+\infty} J'_\lambda(u_m)(u_m - u) - \underbrace{J'_\lambda(u)(u_m - u)}_{=o(1)}.$$

Using again the fact that

$$\mu_m\Phi'_\lambda(u_m)(u_m - u) = \underbrace{\mu_m\Phi'_\lambda(u_m)u_m}_{=o(1)} + \mu_m \underbrace{\Phi'_\lambda(u_m)u}_{=\Phi'_\lambda(u)u+o(1)} = o(1),$$

we get

$$\lim_{m\to+\infty}\left(J'_\lambda(u_m) - J'_\lambda(u)\right)(u_m - u) = \lim_{m\to+\infty}\left[J'_\lambda(u_m) - \mu_m\phi'_\lambda(u_m)\right](u_m - u) = 0.$$

Thus,

$$\begin{aligned} o(1) &= \left(J'_\lambda(u_m) - J'_\lambda(u)\right)(u_m - u) \\ &= \int_M |\nabla^k_g (u_m - u)|^2 dv_g - \int_M f(x)|u_m - u|^N dv_g + o(1). \end{aligned} \tag{4}$$

Then, using Hölder and Sobolev's inequalities, we get

$$o(1) \geq \|\nabla^k_g (u_m - u)\|_2^2 - \max_{x\in M} f(x) \max\left(K_0 + \epsilon, B_\epsilon\right)^{N/2} \|\nabla^k_g (u_m - u)\|_2^N$$

$$\geq \|\nabla^k_g (u_m - u)\|_2^2 \left(1 - \max_{x\in M} f(x) \max\left(K_0 + \epsilon, B_\epsilon\right)^{N/2} \|\nabla^k_g (u_m - u)\|_2^{N-2}\right),$$

Thus, in order to prove that $\|\nabla^k_g (u_m - u)\|_2^2 = o(1)$, we just have to show that

$$\limsup_{m\to\infty} \|\nabla^k_g (u_m - u)\|_2^2 < \frac{1}{(\max_{x\in M} f(x))^{\frac{n}{2k}-1} K_0^{\frac{n}{2k}}}.$$

Using Brezis-Lieb lemma and (4), we find that

$$\begin{aligned} J_\lambda(u_m) - J_\lambda(u) &= \frac{1}{2}\int_M |\nabla^k_g (u_m - u)|^2 dv_g - \frac{1}{N}\int_M f(x)|u_m - u|^N dv_g + o(1) \\ &= \frac{N-2}{N} \|\nabla^k_g (u_m - u)\|_2^2 + o(1). \end{aligned}$$

From which we derive that

$$\limsup_{m\to\infty} \|\nabla^k_g (u_m - u)\|_2^2 \leq \frac{n}{k} \lim_{m\to\infty} \left(J_\lambda(u_m) - J_\lambda(u)\right).$$

As $J_\lambda(u_m) < 0$, we get

$$\limsup_{m\to\infty} \|\nabla^k_g (u_m - u)\|_2^2 \leq -\frac{n}{k} J_\lambda(u).$$

Now, if we are able to prove that

$$-J_\lambda(u) < \frac{k}{n\left(\max_{x\in M} f(x)\right)^{\frac{n}{2k}-1} K_0^{\frac{n}{2k}}},$$

we will be able to conclude that $u_m \to u$ strongly in $H^2_k(M)$.

We know from Lemma 4.1 that $\|u\|_{P^k_g} \leq \rho_\lambda$. As $\Phi_\lambda(u) = 0$, we have

$$J_\lambda(u) = \frac{N-2}{2N} \|u\|^2_{P^k_g} - \lambda \frac{N-q}{qN} \int_M h|u|^q dv_g.$$

Using Hölder and Sobolev's inequalities, (3) and the fact that $\|u\|_{P^k_g} \leq \rho_\lambda$, we get

$$J_\lambda(u) > -\lambda \frac{N-q}{qN} \max_{x\in M} h(x) V(M)^{1-\frac{q}{N}} \max\left(K_0 + \epsilon, B_\epsilon\right)^{q/2} \Lambda^{-q/2} \rho_\lambda^q.$$

As $\rho_\lambda \to 0$ when $\lambda \to 0$, we infer that there exists $\lambda^+_\star \in (0, \lambda_1)$ such that if $\lambda < \lambda^+_\star$, then

$$-J_\lambda(u) < \frac{k}{n\left(\max_{x\in M} f(x)\right)^{\frac{n}{2k}-1} K_0^{\frac{n}{2k}}},$$

and thus,$\|\nabla^k_g (u_m - u)\|_2^2 = o(1)$. In other terms, if $\lambda < \lambda^+_\star$, then $u_m \to u$ strongly in $H^2_k(M)$. The proof is now complete.

□

Theorem 4.4. *There exists $\lambda_\star^+ > 0$ such that if $\lambda \in (0, \lambda_\star^+)$, then equation (1) admits infinitely many pairs of weak solutions with negative energies.*

Proof. It is enough to take $\lambda_\star^+$ as in Proposition 4.3.

We know that $\mathcal{N}_\lambda^+$ is a C^1-submanifold of $H_k^2(M)$, symmetric with respect to the origin and that J_λ is even.

If $\lambda \in (0, \lambda_\star^+)$, then we know from Lemma 3.3 that there exists an odd diffeomorphism $\psi^+ : S \to \mathcal{N}_\lambda^+$. Using Lemma 2.4, we find that

$$\gamma(\mathcal{N}_\lambda^+) = \gamma(\psi^+(S)) \geq \gamma(S) = +\infty,$$

and that

$$\gamma_k(\mathcal{N}_\lambda^+) = \sup\{\gamma(K) \mid K \subset \mathcal{N}_\lambda^+,\ -K = K \text{ and } K \text{ is compact}\} \geq \gamma_k(S) = +\infty.$$

We know from Lemma 4.1 that the restriction of J_λ to $\mathcal{N}_\lambda^+$ is bounded below.

Finally, we know from Proposition 4.3 that $\forall c < \sup_{u \in \mathcal{N}_\lambda^+} J_\lambda(u)$, the restriction of J_λ to $\mathcal{N}_\lambda^+$, satisfies the Palais-Smale condition at level c.

All the conditions of Theorem 2.5 are satisfied. We conclude that the restriction of J_λ to $\mathcal{N}_\lambda^+$, admits infinitely many pairs of critical points. By Lemma 2.6, we infer that those critical points are in fact weak solutions to equation (1). $\square$

We remark that, as $\rho_\lambda \to 0$ when $\lambda \to 0$, the solutions given by Theorem 4.4 go to 0 when $\lambda \to 0$. One cannot construct solutions to the prescribed Q-curvature problem this way.

5. Solutions on $\mathcal{N}_\lambda^-$

Lemma 5.1. *There exists $\lambda_\star^- \in (0, \lambda_\circ)$ (where $\lambda_\circ$ is as in Lemma 3.2) such that such that $\forall \lambda \in (0, \lambda_\star^-)$ the following statements are true.*

- *There exists $\alpha > 0$ such that $\forall u \in \mathcal{N}_\lambda^-$, $J_\lambda(u) \geq \alpha \|u\|_{P_g^k}^2$.*
- $\sup_{u \in \mathcal{N}_\lambda^-} \Phi_\lambda'(u)u < 0$.

Proof. Let $\lambda \in (0, \lambda_\circ)$. Let $u \in \mathcal{N}_\lambda^-$, as $\Phi_\lambda(u) = 0$, we have

$$J_\lambda(u) = \left(\frac{1}{2} - \frac{1}{N}\right)\|u\|_{P_g^k}^2 - \left(\frac{1}{q} - \frac{1}{N}\right)\lambda \int_M h|u|^q dv_g.$$

Using Hölder and Sobolev's inequalities and (3), we get

$$J_\lambda(u) \geq \left(\frac{1}{2} - \frac{1}{N}\right)\|u\|_{P_g^k}^2 - \left(\frac{1}{q} - \frac{1}{N}\right)\lambda \max_{x \in M} h(x) V(M)^{1-\frac{q}{N}} \max{(K_0 + \epsilon, B_\epsilon)}^{q/2} \Lambda^{-q/2} \|u\|_{P_g^k}^q.$$

As $\|u\|_{P_g^k} \geq \rho$, we get

$$J_\lambda(u) \geq \left(\frac{N-2}{2N} - \frac{N-q}{Nq}\lambda \max_{x \in M} h(x) V(M)^{1-\frac{q}{N}} \max{(K_0 + \epsilon, B_\epsilon)}^{q/2} \Lambda^{-q/2} \rho^{q-2}\right).$$

Thus, by taking $\lambda \in (0, \lambda_\star^-)$, where

$$\lambda_\star^- = \min\left(\lambda_\circ, \frac{q}{2} \frac{N-2}{(N-q)\max_{x \in M} h(x) V(M)^{1-\frac{q}{N}} \max{(K_0 + \epsilon, B_\epsilon)}^{q/2} \Lambda^{-q/2} \rho^{q-2}}\right),$$

we get

$$J_\lambda(u) \geq \alpha \|u\|_{P_g^k}^2,$$

where

$$\alpha = \frac{N-2}{2N} - \frac{N-q}{Nq}\lambda \max_{x\in M} h(x) V(M)^{1-\frac{q}{N}} \max(K_0+\epsilon, B_\epsilon)^{q/2}\Lambda^{-q/2}\rho^{q-2} > 0.$$

The first statement is now proved.

Let's prove the second statement. As $\Phi_\lambda(u) = 0$, we have

$$\Phi'_\lambda(u)u = (2-N)\|u\|^2_{P_g^k} + (N-q)\lambda \int_M h|u|^q dv_g.$$

Using Hölder and Sobolev's inequalities and (3), we get

$$\Phi'_\lambda(u)u \le (2-N)\|u\|^2_{P_g^k} + (N-q)\lambda \max_{x\in M} h(x) V(M)^{1-\frac{q}{N}} \max(K_0+\epsilon, B_\epsilon)^{q/2}\Lambda^{-q/2}\|u\|^q_{P_g^k}.$$

As $\|u\|_{P_g^k} \ge \rho$, with $\lambda \in (0, \lambda^-_\star)$, we get

$$\Phi'_\lambda(u)u \le \rho^2 \underbrace{\left((2-N) + (N-q)\lambda \max_{x\in M} h(x) V(M)^{1-\frac{q}{N}} \max(K_0+\epsilon, B_\epsilon)^{q/2}\Lambda^{-q/2}\rho^{q-2}\right)}_{<0} < 0,$$

which proves the last statement. □

Theorem 5.2. *If there exists $v \in H_k^2(M)\backslash\{0\}$ such that*

$$\sup_{t>0} J_0(tv) < \frac{k}{n\left(\max_{x\in M} f(x)\right)^{\frac{n}{2k}-1} K_0^{\frac{n}{2k}}},$$

then there exists $\lambda^\star > 0$ such that $\forall\lambda \in (0, \lambda^\star)$, equation (1) admits a weak solution $u \in H_k^2(M)$ with positive energy.

Proof. First, Notice that there exists $\epsilon > 0$ independent of λ, such that

$$\sup_{t>0} J_0(tv) < \frac{k}{n\left(\max_{x\in M} f(x)\right)^{\frac{n}{2k}-1} K_0^{\frac{n}{2k}}} - \epsilon.$$

Suppose that $\lambda \in (0, \lambda^-_\star)$. Let $c = \inf_{v\in\mathcal{N}^-_\lambda} J_\lambda(v)$, we know from Lemma 5.1 that c is finite and positive. Then consider $(u_m)_{m\in\mathbb{N}}$ a minimizing sequence for J_λ on $\mathcal{N}^+_\lambda$.

By Ekeland's variational principle, we may assume that $J'_\lambda(u_m) - \mu_m\Phi'_\lambda(u_m) \to 0$, where

$$\mu_m = \frac{\langle J'_\lambda(u_m), \Phi'_\lambda(u_m)\rangle}{\|\Phi'_\lambda(u_m)\|^2}.$$

Given that the sequence $(u_m)_{m\in\mathbb{N}}$ is bounded (from Lemma 5.1) and the space $H_k^2(M)$ is reflexive, we can assume $(u_m)_{m\in\mathbb{N}}$ to be weakly convergent to u in $H_k^2(M)$. Thus, $(u_m)_{m\in\mathbb{N}}$ converges strongly to u in $L^p(M)$ for $p < N$.

As $(u_m)_{m\in\mathbb{N}}$ converges weakly to u and $J'_\lambda(u_m) - \mu_m\Phi'_\lambda(u_m) \to 0$ strongly in $(H_k^2(M))^\star$, we have

$$\begin{aligned}\lim_{m\to\infty} \mu_m\Phi'_\lambda(u_m)u_m &= \lim_{m\to\infty} -\underbrace{J'_\lambda(u_m)u_m}_{=0} + \mu_m\Phi'_\lambda(u_m)u_m\\ &= \lim_{m\to\infty} -\left(J'_\lambda(u_m) - \mu_m\Phi'_\lambda(u_m)\right)u_m = 0.\end{aligned}$$

and we know from Lemma 5.1 that $\limsup_{m\to\infty} \Phi'_\lambda(u_m)u_m < 0$. We conclude that $\mu_m \to 0$.

We can now prove that $\Phi_\lambda(u) = 0$, we have

$$\Phi_\lambda(u) = J'_\lambda(u)u = \left(\lim_{m\to\infty}\int_M uP_g^k u_m\, dv_g - \int_M f|u_m|^{N-2}u_m u dv_g - \lambda\int_M h|u_m|^{q-2}u_m u dv_g\right),$$

and as

$$\lim_{m\to\infty} J'_\lambda(u_m)u = \lim_{m\to\infty}\left(J'_\lambda(u_m)u - \underbrace{J'_\lambda(u_m)u_m}_{=0} - \underbrace{\mu_m\Phi'_\lambda(u_m)u_m}_{=o(1)} + \mu_m \underbrace{\Phi'_\lambda(u_m)u}_{=\Phi'_\lambda(u)u+o(1)}\right),$$

we get

$$\Phi_\lambda(u) = \lim_{m\to\infty} J'_\lambda(u_m)u = \lim_{m\to\infty}(J'_\lambda(u_m) - \mu_m\Phi'_\lambda(u_m))(u-u_m) = 0.$$

Let's now prove that there exists $\lambda^\star > 0$ such that if $\lambda < \lambda^\star$, then $u_m \to u$ strongly in $H_k^2(M)$. As $u_m - u \rightharpoonup 0$ in $H_k^2(M)$, we have

$$\lim_{m\to+\infty}\left(J'_\lambda(u_m) - J'_\lambda(u)\right)(u_m-u) = \lim_{m\to+\infty} J'_\lambda(u_m)(u_m-u) - \underbrace{J'_\lambda(u)(u_m-u)}_{=o(1)}.$$

Using again the fact that

$$\mu_m\Phi'_\lambda(u_m)(u_m-u) = \underbrace{\mu_m\Phi'_\lambda(u_m)u_m}_{=o(1)} + \mu_m \underbrace{\Phi'_\lambda(u_m)u}_{=\Phi'_\lambda(u)u+o(1)} = o(1),$$

we get

$$\lim_{m\to+\infty}\left(J'_\lambda(u_m) - J'_\lambda(u)\right)(u_m-u) = \lim_{m\to+\infty}\left[J'_\lambda(u_m) - \mu_m\phi'_\lambda(u_m)\right](u_m-u) = 0.$$

Hence,

$$\begin{aligned} o(1) &= \left(J'_\lambda(u_m) - J'_\lambda(u)\right)(u_m-u) \\ &= \int_M |\nabla_g^k(u_m-u)|^2 dv_g - \int_M f(x)|u_m-u|^N dv_g + o(1). \end{aligned} \tag{5}$$

Then, using Hölder and Sobolev's inequalities, we get

$$\begin{aligned} o(1) &\geq \|\nabla_g^k(u_m-u)\|_2^2 - \max_{x\in M} f(x)\max(K_0+\epsilon, B_\epsilon)^{N/2}\|\nabla_g^k(u_m-u)\|_2^N \\ &\geq \|\nabla_g^k(u_m-u)\|_2^2\left(1 - \max_{x\in M} f(x)\max(K_0+\epsilon, B_\epsilon)^{N/2}\|\nabla_g^k(u_m-u)\|_2^{N-2}\right). \end{aligned}$$

Thus, in order to prove that $\|\nabla_g^k(u_m-u)\|_2^2 = o(1)$, we just have to show that

$$\limsup_{m\to\infty}\|\nabla_g^k(u_m-u)\|_2^2 < \frac{1}{(\max_{x\in M} f(x))^{\frac{n}{2k}-1} K_0^{\frac{n}{2k}}}.$$

Using Brezis-Lieb lemma and (5), we find that

$$\begin{aligned} J_\lambda(u_m) - J_\lambda(u) &= \frac{1}{2}\int_M |\nabla_g^k(u_m-u)|^2 dv_g - \frac{1}{N}\int_M f(x)|u_m-u|^N dv_g + o(1) \\ &= \frac{N-2}{N}\|\nabla_g^k(u_m-u)\|_2^2 + o(1). \end{aligned}$$

From which we derive that

$$\limsup_{m\to\infty}\|\nabla_g^k(u_m-u)\|_2^2 \le \frac{n}{k}\lim_{m\to\infty}(J_\lambda(u_m)-J_\lambda(u)).$$

As $(u_m)_{m\in\mathbb{N}}$ is a minimizing sequence on $\mathcal{N}_\lambda^-$ and, as we know (see 3.2), for a well chosen $t>0$ we have $tv\in\mathcal{N}_\lambda^-$, we have

$$\lim_{m\to\infty} J_\lambda(u_m) \le \sup_{t>0} J_\lambda(tv) \le \sup_{t>0} J_0(tv).$$

Hence,

$$\lim_{m\to\infty} J_\lambda(u_m) < \frac{k}{n\left(\max_{x\in M} f(x)\right)^{\frac{n}{2k}-1} K_0^{\frac{n}{2k}}} - \epsilon.$$

From which we infer that

$$\limsup_{m\to\infty}\|\nabla_g^k(u_m-u)\|_2^2 \le \frac{1}{\left(\max_{x\in M} f(x)\right)^{\frac{n}{2k}-1} K_0^{\frac{n}{2k}}} - \frac{n}{k}\epsilon - \frac{n}{k}J_\lambda(u).$$

Now, if we can prove that $J_\lambda(u) > -\epsilon$, we will be able to conclude that $u_m \to u$ strongly in $H_k^2(M)$. We distinguish two cases.

- In the case where $\Phi_\lambda'(u)u<0$, we have $u\in\mathcal{N}_\lambda^-$ and thus, $J_\lambda(u)>0$.
- In the case where $\Phi'(u)\ge 0$, we know from Lemma 3.1 that $\|u\|_{P_g^k}\le\rho_\lambda$. As $\Phi_\lambda(u)=0$, we have

  $$J_\lambda(u) = \frac{N-2}{2N}\|u\|_{P_g^k}^2 - \lambda\frac{N-q}{qN}\int_M h|u|^q dv_g.$$

  Using Hölder and Sobolev's inequalities, (3) and the fact that $\|u\|_{P_g^k}\le\rho_\lambda$, we get

  $$J_\lambda(u) > -\lambda\frac{N-q}{qN}\max_{x\in M} h(x)V(M)^{1-\frac{q}{N}}\max(K_0+\epsilon, B_\epsilon)^{q/2}\Lambda^{-q/2}\rho_\lambda^q.$$

  As $\rho_\lambda\to 0$ when $\lambda\to 0$, it is clear that, by taking λ small enough, we get $J_\lambda(u)>-\epsilon$.

We conclude that there exists $\lambda^\star>0$ such that if $\lambda\in(0,\lambda^\star)$, then $u_m\to u$ strongly in $H_k^2(M)$ and consequently, we get that:

- $\|u\|_{P_g^k} = \lim_{m\to\infty}\|u_m\|_{P_g^k}\ge\rho>0$.
- $\Phi_\lambda(u) = \lim_{m\to\infty}\Phi_\lambda(u_m)=0$.
- $J_\lambda(u) = \lim_{m\to\infty} J_\lambda(u_m) = c>0$.

From which we infer that $u\in\mathcal{N}_\lambda^-$ and $J_\lambda(u)=c$. Thus, by Lemma 2.6, u is a weak solution of equation (1).

□

Theorem 1.3 follows immediately from Theorem 4.4 and Theorem5.2.

6. Application to the Paneitz-Branson operator

By choosing an appropriate function v in Theorem 1.3, one can obtain more practical corollaries. For instance, we have the following corollary.

Corollary 6.1. *Let* (M, g) *be a compact and connected Riemannian manifold of dimension* $n \geq 7$, f *be a smooth positive function on* M *and* h *a smooth positive function on* M. *We assume that* P_g^2 *is coercive. If there exists* $x_\circ \in M$ *such that* $f(x_\circ) = \max_{x\in M} f(x)$ *and*

$$\frac{\Delta_g f(x_\circ)}{f(x_\circ)} + \frac{4n^2 - 40n - 48}{3(n-6)n(n+2)} S_g(x_\circ) < 0,$$

then there exists $\lambda_\star > 0$ *such that* $\forall \lambda \in (0, \lambda_\star)$, *the equation*

$$P_g^2 u = f|u|^{N-2}u + \lambda h|u|^{q-2}u \tag{6}$$

admits a pair of weak solutions with positive energy and infinitely many pairs of weak solutions with negative energies.

Proof. Let $B_g(x_\circ, \delta)$ be the ball centred at $x_\circ$ of radius δ with $0 < 2\delta < d$ and let η be a smooth function equal to 1 on $B_g(x_o, \delta)$ and equal to 0 on $M \backslash B_g(x_o, 2\delta)$.

Put

$$v_\epsilon(x) = \left(\frac{(n-4)n\left(n^2-4\right)\epsilon^4}{f(x_0)}\right)^{\frac{n-4}{8}} \frac{\eta(r)}{(r^2+\epsilon^2)^{\frac{n-4}{2}}}.$$

We will prove that

$$\sup_{t>0} J_0(tv_\epsilon) < \frac{2}{n\left(\max_{x\in M} f(x)\right)^{\frac{n}{4}-1} K_0^{\frac{n}{4}}}.$$

First, as $J_0(0) = 0$ and $\lim_{t\to+\infty} J_0(tv_\epsilon) = -\infty$, we must have

$$\sup_{t>0} J_0(tv_\epsilon) = J_0(t_{\max} v_\epsilon),$$

where $t_{\max} > 0$ is such that $\frac{d}{dt} J_0(tv_\epsilon)\Big|_{t=t_{\max}} = 0$. By a simple calculation, we find that

$$t_{\max} = \left(\frac{\|v_\epsilon\|^2_{P_g^2}}{\int_M f|v_\epsilon|^N dV_g}\right)^{\frac{1}{N-2}}.$$

Thus, we have

$$\begin{aligned} J_0(t_{\max} v_\epsilon) &= \frac{1}{2}\left(\frac{\|v_\epsilon\|^2_{P_g^2}}{\int_M f|v_\epsilon|^N dv_g}\right)^{\frac{2}{N-2}} \|v_\epsilon\|^2_{P_g^2} - \frac{1}{N}\left(\frac{\|v_\epsilon\|^2_{P_g^2}}{\int_M f|v_\epsilon|^N dV_g}\right)^{\frac{N}{N-2}} \int_M f|v|^N dv_g \\ &= \frac{2}{n}\left(\frac{\|v_\epsilon\|^2_{P_g^2}}{\int_M f|v_\epsilon|^N dV_g}\right)^{\frac{2}{N-2}} \|v_\epsilon\|^2_{P_g^2}. \end{aligned}$$

We will evaluate each factor in this last expression separately. Through the same calculations as in [5, 16], we get

$$\int_M f(x)\,|v_\epsilon(x)|^N\,dv_g = \frac{1}{K_0^{\frac{n}{4}}\left(f(x_\circ)\right)^{\frac{n-4}{4}}}\left(1 - \left(\frac{\Delta_g f(x_\circ)}{2(n-2)f(x_\circ)} + \frac{S_g(x_\circ)}{6(n-2)}\right)\epsilon^2 + o\left(\epsilon^2\right)\right),$$

also,

$$\int_M A(\nabla_g v_\epsilon, \nabla_g v_\epsilon) dv_g = \frac{1}{K_0^{\frac{n}{4}} \left(f(x_\circ)\right)^{\frac{n-4}{4}}} \left(\frac{4(n-1)\,\mathrm{tr}_g A(x_\circ)}{n(n^2-4)(n-6)} \epsilon^2 + o\left(\epsilon^2\right) \right),$$

and

$$\int_M Q_g^2 v_\epsilon^2 dv_g = o\left(\epsilon^2\right),$$

and

$$\int_M \left|\Delta_g v_\epsilon\right|^2 dv_g = \frac{1}{K_0^{\frac{n}{4}} \left(f(x_\circ)\right)^{\frac{n-4}{4}}} \left(1 - \frac{n^2+4n-20}{6(n^2-4)(n-6)} S_g(x_\circ) \epsilon^2 + o\left(\epsilon^2\right) \right).$$

Thus,

$$\int_M \left|\Delta_g v_\epsilon\right|^2 + A(\nabla_g v_\epsilon, \nabla_g v_\epsilon) \left|\nabla_g v_\epsilon\right|^2 + Q_g^2 v_\epsilon^2 dv_g = \frac{1}{K_0^{\frac{n}{4}} \left(f(x_\circ)\right)^{\frac{n-4}{4}}} \cdot$$
$$\left(1 - \left(\frac{n^2+4n-20}{6(n^2-4)(n-6)} S_g(x_\circ) - \frac{4(n-1)}{n(n^2-4)(n-6)}\,\mathrm{tr}_g A(x_\circ) \right) \epsilon^2 + o\left(\epsilon^2\right) \right)$$

Keeping in mind that

$$\mathrm{tr}_g A(x_\circ) = \left(\frac{(n-2)^2+4}{2(n-1)(n-2)} S_g\,\mathrm{tr}_g\, g - \frac{4}{n-2}\,\mathrm{tr}_g Ric_g \right)(x_\circ) = \frac{(n-2)^2-8}{2(n-1)} S_g(x_\circ),$$

we get

$$\|v_\epsilon\|_{P_g^2}^2 = \frac{1}{K_0^{\frac{n}{4}} \left(f(x_\circ)\right)^{\frac{n-4}{4}}} \left(1 - \underbrace{\left(\frac{n^3-8n^2+28n+48}{6(n-6)(n-2)n(n+2)} S_g(x_\circ) \right)}_{>0} \epsilon^2 + o\left(\epsilon^2\right) \right),$$

and

$$\frac{\|v_\epsilon\|_{P_g^2}^2}{\int_M f|v_\epsilon|^N dV_g} = 1 + \frac{1}{2(n-2)} \underbrace{\left(\frac{\Delta_g f(x_\circ)}{f(x_\circ)} + \frac{4n^2-40n-48}{3(n-6)n(n+2)} S_g(x_\circ) \right)}_{<0} \epsilon^2 + o(\epsilon^2).$$

Thus, for $\epsilon > 0$ small enough, we have

$$\|v_\epsilon\|_{P_g^2}^2 < \frac{1}{K_0^{\frac{n}{4}} \left(f(x_\circ)\right)^{\frac{n-4}{4}}}$$

and

$$\frac{\|v_\epsilon\|_{P_g^2}^2}{\int_M f|v_\epsilon|^N dV_g} < 1,$$

and consequently,

$$J_0(t_{\max} v_\epsilon) < \frac{2}{n\left(\max_{x\in M} f(x)\right)^{\frac{n}{4}-1} K_0^{n/4}},$$

which completes the proof. □

7. Application to the GJMS Operator of Sixth Order:

For $k=3$, the sixth order GJMS operator P_g^3 is given by

$$P_g^3 := \Delta_g^3 u + \Delta_g\left(\mathrm{div}_g\left(T_2 u\right)\right) + \mathrm{div}_g\left(T_2\left(\nabla_g u\right)\right)\Delta_g u - \frac{n-2}{2}\Delta_g\left(\sigma_1\left(A_g\right)\Delta_g u\right)$$

$$+\,\mathrm{div}_g\left(T_4\left(\nabla_g u\right)\right) + \frac{n-6}{2}Q_g^3,$$

where

$$T_2 := \frac{8}{n-2}Ric_g + \frac{n^2-4n+12}{2\left(n-2\right)\left(n-1\right)}S_g.g,$$

$$T_4 := -\frac{3n^2-12n-4}{4}\sigma_1\left(A_g\right)^2 g + 4\left(n-4\right)\left|A_g\right|^2 g + 8\left(n-2\right)\sigma_1\left(A_g\right)A_g$$

$$+\left(n-6\right)\Delta_g\sigma_1\left(A_g\right)g - 48A_g^2 - \frac{16}{n-4}B_g,$$

and

$$v_6 = -\frac{1}{8}\sigma_3\left(A_g\right) - \frac{1}{24\left(n-4\right)}\left\langle B,A\right\rangle_g,$$

where A_g and B_g denote respectively the Schouten and the Bach tensors and are defined by

$$A_{ij} = \frac{1}{n-2}\left(R_{ij} - \frac{S_g}{2\left(n-1\right)}g_{ij}\right),$$

$$B_{ij} = \Delta_g\left(A_{ij}\right) - \nabla^k\nabla_j A_{ik} - A^{kl}W_{kijl},$$

and

$$Q_g^3 = -3!2^6 v_6 - \frac{n+2}{2}\Delta_g\left(\sigma_1\left(A_g\right)^2\right) + 4\Delta_g\left(\left|A_g\right|^2\right) + 8\,\mathrm{div}_g\left(A_g\nabla_g\sigma_1\left(A_g\right)\right) + \Delta_g^2\left(\sigma_1\left(A_g\right)\right)$$

$$-\frac{n-6}{2}\sigma_1\left(A_g\right)\Delta_g\left(\sigma_1\left(A_g\right)\right) - 4\left(n-6\right)\sigma_1\left(A_g\right)\left|A_g\right|^2 + \frac{\left(n-6\right)\left(n+6\right)}{4}\sigma_1\left(A_g\right)^3,$$

while $\sigma_k\left(A_g\right)$ is the k^{th} symmetric function of the eigenvalues of the Schouten tensor A_g.

For every $\epsilon > 0$, we define the test function

$$v_\epsilon\left(x\right) := \left(\frac{n\left(n^2-4\right)\left(n^2-16\right)\left(n-6\right)\epsilon^6}{f\left(x_o\right)}\right)^{\frac{n-6}{12}} \frac{\eta\left(x\right)}{\left(\epsilon^2+r^2\right)^{\frac{n-6}{2}}},$$

where $\eta \in C_o^\infty\left(M\right)$ with $0 \le \eta\left(x\right) \le 1$, such that

$$\eta\left(x\right) := \begin{cases} 1 & \text{if } x \in B\left(x_o,\delta\right), \\ 0 & \text{if } x \in M \setminus B\left(x_o, 2\delta\right). \end{cases}$$

The aim of this section is to compute the expansions of

$$\int_M v_\epsilon.P_g^3\left(v_\epsilon\right)dv_g \text{ and } \int_M f\left(x\right)\left|v_\epsilon\right|^{2^\sharp}dv_g.$$

We will analyse these computations for the cases $n > 10$ and for $n = 10$. For instance, we have the following corollary.

Corollary 7.1. *Let (M, g) be a compact and connected Riemannian manifold of dimension $n \geq 10$, f be a smooth positive function on M. Assuming that P_g^3 is coercive and if there exists $x_o \in M$ such that $f(x_o) := \max_{x \in M} f(x)$, the Weyl tensor is nonzero $\left(i.e.\ W_g(x_o) \neq 0\right)$, and $\nabla_g f(x_o) = \Delta_g f(x_o) = \Delta_g^2 f(x_o) = 0$, then there exists $\lambda_* > 0$, such that for all $\lambda \in (0, \lambda_*)$, the equation*

$$P_g^3(u) = f|u|^{N-2} u + \lambda h |u|^{q-2} u,$$

admits a pair of weak solutions with positive energy and infinitely many pairs of weak solutions with negative energies.

7.1. Taylor's Expansion of J_o for $n > 10$:

Proof. We compute each terms separately, we strat with the important term $\int_M v_\epsilon . P_g^3(v_\epsilon) dv_g$, and taking into account that

$$w_n = \left[\frac{2^6}{n(n^2-4)(n^2-16)(n-6)K_0}\right]^{\frac{n}{6}}$$

and

$$w_n = w_{n-1} 2^{n-1} I_n^{\frac{n}{2}-1},$$

we have

$$\int_M v_\epsilon . P_g^3(v_\epsilon) dv_g = \int_M v_\epsilon \Big[\Delta_g^3 v_\epsilon + \Delta_g\left(\mathrm{div}_g(T_2 v_\epsilon)\right) + \mathrm{div}_g\left(T_2\left(\nabla_g v_\epsilon\right)\right)\Delta_g v_\epsilon$$

$$-\frac{n-2}{2}\Delta_g\left(\sigma_1\left(A_g\right)\Delta_g v_\epsilon\right) + \mathrm{div}_g\left(c\left(\nabla_g v_\epsilon\right)\right) + \frac{n-6}{2}Q_g^3\Big] dv_g.$$

Then,

$$\int_M \left|\nabla_g\left(\Delta_g v_\epsilon\right)\right|^2 dv_g = \frac{1}{(\max_{x\in M} f(x))^{\frac{n-6}{6}} K_0^{\frac{n}{6}}}\left(1 + o\left(\epsilon^4\right)\right), \tag{7}$$

$$-\frac{n-2}{2}\int_M \sigma_1\left(A_g\right)\left(\Delta_g v_\epsilon\right)^2 dv_g = \frac{\left|W_g(x_o)\right|^2}{(\max_{x\in M} f(x))^{\frac{n-6}{6}} K_0^{\frac{n}{6}}} \times A_{n,1}\left(\epsilon^4 + o\left(\epsilon^4\right)\right), \tag{8}$$

where

$$A_{n,1} = \frac{2^{2n-15}(n-6)\left(n^3 - 6n^2 + 8n - 32\right)}{6n^2(n-1)(n-8)(n-10)(n+2)(n+4)} \frac{I_{n-2}^{\frac{n}{2}}}{I_n^{\frac{n}{2}-1}},$$

$$-2\int_M T_2\left(\nabla_g v_\epsilon, \nabla_g \Delta_g v_\epsilon\right) dv_g = \frac{\left|W_g(x_o)\right|^2}{(\max_{x\in M} f(x))^{\frac{n-6}{6}} K_0^{\frac{n}{6}}} \times A_{n,2}\left(\epsilon^4 + o\left(\epsilon^4\right)\right), \tag{9}$$

where

$$A_{n,2} := -\frac{16\left(n^2 - 4n - 4\right)\left(n^2 - 28\right)(n-6)}{3(n-10)(n-8)(n-1)n^2(n^2-4)(n+4)} \times \frac{I_{n-2}^{\frac{n}{2}}}{2^{n-1} I_n^{\frac{n}{2}-1}},$$

And

$$-\int_M T_4\left(\nabla_g v_\epsilon, \nabla_g v_\epsilon\right) dv_g = \frac{\left|W_g(x_o)\right|^2}{(\max_{x\in M} f(x))^{\frac{n-6}{6}} K_0^{\frac{n}{6}}} \times A_{n,3}\left(\epsilon^4 + o\left(\epsilon^4\right)\right), \tag{10}$$

where

$$A_{n,3} := \frac{(n-6)^2 (n-3)}{3(n-10)(n-8)(n^2-4)(n-1)n(n+4)} \frac{I_{n-2}^{\frac{n}{2}}}{I_n^{\frac{n}{2}-1}}.$$

And also,

$$\int_M Q_g^3 v_\epsilon^2 dv_g = o\left(\epsilon^4\right). \tag{11}$$

Plugging (7), (8),(9), (10) and (11), it comes that

$$\int_M v_\epsilon . P_g^3 (v_\epsilon) dv_g = \frac{1}{(\max_{x\in M} f(x))^{\frac{n-6}{6}} K_0^{\frac{n}{6}}} \times$$

$$\left[1 + (A_{n,1} + A_{n,2} + A_{n,3}) \left|W_g(x_o)\right|^2 \epsilon^4 + o\left(\epsilon^4\right)\right]. \tag{12}$$

Using the same computation and taking into account that $\nabla_g f(x_o) = \Delta_g f(x_o) = \Delta_g^2 f(x_o) = 0$, we have that

$$\int_M f(x) |v_\epsilon|^{2^\sharp} dv_g = \frac{1}{(\max_{x\in M} f(x))^{\frac{n-6}{6}} K_0^{\frac{n}{6}}} \left(1 + o\left(\epsilon^4\right)\right). \tag{13}$$

Then, we obtain from (12) and (13),

$$\frac{1}{2}\int_M v_\epsilon . P_g^3 (v_\epsilon) dv_g - \frac{1}{2^\sharp}\int_M f(x) |v_\epsilon|^{2^\sharp} dv_g =$$

$$\frac{3}{n(\max_{x\in M} f(x))^{\frac{n-6}{6}} K_0^{\frac{n}{6}}} \times \left(1 + \frac{1}{2}(A_{n,1} + A_{n,2} + A_{n,3}) \left|W_g(x_o)\right|^2 \epsilon^4 + o\left(\epsilon^4\right)\right).$$

Since the constant $A_{n,1} + A_{n,2} + A_{n,3} < 0$, consequently

$$J_o(v_\epsilon) < \frac{3}{n(\max_{x\in M} f(x))^{\frac{n-6}{6}} K_0^{\frac{n}{6}}}.$$

□

7.2. Taylor's Expansion of J_o for $n = 10$:

Proof. We the same technique, we obtain

$$\int_M \left|\nabla_g\left(\Delta_g v_\epsilon\right)\right|^2 dv_g = \frac{1}{(\max_{x\in M} f(x))^{\frac{n-6}{6}} K_0^{\frac{n}{6}}} \left(1 + o\left(\epsilon^4\right)\right),$$

and

$$-\frac{n-2}{2}\int_M \sigma_1\left(A_g\right)\left(\Delta_g v_\epsilon\right)^2 dv_g = \frac{\left|W_g(x_o)\right|^2}{(\max_{x\in M} f(x))^{\frac{n-6}{6}} K_0^{\frac{n}{6}}} \times B_{n,1}\left(16\epsilon^4 \ln\frac{1}{\epsilon^2} + o\left(\epsilon^4\right)\right),$$

where

$$B_{n,1} := -\frac{(n-2)\left(n^2-4\right)\left(n^2-16\right)(n-6)^3}{48(n-1) I_n^{\frac{n}{2}-1}}.$$

And also,

$$-2\int_M T_2\left(\nabla_g v_\epsilon, \nabla_g \Delta_g v_\epsilon\right) dv_g = \frac{\left|W_g\left(x_o\right)\right|^2}{\left(\max_{x\in M} f\left(x\right)\right)^{\frac{n-6}{6}} K_0^{\frac{n}{6}}} \times B_{n,2}\left(4\epsilon^4 \ln\frac{1}{\epsilon^2} + o\left(\epsilon^4\right)\right),$$

where

$$B_{n,2} := -\frac{\left(n^2-28\right)(n-4)(n-6)^3(n-2)\left(n^2-16\right)}{12(n-1) I_n^{\frac{n}{2}-1}}.$$

And

$$-\int_M T_4\left(\nabla_g v_\epsilon, \nabla_g v_\epsilon\right) dv_g = \frac{\left|W_g\left(x_o\right)\right|^2}{\left(\max_{x\in M} f\left(x\right)\right)^{\frac{n-6}{6}} K_0^{\frac{n}{6}}} \times B_{n,3}\left(\epsilon^4 \ln\frac{1}{\epsilon^2} + o\left(\epsilon^4\right)\right),$$

$$B_{n,3} := \frac{(n-6)^2}{12(n-1)\, n\left(n^2-4\right)\left(n^2-16\right) I_n^{\frac{n}{2}-1}}.$$

And also,

$$\int_M Q_g^3 v_\epsilon^2 dv_g = o\left(\epsilon^4\right).$$

$$\int_M f(x)\left|v_\epsilon\right|^{2^\sharp} dv_g = \frac{1}{\left(\max_{x\in M} f\left(x\right)\right)^{\frac{n-6}{6}} K_0^{\frac{n}{6}}}\left[1 + \epsilon^4 \ln\frac{1}{\epsilon^2} + o\left(\epsilon^4\right)\right].$$

Summing the above estimations, we get that

$$\frac{1}{2}\int_M v_\epsilon . P_g^6\left(v_\epsilon\right) dv_g - \frac{1}{2^\sharp}\int_M f(x)\left|v_\epsilon\right|^{2^\sharp} dv_g = \frac{3}{n\left(\max_{x\in M} f\left(x\right)\right)^{\frac{n-6}{6}} K_0^{\frac{n}{6}}} \times$$

$$\left[1 + \left(B_{n,4} - \frac{n-6}{2n}\right)\epsilon^4 \ln\frac{1}{\epsilon^2} + o\left(\epsilon^4\right)\right].$$

where

$$B_{n,4} := \left(8 \times B_{n,1} + 2 \times B_{n,2} + \frac{1}{2} B_{n,3}\right)\left|W_g\left(x_o\right)\right|^2.$$

Since $B_{n,4} - \frac{n-6}{2n} < 0$ when $n = 10$, it is clear that

$$J_o\left(v_\epsilon\right) < \frac{3}{n\left(\max_{x\in M} f\left(x\right)\right)^{\frac{n-6}{6}} K_0^{\frac{n}{6}}},$$

which completes the proof. □

References

[1] S. Agmon, A. Douglis, L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*, I, Comm. Pure Appl. Math. **12** (1959), 623—727.

[2] A. Ambrosetti, A. Malchiodi, *Nonlinear Analysis and Semilinear Elliptic Problems*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2007, 157–176.

[3] T. Aubin, *Some Nonlinear Problems in Riemannian Geometry*, Springer Berlin, Heidelberg, 1998, 145–250.

[4] M. Benalili, K. Tahri, *Multiple solutions to singular fourth order elliptic equations on compact manifolds*, Complex Variables and Elliptic equations. **60** (2015), 1621–1646.

[5] M. Benalili, K. Tahri, *Nonlinear elliptic fourth order equations existence and multiplicity results*, Nonlinear Difer. Equ. Appl. **18** (2011), 539–556.
[6] T.P. Branson, *Group representations arising from Lorentz conformal geometry*, J. Funct. Anal. **74** (1987), 199–291.
[7] T.P. Branson, *Origins, applications and generalisations of the Q-curvature*, Acta Appl. Math. **102, no. 2-3** (2008), 131–146.
[8] Z. Djadli, E. Hebey, M. Ledoux, *Paneitz-type operators and applications*, Duke Math. J. **104** (2000), 129–169.
[9] P. Esposito, F. Robert, *Mountain pass critical points for Paneitz-Branson operators*, Calc. Var. **15** (2002), 493–517.
[10] C.R. Graham, R. Jenne, L.J. Mason, G.A.J. Sparling, *Conformally invariant powers of the Laplacian*, I. Existence. J. London Math. Soc. **46** (1992), 557–565.
[11] S. Mazumdar, *GJMS-type operators on a compact Riemannian manifold: best constants and Coron-type solutions*, J. Differential equations **261, no. 9** (2016), 4997–5034.
[12] F. Robert, *Admissible Q-Curvatures under isometries for the conformal GJMS Operator*, arXiv:1003.0612 (2010).
[13] A. Szulkin, *Ljusternik-Schnirelmann theory on C^1-manifolds*, Annales de l'I.H.P. Analyse non linéaire **5.2** (1988), 119–139.
[14] C.A. Swanson, *The best Sobolev constant*, Applicable Analysis **47(1-4)** (1992), 1227–239.
[15] Xuezhang Chen, Fei Hou, *Remarks on GJMS operator of order six*, Pacific Journal of Mathematics **289, no. 1** (2017), 35–70.
[16] D. Caraffa, *Equations elliptiques du quatrième ordre avec un exposent critiques sur les variétés Riemanniennes compactes*, J. Math. Pures appl. **80(9)** (2001), 941–960.