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On the $L_{YJ}(\lambda, \mu, X)$ constant for the regular octagon space

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Abstract. In this paper, we will investigate a class of geometric constants for a non-Hilbert space X satisfying James constant $J(X) = \sqrt{2}$. Specifically, the exact values of $L_{YJ}(\lambda, \mu, X)$ and $L'_{YJ}(\lambda, \mu, X)$ will be calculated for the regular octagon space X, which improve the estimation obtained by Q. Liu et al.

1. Introduction

Let S_X (resp. B_X) be the unit sphere (resp. the unit closed ball) of a real Banach space X. Throughout this paper, we will use $ex(B_X)$ to denote the set of extreme points of B_X.

Recall that the von Neumann-Jordan constant $C_{NJ}(X)$ of a Banach space X was introduced by Clarkson [1] as the smallest constant C for which,

$$\frac{1}{C} \leq \frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} \leq C$$

holds for all $x, y \in X$ with $\|x\|^2 + \|y\|^2 \neq 0$.

An equivalent definition of this constant is found in the following form:

$$C_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{2(\|x\|^2 + \|y\|^2)} : x, y \in X, (x, y) \neq (0, 0)\right\}.$$

For finite-dimensional spaces, we can use the extreme points of B_X to simplify the above formula as follows

$$C_{NJ}(X) = \sup\left\{\frac{\|x+ty\|^2 + \|x-ty\|^2}{2(1+t^2)} : x, y \in ex(B_X), t \in [0, 1]\right\}.$$

A smaller constant than $C_{NJ}(X)$ is the following constant

$$C'_{NJ}(X) = \sup\left\{\frac{\|x+y\|^2 + \|x-y\|^2}{4} : x, y \in S_X\right\},$$

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and we know that a relationship between them is as follows

$$C_{NJ}(X) \leq 2\Big[1 + C'_{NJ}(X) - \sqrt{2C'_{NJ}(X)}\Big].$$

Several studies on these two constants have been conducted by many authors for more details see (for example, [1–4, 7, 10–12]. It is worth mentioning that geometric constants play a vital role as a tool for solving other problems, such as in the study of Banach-Stone theorem, Bishop-Phelps-Bollobás theorem, and Tingley's problem. These are important research topics in functional analysis.

It is well known that a necessary and sufficient condition for a Banach space to be an inner product space is the parallelogram law. Instead of the parallelogram law, Moslehian and Rassias [6] have recently proved a new equivalent characterization of inner space which is called the Euler-Lagrange type identity. Motivated by the new characterization of inner space by Moslehian and Rassias, the authors in [11] introduced a new geometric constant $L_{YJ}(\lambda, \mu, X)$ for $\lambda, \mu > 0$ as follows

$$L_{YJ}(\lambda, \mu, X) = \sup\Big\{\frac{\|\lambda x + \mu y\|^2 + \|\mu x - \lambda y\|^2}{(\lambda^2 + \mu^2)(\|x\|^2 + \|y\|^2)}, x, y \in X, (x, y) \neq (0, 0)\Big\}. \tag{1.2}$$

A similar constant

$$L'_{YJ}(\lambda, \mu, X) = \sup\Big\{\frac{\|\lambda x + \mu y\|^2 + \|\mu x - \lambda y\|^2}{2(\lambda^2 + \mu^2)}, x, y \in S_X\Big\} \tag{1.3}$$

was also introduced by Liu et al. [12].

Now let us collect some properties of these constants (see [11, 12]). For a Banach space X , then

(i) $1 \leq L_{YJ}(\lambda, \mu, X) \leq 2$ and $1 \leq L'_{YJ}(\lambda, \mu, X) \leq 1 + \frac{2\lambda\mu}{\lambda^2+\mu^2}$;

(ii) X is Hilbert space if and only if $L_{YJ}(\lambda, \mu, X) = 1$, and if and only if $L'_{YJ}(\lambda, \mu, X) = 1$;

(iii) $L_{YJ}(\lambda, \mu, X) \leq \frac{2\lambda^2}{\lambda^2+\mu^2} C_{NJ}(X) + \frac{2\sqrt{2}\lambda|\lambda-\mu|}{\lambda^2+\mu^2}\sqrt{C_{NJ}(X)} + \frac{|\lambda-\mu|^2}{\lambda^2+\mu^2}$;

(iv) X is uniformly non-square if and only if $L'_{YJ}(\lambda, \mu, X) < 1 + \frac{2\lambda\mu}{\lambda^2+\mu^2}$.

Consider the regular octagon space $X = R^2$, which is equipped with the norm defined by

$$\|(x_1, x_2)\| = \max\{|x_1| + (\sqrt{2} - 1)|x_2|, |x_2| + (\sqrt{2} - 1)|x_1|\}.$$

In [11], the authors have shown that the following estimate

$$L_{YJ}(\lambda, \mu, X) \leq (4 - 2\sqrt{2})\frac{2\max\{\lambda^2, \mu^2\}}{\lambda^2 + \mu^2}.$$

In this paper, we give the exact value of the constant $L_{YJ}(\lambda, \mu, X)$ and the constant $L'_{YJ}(\lambda, \mu, X)$ for the regular octagon space, and our results improve the above estimation.

2. Main results

Before describing our main results, we give some lemmas.

Lemma 2.1. *Let $\lambda > \mu > 0$ and $\varphi(t) = \frac{2\lambda\mu}{\lambda^2+\mu^2}\frac{t}{1+t^2} - \frac{t^2}{1+t^2}$, then*

$$\max\Big\{\varphi(t) : t \in [0, \frac{\mu}{\lambda}]\Big\} = \frac{\sqrt{(\lambda^2 + \mu^2)^2 + 4\lambda^2\mu^2} - (\lambda^2 + \mu^2)}{2(\lambda^2 + \mu^2)}. \tag{2.1}$$

Proof. Let $A = \frac{2\lambda\mu}{\lambda^2+\mu^2}$, and let $\varphi'(t) = \frac{A(1-t^2)-2t}{(1+t^2)^2} = 0$, we can get $t^2 = 1 - \frac{2t}{A}$ and $t = \frac{\sqrt{A^2+1}-1}{A} =: t_0 \in (0, \frac{\mu}{\lambda})$. By $\varphi'(t) > 0$ in $(0, t_0)$ and $\varphi'(t) < 0$ in $(t_0, \frac{\mu}{\lambda})$, we know that $\max\{\varphi(t) : t \in [0, \frac{\mu}{\lambda}]\} = \varphi(t_0)$. Now from $\varphi(t_0) = \frac{At_0+1}{1+t_0^2} - 1 = \frac{\sqrt{A^2+1}}{2-\frac{2t_0}{A}} - 1 = \frac{A^2\sqrt{A^2+1}}{2A^2+2-2\sqrt{A^2+1}} - 1 = \frac{\sqrt{A^2+1}+1}{2} - 1$, we have (2.1). □

Lemma 2.2. *(i) Let $\lambda > \mu > 0$ and $\psi(t) = \frac{\sqrt{2}\lambda\mu t}{1+t^2} - \frac{t^2\lambda^2}{1+t^2}$, then*

$$\max\left\{\psi(t) : t \in [0, \frac{\mu}{\sqrt{2}\lambda}]\right\} = \frac{\lambda\sqrt{\lambda^2+2\mu^2} - \lambda^2}{2}. \tag{2.2}$$

(ii) Let $\lambda > \mu > 0$ and $\phi(t) = \frac{\sqrt{2}\lambda\mu t}{1+t^2} - \frac{t^2\mu^2}{1+t^2}$, then

$$\max\left\{\phi(t) : t \in [0, \min\{\frac{\lambda}{\sqrt{2}\mu}, 1\}]\right\} = \frac{\mu\sqrt{\mu^2+2\lambda^2} - \mu^2}{2}. \tag{2.3}$$

(iii) Let $\lambda, \mu > 0$ and $h(t) = \frac{2\sqrt{2}\lambda\mu t}{1+t^2} - \frac{\lambda^2t^2+\mu^2}{1+t^2}$, then

$$\max\left\{h(t) : t \in [\frac{\mu}{\sqrt{2}\lambda}, \min\{\frac{\sqrt{2}\mu}{\lambda}, 1\}]\right\} = \frac{\sqrt{(\lambda^2+\mu^2)^2 + 4\lambda^2\mu^2} - (\lambda^2+\mu^2)}{2}. \tag{2.4}$$

Proof. (i) Similar to the proof of Lemma 2.1, we know that the function $\psi(t)$ reaches its maximum at $t_0 = \frac{-\lambda+\sqrt{\lambda^2+2\mu^2}}{\sqrt{2}\mu} \in (0, \frac{\mu}{\sqrt{2}\lambda})$ and $\psi(t_0) = \frac{\lambda\sqrt{\lambda^2+2\mu^2}-\lambda^2}{2}$.

(ii) Similar to the proof of (i), we know that the function $\phi(t)$ reaches its maximum at $t_0 = \frac{-\mu+\sqrt{\mu^2+2\lambda^2}}{\sqrt{2}\lambda} \in t \in [0, \min\{\frac{\lambda}{\sqrt{2}\mu}, 1\}]$ and $\phi(t_0) = \frac{\mu\sqrt{\mu^2+2\lambda^2}-\mu^2}{2}$.

(iii) Similar to the proof of Lemma 2.1, we also know that the function $h(t)$ reaches its maximum at $t_0 = \frac{\sqrt{(\lambda^2+\mu^2)^2+4\lambda^2\mu^2}-(\lambda^2-\mu^2)}{2\sqrt{2}\lambda\mu} \in (\frac{\mu}{\sqrt{2}\lambda}, \min\{\frac{\sqrt{2}\mu}{\lambda}, 1\})$ and $h(t_0) = \frac{\sqrt{(\lambda^2+\mu^2)^2+4\lambda^2\mu^2}-(\lambda^2+\mu^2)}{2}$. □

Lemma 2.3. *(i) If $0 < \mu \le \lambda \le \sqrt{\frac{3}{2}}\mu$, then*

$$2\mu(\lambda-\mu) \le (\sqrt{\mu^2+2\lambda^2} - \mu)\mu \le \sqrt{\lambda^4+\mu^4+6\lambda^2\mu^2} - \lambda^2 - \mu^2,$$

and

$$(\sqrt{\lambda^2+2\mu^2} - \lambda)\lambda \le \sqrt{\lambda^4+\mu^4+6\lambda^2\mu^2} - \lambda^2 - \mu^2.$$

(ii) If $\sqrt{\frac{3}{2}}\mu \le \lambda \le 2\mu$, then

$$(\sqrt{\lambda^2+2\mu^2} - \lambda)\lambda \le \sqrt{\lambda^4+\mu^4+6\lambda^2\mu^2} - \lambda^2 - \mu^2 \le (\sqrt{\mu^2+2\lambda^2} - \mu)\mu,$$

and

$$2\mu(\lambda-\mu) \le (\sqrt{\mu^2+2\lambda^2} - \mu)\mu.$$

(iii) If $2\mu \le \lambda$, then

$$(\sqrt{\lambda^2+2\mu^2} - \lambda)\lambda \le \sqrt{\lambda^4+\mu^4+6\lambda^2\mu^2} - \lambda^2 - \mu^2 \le (\sqrt{\mu^2+2\lambda^2} - \mu)\mu \le 2\mu(\lambda-\mu).$$

Proof. We only need to note that the following facts is easy to be checked.

(a) $(\sqrt{\mu^2+2\lambda^2}-\mu)\mu \leq \sqrt{\lambda^4+\mu^4+6\lambda^2\mu^2}-\lambda^2-\mu^2$ is equivalent to $2\lambda^2 \leq 3\mu^2$;

(b) $2\mu(\lambda-\mu) \leq (\sqrt{\mu^2+2\lambda^2}-\mu)\mu$ is equivalent to $\lambda \leq 2\mu$;

(c) $(\sqrt{\lambda^2+2\mu^2}-\lambda)\lambda \leq \sqrt{\lambda^4+\mu^4+6\lambda^2\mu^2}-\lambda^2-\mu^2$ is equivalent to $2\mu^2 \leq 3\lambda^2$;

and

(d) $\sqrt{\lambda^4+\mu^4+6\lambda^2\mu^2}-\lambda^2-\mu^2 \leq (\sqrt{\mu^2+2\lambda^2}-\mu)\mu$ is equivalent to $3\mu^2 \leq 2\lambda^2$. □

Inspired by using extreme points to calculate the von Neumann-Jordan constant, we can also use it to calculate L_{YJ} constant for the regular octagon space.

Theorem 2.4. *Let X be the regular octagon space, which is $\mathbb{R}^2$ endowed with the norm*

$$\|x\| = \max\{|x_1| + (\sqrt{2}-1)|x_2|, |x_2| + (\sqrt{2}-1)|x_1|\}.$$

(i) If $\mu \leq \lambda \leq \sqrt{\frac{3}{2}}\mu$, then

$$L_{YJ}(\lambda,\mu,X) = 1 + \frac{\sqrt{2}-1}{\lambda^2+\mu^2}\Big[\sqrt{\lambda^4+\mu^4+6\lambda^2\mu^2}-(\lambda^2+\mu^2)\Big].$$

(ii) If $\sqrt{\frac{3}{2}}\mu \leq \lambda \leq 2\mu$, then

$$L_{YJ}(\lambda,\mu,X) = 1 + \frac{\sqrt{2}-1}{\lambda^2+\mu^2}\Big[\sqrt{2\lambda^2+\mu^2}-\mu)\Big]\mu.$$

(iii) If $\lambda \geq 2\mu$, then

$$L_{YJ}(\lambda,\mu,X) = 1 + \frac{\sqrt{2}-1}{\lambda^2+\mu^2}[2\mu(\lambda-\mu)].$$

(iv) If $\sqrt{\frac{2}{3}}\mu \leq \lambda \leq \mu$, then

$$L_{YJ}(\lambda,\mu,X) = 1 + \frac{\sqrt{2}-1}{\lambda^2+\mu^2}\Big[\sqrt{\lambda^4+\mu^4+6\lambda^2\mu^2}-(\lambda^2+\mu^2)\Big].$$

(v) If $\frac{\mu}{2} \leq \lambda \leq \sqrt{\frac{2}{3}}\mu$, then

$$L_{YJ}(\lambda,\mu,X) = 1 + \frac{\sqrt{2}-1}{\lambda^2+\mu^2}[\sqrt{2\mu^2+\lambda^2}-\lambda)]\lambda.$$

(vi) If $0 < \lambda \leq \frac{\mu}{2}$, then

$$L_{YJ}(\lambda,\mu,X) = 1 + \frac{\sqrt{2}-1}{\lambda^2+\mu^2}[2\lambda(\mu-\lambda)].$$

Proof. If $\lambda = \mu$, then the result is clear by $C_{NJ}(X) = 4-2\sqrt{2}$. Because $(iv)-(vi)$ can be obtained from $(i)-(iii)$ by changing λ and μ, we only need to prove the case for $\lambda > \mu$. From the following formula

$$L_{YJ}(\lambda,\mu,X) = \sup\Big\{\frac{\|\lambda x+\mu ty\|^2+\|\mu x-\lambda ty\|^2}{(\lambda^2+\mu^2)(1+t^2)} : x, y \in ext(B_X), 0 \leq t \leq 1\Big\},$$

we also have

$$L_{YJ}(\lambda,\mu,X)=\sup\Big\{\sup_{0\le t\le 1}\frac{\|\lambda x+\mu ty\|^2+\|\mu x-\lambda ty\|^2}{(\lambda^2+\mu^2)(1+t^2)}:x,y\in ext(B_X)\Big\},$$

where $ext(B_X)=\Big\{(1,0),(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}),(\frac{-\sqrt{2}}{2},\frac{\sqrt{2}}{2}),(0,1),(-1,0),(0,-1),(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}),(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2})\Big\}$.

Now letting $x_1=y_1=(1,0),x_2=y_2=(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}),x_3=y_3=(\frac{-\sqrt{2}}{2},\frac{\sqrt{2}}{2}),y_4=(0,1),y_5=(-1,0),y_6=(0,-1),y_7=(-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2})$ and $y_8=(\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2})$.

Taking $B(x,y,t)=\|\lambda x+\mu ty\|^2+\|\mu x-\lambda ty\|^2$ and $M(x,y)=\sup\Big\{\frac{B(x,y,t)}{(\lambda^2+\mu^2)(1+t^2)}:0\le t\le 1\Big\}$, we can see that $B(x,y,t)=B(-x,-y,t)$ and $B(x,y,t)=B(x',y',t)$, where $z'=(a_2,a_1)$ for any $z=(a_1,a_2)\in X$.

Hence, we only need to consider the following three cases.

Case I. $x=x_1=(1,0)$.

($\mathbf{I_1}$). If $y=y_1$ or y_5, then $B(x_1,y_1,t)=B(x_1,y_5,t)=(1+t^2)(\lambda^2+\mu^2)$ and $M(x_1,y_1)=M(x_1,y_5)=1$.

($\mathbf{I_2}$). If $y=y_4$ or y_6, then

$$\begin{aligned}B(x_1,y_4,t)=B(x_1,y_6,t)&=\|(\lambda,\pm\mu t)\|^2+\|(\mu,\mp\lambda t)\|^2\\&=(\lambda+(\sqrt{2}-1)\mu t)^2+\max\Big\{(\lambda t+(\sqrt{2}-1)\mu)^2,(\mu+(\sqrt{2}-1)\lambda t)^2\Big\}\\&=\begin{cases}\left(\lambda^2+\mu^2\right)\left(1+t^2\right)+(2\sqrt{2}-2)\left[2\lambda\mu t-\mu^2\left(1+t^2\right)\right]; & \frac{\mu}{\lambda}\le t\le 1,\\ \left(\lambda^2+\mu^2\right)\left(1+t^2\right)+(2\sqrt{2}-2)\left[2\lambda\mu t-\left(\lambda^2+\mu^2\right)t^2\right]; & 0\leqslant t\leqslant\frac{\mu}{\lambda}.\end{cases}\end{aligned}$$

So, by Lemma 2.1(when $0\leqslant t\leqslant\frac{\mu}{\lambda}$), we have

$$\begin{aligned}M(x_1,y_4)&=M(x_1,y_6)\\&=\max\left\{1+\frac{2(\sqrt{2}-1)}{\lambda^2+\mu^2}(\lambda-\mu)\mu,1+\frac{\sqrt{2}-1}{\lambda^2+\mu^2}\left(\sqrt{(\lambda^2+\mu^2)^2+4\lambda^2\mu^2}-\lambda^2-\mu^2\right)\right\}.\end{aligned}$$

($\mathbf{I_3}$). If $y=y_2$ or y_8, then

$$\begin{aligned}B(x_1,y_2,t)=B(x_1,y_8,t)&=\left\|\left(\lambda+\frac{\mu t}{\sqrt{2}},\pm\frac{\mu t}{\sqrt{2}}\right)\right\|^2+\left\|\left(\mu-\frac{\lambda t}{\sqrt{2}},\mp\frac{\lambda t}{\sqrt{2}}\right)\right\|^2\\&=(\lambda+\mu t)^2+\max\left\{\left[\left|\mu-\frac{\lambda t}{\sqrt{2}}\right|+\frac{\lambda t}{\sqrt{2}}(\sqrt{2}-1)\right]^2,\left[\frac{t\lambda}{\sqrt{2}}+(\sqrt{2}-1)\left|\mu-\frac{\lambda t}{\sqrt{2}}\right|\right]^2\right\}\\&=(\lambda+\mu t)^2+\begin{cases}(\mu+(1-\sqrt{2})\lambda t)^2; & \mu\geqslant\sqrt{2}t\lambda\\(\lambda t-(\sqrt{2}-1)\mu)^2; & 0\leqslant\mu\leqslant\frac{\lambda t}{\sqrt{2}}\\(\sqrt{2}-1)^2(t\lambda+\mu)^2; & \frac{\lambda t}{\sqrt{2}}\leqslant\mu\leqslant\sqrt{2}\lambda t\end{cases}\\&\overset{if\lambda\geq\sqrt{2}\mu}{=}\begin{cases}\left(1+t^2\right)\left(\lambda^2+\mu^2\right)+(2-2\sqrt{2})\lambda^2t^2+(4-2\sqrt{2})\lambda\mu t; & 0\leqslant t\leqslant\frac{\mu}{\sqrt{2}\lambda}\\\left(1+t^2\right)\left(\lambda^2+\mu^2\right)+(2-2\sqrt{2})\mu^2+(4-2\sqrt{2})\lambda\mu t; & \frac{\sqrt{2}\mu}{\lambda}\leqslant t\leqslant 1\\\left(1+t^2\right)\left(\lambda^2+\mu^2\right)+(2-2\sqrt{2})\left(\mu^2+\lambda^2t^2\right)+(8-4\sqrt{2})\lambda\mu t; & \frac{\mu}{\sqrt{2}\lambda}\leqslant t\leqslant\frac{\sqrt{2}\mu}{\lambda}\end{cases}\end{aligned}$$

or

$$\overset{if\lambda\leqslant\sqrt{2}\mu}{=}\begin{cases}\left(1+t^2\right)\left(\lambda^2+\mu^2\right)+(2-2\sqrt{2})\lambda^2t^2+(4-2\sqrt{2})\lambda\mu t; & 0\leqslant t\leqslant\frac{\mu}{\sqrt{2}\lambda}\\\left(1+t^2\right)\left(\lambda^2+\mu^2\right)+(2-2\sqrt{2})\left(\mu^2+\lambda^2t^2\right)+(8-4\sqrt{2})\lambda\mu t; & \frac{\mu}{\sqrt{2}\lambda}\leqslant t\leqslant 1.\end{cases}$$

When $\lambda > \sqrt{2}\mu$, by (2.2) and (2.4), we have

$$\begin{aligned} &M(x_1,y_2) = M(x_1,y_8) \\ &= \max\left\{1+\frac{\sqrt{2}-1}{\lambda^2+\mu^2}\left(\sqrt{\lambda^2+2\mu^2}-\lambda\right)\lambda\ ,\ 1+\frac{\sqrt{2}-1}{\lambda^2+\mu^2}(\sqrt{2}\lambda-\mu)\mu\ ,\right. \\ &\left. 1+\frac{\sqrt{2}-1}{\lambda^2+\mu^2}\left(\sqrt{\left(\lambda^2+\mu^2\right)^2+4\lambda^2\mu^2}-\lambda^2-\mu^2\right)\right\}. \end{aligned}$$

If $\mu \leq \lambda \leq \sqrt{2}\mu$, then we have

$$\begin{aligned} M(x_1,y_2) &= M(x_1,y_8) \\ &= \max\left\{1+\frac{\sqrt{2}-1}{\lambda^2+\mu^2}\left(\sqrt{\lambda^2+2\mu^2}-\lambda\right)\lambda, 1+\frac{\sqrt{2}-1}{\lambda^2+\mu^2}\left(\sqrt{\left(\lambda^2+\mu^2\right)^2+4\lambda^2\mu^2}-\lambda^2-\mu^2\right)\right\}. \end{aligned}$$

($\mathbf{I_4}$). If $y = y_3$ or y_7 , then

$$\begin{aligned} B(x_1,y_3,t) = B(x_1,y_7,t) &= \|(\lambda-\frac{\mu t}{\sqrt{2}},\pm\frac{\mu t}{\sqrt{2}})\|^2+\|(\mu+\frac{\lambda t}{\sqrt{2}},\mp\frac{\lambda t}{\sqrt{2}})\|^2 \\ &\overset{if\lambda\leq\sqrt{2}\mu}{=}\begin{cases}\left(\lambda^2+\mu^2\right)\left(1+t^2\right)+(2-2\sqrt{2})\mu^2t^2+(4-2\sqrt{2})\lambda\mu t; & 0\leq t\leq\frac{\lambda}{\sqrt{2}\mu},\\ \left(\lambda^2+\mu^2\right)\left(1+t^2\right)+(2-2\sqrt{2})(\lambda^2+t^2\mu^2)+(8-4\sqrt{2})\lambda\mu t; & \frac{\lambda}{\sqrt{2}\mu}\leq t\leq 1,\end{cases} \end{aligned}$$

or

$$\overset{if\lambda>\sqrt{2}\mu}{=}\left(\lambda^2+\mu^2\right)\left(1+t^2\right)+(2-2\sqrt{2})\mu^2t^2+(4-2\sqrt{2})\lambda\mu t;\quad 0\leq t\leq 1$$

and by (2.3)

$$\begin{aligned} &M(x_1,y_3) = M(x_1,y_7) \\ &= \begin{cases}\max\left\{1+\frac{\sqrt{2}-1}{\lambda^2+\mu^2}\left(\sqrt{\mu^2+2\lambda^2}-\mu\right)\mu, 1+\frac{\sqrt{2}-1}{\lambda^2+\mu^2}\left(\sqrt{(\lambda^2+\mu^2)^2+4\lambda^2\mu^2}-\lambda^2-\mu^2\right)\right\}; & \lambda\leqslant\sqrt{2}\mu\\ 1+\frac{\sqrt{2}-1}{\lambda^2+\mu^2}\left(\sqrt{\mu^2+2\lambda^2}-\mu\right)\mu; & \lambda\geq\sqrt{2}\mu\end{cases} \end{aligned}$$

Case II. $x = x_2 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

($\mathbf{II_1}$). If $y = y_2$ or y_7 , then $B(x_2,y_2,t) = B(x_2,y_7,t) = \left(\lambda^2+\mu^2\right)\left(1+t^2\right)$, and $M(x_2,y_2) = M(x_2,y_7) = 1$.

($\mathbf{II_2}$). If $y = y_1$ or y_4 , then

$$\begin{aligned} &B(x_2,y_1,t) = B(x_2,y_4,t) \\ &= \left\|\left(\frac{\lambda}{\sqrt{2}}+\mu t, \frac{\lambda}{\sqrt{2}}\right)\right\|^2+\left\|\left(\frac{\mu}{\sqrt{2}}-\lambda t, \frac{\mu}{\sqrt{2}}\right)\right\|^2 \\ &\overset{if\lambda\geq\sqrt{2}\mu}{=}\begin{cases}\left(\lambda^2+\mu^2\right)\left(1+t^2\right)+(2-2\sqrt{2})\lambda^2t^2+(4-2\sqrt{2})\lambda\mu t; & 0\leqslant t\leqslant\frac{\mu}{\sqrt{2}\lambda}\\ \left(\lambda^2+\mu^2\right)\left(1+t^2\right)+(2-2\sqrt{2})\left(\mu^2+\lambda^2t^2\right)+(8-4\sqrt{2})\lambda\mu t; & \frac{\mu}{\sqrt{2}\lambda}\leq t\leq\frac{\sqrt{2}\mu}{\lambda}\\ \left(\lambda^2+\mu^2\right)\left(1+t^2\right)+(2-2\sqrt{2})\mu^2+(4-2\sqrt{2})\lambda\mu t; & \frac{\sqrt{2}\mu}{\lambda}\leqslant t\leqslant 1\end{cases} \end{aligned}$$

or

$$\begin{aligned}
& B(x_2, y_1, t) \\
= \; & B(x_2, y_4, t) \\
= \; & \left\|\left(\frac{\lambda}{\sqrt{2}} + \mu t, \frac{\lambda}{\sqrt{2}}\right)\right\|^2 + \left\|\left(\frac{\mu}{\sqrt{2}} - \lambda t, \frac{\mu}{\sqrt{2}}\right)\right\|^2 \\
\overset{if\,\lambda < \sqrt{2}\mu}{=} & \begin{cases} \left(\lambda^2+\mu^2\right)\left(1+t^2\right) + (2-2\sqrt{2})\lambda^2 t^2 + (4-2\sqrt{2})\lambda\mu t; & 0 \leqslant t \leqslant \frac{\mu}{\sqrt{2}\lambda}, \\ \left(\lambda^2+\mu^2\right)\left(1+t^2\right) + (2-2\sqrt{2})\left(\mu^2+\lambda^2 t^2\right) + (8-4\sqrt{2})\lambda\mu t; & \frac{\mu}{\sqrt{2}\lambda} \leq t \leq 1. \end{cases}
\end{aligned}$$

Hence, the case is as same as (I_3).

(II$_3$). If $y = y_3$ or y_8 ,

$$\begin{aligned}
B(x_2, y_3, t) &= B(x_2, y_8, t) \\
&= \left\|\left(\frac{\lambda - \mu t}{\sqrt{2}}, \frac{\lambda + \mu t}{\sqrt{2}}\right)\right\|^2 + \left\|\left(\frac{\mu + \lambda t}{\sqrt{2}}, \frac{\mu - \lambda t}{\sqrt{2}}\right)\right\|^2 \\
&= \frac{1}{2}\left\{[\lambda + \mu t + (\sqrt{2}-1) \mid \lambda - \mu t|]^2 + [\mu + \lambda t + (\sqrt{2}-1)|\mu - \lambda t|]^2\right\} \\
&= \begin{cases} \left(\lambda^2+\mu^2\right)\left(1+t^2\right) + (2-2\sqrt{2})\left(\lambda^2+\mu^2\right)t^2 + (4\sqrt{2}-4)\lambda\mu t; & \mu \geqslant \lambda t, \\ \left(\lambda^2+\mu^2\right)\left(1+t^2\right) + (2-2\sqrt{2})\mu^2\left(1+t^2\right) + (4\sqrt{2}-4)\lambda\mu t; & \mu \leqslant \lambda t, \end{cases}
\end{aligned}$$

which is just as same as the case (I_2).

(II$_4$). If $y = y_5$ or y_6 , then

$$\begin{aligned}
B(x_2, y_5, t) &= B(x_2, y_6, t) \\
&= \left\|\left(\frac{\lambda}{\sqrt{2}} - \mu t, \frac{\lambda}{\sqrt{2}}\right)\right\|^2 + \left\|\left(\frac{\mu}{\sqrt{2}} + \lambda t, \frac{\mu}{\sqrt{2}}\right)\right\|^2 \\
&\overset{\text{if } \lambda < \sqrt{2}\mu}{=} \begin{cases} \left(1+t^2\right)\left(\lambda^2+\mu^2\right) + (2-2\sqrt{2})\left(\lambda^2+\mu^2 t^2\right) + (8-4\sqrt{2})\lambda\mu t; & \frac{\lambda}{\sqrt{2}\mu} \leqslant t \leqslant 1, \\ \left(1+t^2\right)\left(\lambda^2+\mu^2\right) + (2-2\sqrt{2})\mu^2 t^2 + (4-2\sqrt{2})\lambda\mu t; & 0 \leqslant t \leqslant \frac{\lambda}{\sqrt{2}\mu}, \end{cases} \\
(or) \; &\overset{(1 \geqslant \sqrt{2}\mu t)}{=} \left(1+t^2\right)\left(\lambda^2+\mu^2\right) + (2-2\sqrt{2})\mu^2 t^2 + (4-2\sqrt{2})\lambda\mu t; \quad 0 \leq t \leq 1.
\end{aligned}$$

So, by (2.3) and (2.4), we can get

(1) For $\lambda < \sqrt{2}\mu$,

$$\begin{aligned}
M(x_2, y_5) &= M(x_2, y_6) \\
&= \max\left\{1 + \frac{\sqrt{2}-1}{\lambda^2+\mu^2}(\sqrt{\mu^2+2\lambda^2} - \mu)\mu, 1 + \frac{\sqrt{2}-1}{\lambda^2+\mu^2}(\sqrt{(\lambda^2+\mu^2)^2 + 4\lambda^2\mu^2} - \lambda^2 - \mu^2)\right\}.
\end{aligned}$$

(2) For $\lambda \geq \sqrt{2}\mu$,

$$M(x_2, y_5) = M(x_2, y_6) = 1 + \frac{\sqrt{2}-1}{\mu^2+\lambda^2}\left(\sqrt{\mu^2+2\lambda^2} - \mu\right)\mu.$$

Case III. $x = x_3 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

(III$_1$). If $y = y_3$ or y_8, then $M(x_3, y_3) = M(x_3, y_8) = 1$.

(III$_2$). If $y = y_1$ or y_6 , then

$$B(x_3, y_1, t) = B(x_3, y_6, t) = \left\|\left(\tfrac{-\lambda}{\sqrt{2}} + \mu t, \tfrac{\lambda}{\sqrt{2}}\right)\right\|^2 + \left\|\left(-\lambda t - \tfrac{\mu}{\sqrt{2}}, \tfrac{\mu}{\sqrt{2}}\right)\right\|^2$$
$$= B(x_2, y_5, t) = B(x_2, y_6, t).$$

(III$_3$). If $y = y_4$ or y_5, then

$$B(x_3, y_4, t) = B(x_3, y_5, t) = \left\|\left(-\frac{\lambda}{\sqrt{2}}, \frac{\lambda}{\sqrt{2}} + \mu t\right)\right\|^2 + \left\|\left(-\frac{\mu}{\sqrt{2}}, \frac{\mu}{\sqrt{2}} - \lambda t\right)\right\|^2$$
$$= B(x_2, y_1, t) = B(x_2, y_4, t).$$

(III$_4$). If $y = y_2$ or y_7, then

$$B(x_3, y_2, t) = B(x_3, y_7, t) = \left\|\left(-\frac{\lambda}{\sqrt{2}} + \frac{\mu t}{\sqrt{2}}, \frac{\lambda}{\sqrt{2}} + \frac{\mu t}{\sqrt{2}}\right)\right\|^2 + \left\|\left(-\frac{\mu}{\sqrt{2}} - \frac{\lambda t}{\sqrt{2}}, \frac{\mu}{\sqrt{2}} - \frac{\lambda t}{\sqrt{2}}\right)\right\|^2$$
$$= B(x_2, y_3, t) = B(x_3, y_8, t).$$

Therefore, we have

(a) If $\mu \le \lambda \le \sqrt{2}\mu$, then

$$L_{YJ}(X) = \max\{l_1, l_2, l_3, l_4\}.$$

(b) If $\lambda \geqslant \sqrt{2}\mu$, then

$$L_{YJ}(X) = \max\{l_1, l_2, l_3, l_4, l_5\},$$

Where

$$l_1 = 1 + \frac{2(\sqrt{2}-1)(\lambda - \mu)\mu}{\lambda^2 + \mu^2},$$
$$l_2 = 1 + \frac{\sqrt{2}-1}{\lambda^2 + \mu^2}\left[\sqrt{\lambda^4 + \mu^4 + 6\lambda^2\mu^2} - \lambda^2 - \mu^2\right],$$
$$l_3 = 1 + \frac{\sqrt{2}-1}{\lambda^2 + \mu^2}\left(\sqrt{\lambda^2 + 2\mu^2} - \lambda\right)\lambda,$$
$$l_4 = 1 + \frac{\sqrt{2}-1}{\lambda^2 + \mu^2}\left(\sqrt{\mu^2 + 2\lambda^2} - \mu\right)\mu,$$
$$l_5 = 1 + \frac{\sqrt{2}-1}{\lambda^2 + \mu^2}(\sqrt{2}\lambda - \mu)\mu.$$

Hence, by Lemma 2.3, we can obtain
(1) If $\mu \le \lambda \le \sqrt{\frac{3}{2}}\mu$, then $L_{YJ}(\lambda, \mu, X) = l_2$.
(2) If $\sqrt{\frac{3}{2}}\mu \leqslant \lambda \leqslant \sqrt{2}\mu$, then $L_{YJ}(\lambda, \mu, X) = l_4$.
(3) If $\sqrt{2}\mu \leqslant \lambda \leqslant 2\mu$, then $L_{YJ}(\lambda, \mu, X) = l_4$.
(4) If $2\mu \le \lambda$, then $L_{YJ}(\lambda, \mu, X) = l_1$. □

From the proof of Theorem 2.4, we can also get the following corollary.

Corollary 2.5. *Let X be the regular octagon space, which is* $\mathbb{R}^2$ *endowed with the norm*

$$\|x\| = \max\{|x_1| + (\sqrt{2}-1)|x_2|, |x_2| + (\sqrt{2}-1)|x_1|\}.$$

(i) If $\mu \le \lambda \le \sqrt{2}\mu$*, then*

$$L'_{YJ}(\lambda,\mu,X) = (2-\sqrt{2})\frac{(\lambda+\mu)^2}{\lambda^2+\mu^2}.$$

(ii) If $\sqrt{2}\mu \le \lambda \le (1+\frac{\sqrt{2}}{2})\mu$*, then*

$$L'_{YJ}(\lambda,\mu,X) = \frac{\lambda^2 + (2-\sqrt{2})(\mu^2+\lambda\mu)}{\lambda^2+\mu^2}.$$

(iii) If $(1+\frac{\sqrt{2}}{2})\mu \le \lambda$*, then*

$$L'_{YJ}(\lambda,\mu,X) = \frac{[\lambda + (\sqrt{2}-1)\mu]^2}{\lambda^2+\mu^2}.$$

(iv) If $\frac{\sqrt{2}}{2}\mu \le \lambda \le \mu$*, then*

$$L'_{YJ}(\lambda,\mu,X) = .(2-\sqrt{2})\frac{(\lambda+\mu)^2}{\lambda^2+\mu^2}.$$

(v) If $(2-\sqrt{2})\mu \le \lambda \le \frac{\mu}{\sqrt{2}}$*, then*

$$L'_{YJ}(\lambda,\mu,X) = \frac{\mu^2 + (2-\sqrt{2})(\lambda^2+\lambda\mu)}{\lambda^2+\mu^2}.$$

(vi) If $\lambda \le (2-\sqrt{2})\mu$*, then*

$$L'_{YJ}(\lambda,\mu,X) = 1 + \frac{[\mu + (\sqrt{2}-1)\lambda]^2}{\lambda^2+\mu^2}.$$

Proof. By symmetry, we just have to prove (i)-(iii).

(i) If $\mu \le \lambda \le \sqrt{2}\mu$, then

$$\begin{aligned}
& \{(2-2\sqrt{2})(\lambda^2+\mu^2) + (8-4\sqrt{2})\lambda\mu\} - 4(\sqrt{2}-1)(\lambda\mu-\mu^2) \\
= \ & (2\sqrt{2}-2)[\mu^2 + (2\sqrt{2}-2)\lambda\mu - \lambda^2] \\
\ge \ & (2\sqrt{2}-2)[\frac{1}{2}\lambda^2 + (2-\sqrt{2})\lambda^2 - \lambda^2] \\
\ge \ & 0.
\end{aligned}$$

On the other hand, by the proof of Theorem 2.4, we can get

$$\begin{aligned}
& \sup\{\|\lambda x + \mu y\|^2 + \|\mu x - \lambda y\|^2 : x, y \in S_X\} \\
= \ & \max\{2(\lambda^2+\mu^2) + 4(\sqrt{2}-1)(\lambda\mu-\mu^2), 2(\lambda^2+\mu^2) + (2-2\sqrt{2})(\lambda^2+\mu^2) + (8-4\sqrt{2})\lambda\mu\} \\
= \ & 2(\lambda^2+\mu^2) + (2-2\sqrt{2})(\lambda^2+\mu^2) + (8-4\sqrt{2})\lambda\mu \\
= \ & (4-2\sqrt{2})(\lambda+\mu)^2.
\end{aligned}$$

Hence,

$$L'_{YJ}(\lambda,\mu,X) = (2-\sqrt{2})\frac{(\lambda+\mu)^2}{\lambda^2+\mu^2}.$$

(ii) If $\sqrt{2}\mu \leq \lambda \leq \frac{2+\sqrt{2}}{2}\mu$, then

$$4(\sqrt{2}-1)(\lambda\mu-\mu^2)-[(2-2\sqrt{2})\mu^2+(4-2\sqrt{2})\lambda\mu]=(6\sqrt{2}-8)\mu[\lambda-\frac{2+\sqrt{2}}{2}\mu]\leq 0.$$

Also, by the proof of Theorem 2.4, we can get

$$\begin{aligned}
&\sup\{\|\lambda x+\mu y\|^2+\|\mu x-\lambda y\|^2 : x, y\in S_X\}\\
=\;&\max\{2(\lambda^2+\mu^2)+4(\sqrt{2}-1)(\lambda\mu-\mu^2), 2(\lambda^2+\mu^2)+(2-2\sqrt{2})\mu^2+(4-2\sqrt{2})\lambda\mu\}\\
=\;&2(\lambda^2+\mu^2)+(2-2\sqrt{2})\mu^2+(4-2\sqrt{2})\lambda\mu.
\end{aligned}$$

Hence,

$$L'_{YJ}(\lambda,\mu,X)=\frac{\lambda^2+(2-\sqrt{2})(\mu^2+\lambda\mu)}{\lambda^2+\mu^2}.$$

(iii) If $\frac{2+\sqrt{2}}{2}\mu \leq \lambda$, then

$$4(\sqrt{2}-1)(\lambda\mu-\mu^2)-[(2-2\sqrt{2})\mu^2+(4-2\sqrt{2})\lambda\mu]=(6\sqrt{2}-8)\mu[\lambda-\frac{2+\sqrt{2}}{2}\mu]\geq 0.$$

Also, by the proof of Theorem 2.4, we can get

$$\begin{aligned}
&\sup\{\|\lambda x+\mu y\|^2+\|\mu x-\lambda y\|^2 : x, y\in S_X\}\\
=\;&\max\{2(\lambda^2+\mu^2)+4(\sqrt{2}-1)(\lambda\mu-\mu^2), 2(\lambda^2+\mu^2)+(2-2\sqrt{2})\mu^2+(4-2\sqrt{2})\lambda\mu\}\\
=\;&2(\lambda^2+\mu^2)+4(\sqrt{2}-1)(\lambda\mu-\mu^2).
\end{aligned}$$

Hence,

$$L'_{YJ}(\lambda,\mu,X)=\frac{[\lambda+(\sqrt{2}-1)\mu]^2}{\lambda^2+\mu^2}.$$

□

Remark 2.6. *The regular octagonal space is very important in the theory of geometric constants of spaces. For example, it is a non-Hilbert space satisfying James constant $J(X)=\sqrt{2}$. Therefore, it is very meaningful to calculate the values of some geometric constants in this space. The conclusion of Theorem 2.4 is a generalization of the classical result $C_{NJ}(X)=4-2\sqrt{2}$ for the regular octagonal space X.*

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