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Deferred statistical convergence of A-transformation of real valued sequences

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Abstract. This paper introduces the concept of deferred statistical A-convergence, which combines the deferred density of subsets of natural numbers with an infinite regular matrix $A=(a_{mk})$. We explore the fundamental properties of deferred statistical A-convergent sequences and investigate the relationships between deferred statistical A-convergence, strongly deferred statistical A-convergence, and deferred statistical convergence. Additionally, we present a Korovkin-type approximation theorem utilizing deferred statistical A-convergence and provide a counter-example to demonstrate its limitations. The findings contribute to the understanding of statistical convergence, deferred statistical convergence, matrix transformations, and the development of Korovkin-type theorems.

1. Introduction

The theory of statistical convergence, which is an active area of research and a generalization of ordinary convergence for both real and complex sequences, was independently introduced by Fast [15] and Steinhaus [38] in the same year. The idea can be traced back to Zygmund's studies [39]. Since then, numerous researchers have studied this concept, especially in the last twenty years. The basis of statistical convergence lies in the asymptotic density of subsets of natural numbers, which gives rise to different versions of convergence. Examples of these include logarithmic density and uniform density defined by any regular matrix.

Let's consider a subset K of $\mathbb{N}$. The natural density of K, denoted by $\delta(K)$, is defined as the limit:

$$\lim_{n\to\infty}\frac{1}{n}|\{k\leq n : k\in K\}|$$

where $|\cdot|$ represents the cardinality of the given set. For a sequence $x=(x_n)$, if the limit exists:

$$\lim_{n\to\infty}\frac{1}{n}|\{k\leq n:\ |x_k-l|>\varepsilon\}|=0,$$

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for any arbitrary $\varepsilon > 0$, then (x_n) statistically converges to l.

The exploration of statistical convergence, as opposed to the generalization of classical convergence, has yielded interesting results in various branches of mathematics. Notable studies in this direction include: approximation theory [2, 3, 5, 8, 20, 32]; probability theory, metric spaces, and topological spaces [6, 7, 10, 14, 18, 25, 26]; strong summability and matrix transformations [4, 16, 22, 35]; summability theory [1, 9, 12, 13, 17, 19, 21, 23, 24, 30, 34, 36]; statistical convergence in abstract spaces [27]; generalization of (λ, A)-statistical convergence [28]; statistical C_1 convergence [29].

In this paper, we use the symbols $\mathbb{N}$ and $\mathbb{R}$ to represent natural and real numbers, respectively. The set c_0 denotes the collection of null real sequences, c represents the set of convergent real sequences, and ℓ_∞ represents the set of bounded real sequences.

Let $A = (a_{mk})$ be an infinite matrix and $x = (x_k)$ be real valued sequence. The A-transformation of the sequence $x = (x_k)$ is defined as

$$(Ax)_m = \left(\sum_{k=1}^{\infty} a_{mk} x_k\right)$$

when $\sum_{k=1}^{\infty} a_{mk} x_k$ is convergent for all $m \in \mathbb{N}$. An infinite matrix $A = (a_{mk})$ is called regular if $\lim_{m\to\infty} (Ax)_m = \lim_{k\to\infty} x_k$ for all $x \in c$. We will consider an infinite regular matrix, and the following well-known theorem holds (see, for example, [8]).

Theorem 1.1. *(*Silverman-Toeplitz Theorem*) A matrix $A = (a_{mk})$ is regular if and only if*

(1) $\lim_{m\to\infty} a_{mk} = 0$, *for all* $k \in \mathbb{N}$,

(2) $\sup_m \sum_{k=1}^{\infty} |a_{mk}| < \infty$,

(3) $\lim_{m\to\infty} \sum_{k=1}^{\infty} a_{mk} = 1$.

Consider sequences of non-negative integers denoted as $p := (p_n)$ and $q := (q_n)$ such that $p_n < q_n$ for all $n \in \mathbb{N}$, and q_n approaches infinity as n approaches infinity. Unless explicitly stated otherwise, whenever we refer to sequences (p_n) and (q_n), they will always satisfy the aforementioned properties. These properties are commonly referred to as the deferred property. For a real-valued sequence denoted as $x := (x_k)$, the sequence $\left(D_{p,q}x\right)_n$ is referred to as the deferred Cesáro mean and is defined as follows:

$$\left(\mathrm{D_{p,q}x}\right)_\mathrm{n} := \frac{1}{\mathrm{q_n} - \mathrm{p_n}} \sum_{\mathrm{k}=\mathrm{p_n}+1}^{\mathrm{q_n}} \mathrm{x_k},$$

for each n$\in\mathbb{N}$. If the deferred Cesàro mean $\left(\mathrm{D_{p,q}x}\right)_\mathrm{n}$ converges to a limit l, we denote this by $x_k \to l\left(D_{p,q}\right)$. The matrix representation of the deferred Cesàro mean, $D_{(p,q)}$, is denoted as $D_{p,q} = (d_{nk})_{n,k\geq 1}$, where the entries are defined as follows:

$$d_{nk} := \begin{cases} \frac{1}{q_n - p_n}, & \text{if } p_n < k \leq q_n \\ 0, & \text{if } k \leq p_n \textit{ and } k > q_n \end{cases}$$

for all $n, k \in \mathbb{N}$. It is important to note that the entries of the deferred Cesàro matrix depend on the specific values of (p_n) and (q_n) satisfying the deferred property. The deferred Cesàro matrix, as given in above, satisfies the Silverman-Toeplitz Theorem.

Example 1.2. *Consider the sequence $x = (x_k)$ defined as $x_k = 0$ if k is odd, and $x_k = 1$ if k is even, for each value of k. It is evident that x_k converges to $1/2$ within the subset $D_{2n-1,4n-1}$. However, when considering the concept of usual convergence, the sequence $x = (x_n)$ does not converge to 1/2.*

We introduce the notion of strong $D_{(p,q)}$-convergence for a sequence (x_k) with respect to the limit l. This occurs when the following limit exists

$$\lim_{n\to\infty}\frac{1}{q_n-p_n}\sum_{k=p_n+1}^{q_n}|x_k-l|=0,$$

We denote this condition as $x_k\to l\left(SD_{p,q}\right)$, and the collection of sequences that exhibit strong $D_{(p,q)}$-convergence is denoted as $SD_{p,q}$. Furthermore, we define the sets of sequences as follows:

$$SD^0_{p,q}:=\{x=(x_k):D[p,q]\,|x|\in c_0\}$$

and

$$SD^\infty_{p,q}=\{x=(x_k):D[p,q]\,|x|\in\ell_\infty\}.$$

In the special case where $q_n=n$ and $p_n=0$ for all values of n, the resulting sequence spaces are known as $SD^0_{0,n}$ which corresponds to the sequence space w_0,, and $SD^\infty_{0,n}$, which corresponds to the sequence space w_∞. Additionally, we define a semi-norm $\|\cdot\|_{D_{(p,q)}}$ on the set $SD^0_{p,q}$, denoted by

$$\|x\|_{p,q}:=\|D_{p,q}\,|x|\,\|_\infty=\sup_n\left(\frac{1}{q_n-p_n}\sum_{k=p_n+1}^{q_n}|x_k|\right).$$

2. MAIN PROPERTIES OF THE SET $D_{p,q}S\,(A)$

Recall that a sequence $x:=(x_k)$ is said to be deferred statistically convergent to $l\in\mathbb{R}$, denoted as $x_k\to l\left(D_{p,q}S\right)$), if for every $\varepsilon>0$, the following limit exists

$$\lim_{n\to\infty}\frac{1}{q_n-p_n}|\{p_n<k\le q_n:|x_k-l|\ge\varepsilon\}|=0.$$

We introduce the concept of deferred statistical A-convergence as an extension of deferred statistical convergence. This concept plays a pivotal role in the context of this paper.

Definition 2.1. *Consider an infinite matrix $A:=(a_{mk})_{m,k\ge1}$ with indices m and k. Let p and q be sequences that satisfy the deferred property. Then, a sequence $x:=(x_k)$ is said to be deferred statistically A-convergent to l if, for every $\varepsilon>0$, the limit*

$$\lim_{n\to\infty}\frac{1}{q_n-p_n}|\{p_n<m\le q_n:\left|(Ax)_m-l\right|\ge\varepsilon\}|=0\qquad(*)$$

exists. We denote this condition as $x_n\to l\left(D_{p,q}S\,(A)\right)$.

The set of all deferred statistically A-convergent sequences is denoted by $D_{(p,q)}S\,(A)$. If we consider the sequences $q_n=n$ and $p_n=n-\lambda_n+1$, where $\lambda:=(\lambda_n)$ is a sequence satisfying $\lambda_1=1$, $\lambda_{n+1}\le\lambda_n+1$ and $I_n=[n-\lambda_n+1,n]$ for all $n\in\mathbb{N}$, then Definition 1 corresponds to the concept of (λ,A)-statistical convergence, which was introduced and studied by Malafosse and Rakocevic [28]. Additionally, it was that (λ,A)-statistical convergence generalizes the notion of statistical convergence. However, it is important to note that this generalization does not hold in all cases, as the following examples will illustrate.

Example 2.2. *Let us consider the Cesàro matrix C_1 and the sequences $(p_n)=\left(2n^2\right)$ and $(q_n)=\left(4n^2\right)$ for all $n\in\mathbb{N}$. We define the sequence $x=(x_n)$ as follows:*

$$x_n=\begin{cases} n, & n=k^2,\\ 0, & n\ne k^2,\end{cases}$$

for all $n\in\mathbb{N}$. It is evident that the sequence $x=(x_n)$ is statistically convergent to 0. By applying Theorem 2.2.1 [8], we conclude that $x=(x_n)$ is also deferred statistically convergent to 0 with respect to the sequences $(p_n)=\left(2n^2\right)$ and $(q_n)=\left(4n^2\right)$. However, we aim to investigate whether x is deferred C_1-statistically convergent to 0. Upon observing that $[C_1x]_n=\frac{1}{n}\sum_{m=1}^{n}x_k$. We take $\varepsilon=1/10$. Notably, the set $\{n:|[C_1x]_n-0|\geq 1/10\}$ corresponds to the entire set of natural numbers $\mathbb{N}$, which possesses a deferred density of 1. Consequently, we deduce that $x=(x_n)$ is not deferred statistically C_1-convergent to 0.

Example 2.3. *Let p and q be sequences satisfying the deferred property. Consider the Cesàro matrix C_1, and let (x_n) be the sequence defined as $(x_n):=((-1)^n)$. It is evident that the following limit exists*

$$\lim_{n\to\infty}\frac{1}{q_n-p_n}\left|\left\{p_n<k\leq q_n:\left|(-1)^n-0\right|\geq\varepsilon\right\}\right|\neq 0.$$

Furthermore, since

$$C_1(-1)^n=\begin{cases}0, & n=2k,\\ -\frac{1}{n}, & n=2k+1,\end{cases},$$

we can observe

$$\lim_{n\to\infty}\frac{1}{q_n-p_n}|\{p_n<k\leq q_n:\left|C_1(-1)^n-0\right|\geq\varepsilon\}|=0.$$

The examples presented above demonstrate that deferred statistical C_1-convergence and deferred statistical convergence do not necessarily imply each other. In general, we have $c\subset D_{p,q}S\subsetneq D_{p,q}S\,(C_1)$ and $c\subset D_{p,q}S\,(C_1)\subsetneq D_{p,q}S$. These results hold for any sequences (p_n) and (q_n) that satisfy the deferred property. Therefore, it is mathematically significant to examine the fundamental properties of deferred statistical A-convergence for general (p_n) and (q_n) sequences, or their special cases, which satisfy the deferred property. Notably, some results have been obtained for certain infinite matrices such as Cesàro, $(H,1)$, and (N,p), where $p_n=0$ and $q_n=0$ [5, 24, 28, 29]. In the following, we present the main findings.

Theorem 2.4. *Let p and q be sequences satisfying the deferred property, and $A=(a_{mk})$ be a regular matrix. Then, the deferred statistical A-limit of a sequence is uniquely determined.*

Proof. Suppose that $x_k\to l_1\left(D_{p,q}S\,(A)\right)$ and $x_k\to l_2\left(D_{p,q}S\,(A)\right)$, where $l_1\neq l_2$. For any $\varepsilon>0$, we have:

$$\frac{1}{q_n-p_n}|\{p_n<m\leq q_n:\left|(Ax)_m-l_1\right|\geq\varepsilon\}|\to 0$$

and

$$\frac{1}{q_n-p_n}|\{p_n<m\leq q_n:\left|(Ax)_m-l_2\right|\geq\varepsilon\}|\to 0.$$

By applying the triangle inequality:

$$|l_1-l_2|\leq\left|(Ax)_m-l_1\right|+\left|(Ax)_m-l_2\right|,$$

we can conclude that the desired result is obtained. Thus, the limit is uniquely determined. □

Theorem 2.5. *Let p and q be sequences satisfying the deferred property, $A=(a_{nk})$ be a regular matrix, $x_n\to l_1\left(D_{p,q}S\,(A)\right)$, $y_n\to l_2\left(D_{p,q}S\,(A)\right)$, and $\beta\in\mathbb{R}$. Then, the following statements hold:*

1. $x_n+y_n\to l_1+l_2\left(D_{p,q}S\,(A)\right)$;

2. $\beta x_n \to \beta l_1 \left(D_{p,q}S(A)\right)$.

Proof. From the assumption and the linearity property of the regular matrix $A=(a_{nk})$, for any $\varepsilon > 0$, we have:

$$\left\{p(n)+1\le m\le q(n):\left|(A(x+y))_m-(l_1+l_2)\right|\ge\varepsilon\right\}$$

$$=\left\{p(n)+1\le m\le q(n):\left|(Ax)_m-l_1\right|\ge\frac{\varepsilon}{2}\right\}\cup\left\{p(n)+1\le m\le q(n):\left|(Ay)_m-l_2\right|\ge\frac{\varepsilon}{2}\right\}$$

and for any $\beta\neq0$

$$\left\{p(n)+1\le m\le q(n):\left|(A(\beta x))_m-\beta l_1\right|\ge\varepsilon\right\}=\left\{p(n)+1\le m\le q(n):\left|(Ax)_m-l_1\right|\ge\frac{\varepsilon}{|\beta|}\right\}.$$

Hence, we can easily prove (i) and (ii) by following suitable steps. □

Corollary 2.6. *Under the assumption of Theorem 3, the set $D_{p,q}S(A)$ is a real (or complex) vector space.*

Theorem 2.7. *Let p and q be sequences satisfying the deferred property, and $A=(a_{mk})$ be a regular summability matrix. Then, deferred statistical A-convergence is a regular summability method.*

Proof. Let (x_k) be a sequence such that $\lim_{k\to\infty} x_k=l$ holds. Since the matrix $A=(a_{mk})$ is a regular matrix, then the transformation sequence

$$(y_m):=(Ax)_m=\left(\sum_{k=1}^{\infty}a_{mk}x_k\right)$$

is also convergent to the same limit l. So, the regularity of deferred statistical convergence (see [8]) implies the desired result. □

For a matrix $A=(a_{mk})$, denote by $C(A)$ as the set of all sequences from ω which is A-convergent to any real numbers:

$$C(A):=\{(x_n)\in\omega:\exists l\in\mathbb{R},\ (Ax)_m=\sum_{k=1}^{\infty}a_{mk}x_k\to l \text{ as } m\to\infty\}.$$

Corollary 2.8. $C(A)\subseteq D_{p,q}S(A)$.

Remark 2.9. *The converse of Corollary 6 is not true, in general.*

Example 2.10. *Consider sequences (p_n) and (q_n) satisfying the deferred property, and let $A:=(C_1)$ be the infinite Cesàro matrix. Define the sequence $x=(x_k)$ as follows:*

$$x_k:=\begin{cases} k, & k=m^2,\\ -k+1, & k-1\neq m^2,\\ 0, & \text{else.}\end{cases}$$

It is evident that

$$[C_1x]_n=\begin{cases}1, & n \text{ is a square},\\ 0, & \text{else},\end{cases}$$

does not converge in the usual sense, indicating that x does not belong to $C(C_1)$. However, we observe that for any $\varepsilon > 0$, the inequality

$$\frac{1}{q_n-p_n}\left|\{p_n<m\le q_n:\left|[C_1x]_m\right|\ge\varepsilon\}\right|\le\frac{\sqrt{q_n}-\sqrt{p_n}+1}{q_n-p_n}$$

holds. Consequently, we conclude that $x=(x_n)$ is deferred statistically C_1-convergent to 0, i.e., $x\in D_{p,q}S(C_1)$.

An interesting question arises from Example 4.

Question 2.11. *Is a subsequence of a deferred statistically convergent sequence deferred statistically A-convergent?*

Theorem 2.12. *Let p and q be sequences satisfying the deferred property, and let $A=(a_{mk})$be an infinite regular matrix. If $x_k \to l\left(D_{p,q}S\right)$, then there exists a sequence $y=(y_k)$ with $\delta_{p,q}(\{k : x_k \neq y_k\})=0$ such that $y_k \to l\left(D_{p,q}S(A)\right)$.*

Proof. Suppose $x_k \to l\left(D_{p,q}S\right)$ holds for any sequences p and q satisfying the deferred property. This implies that for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \left|\{p_n+1 \leq m \leq q_n : |x_m - l| \geq \varepsilon\}\right| = 0.$$

By considering the index set $K=\{m_k : k \in \mathbb{N}\}$, where $x_{m_k} \to l$ as $n \to \infty$, we construct the sequence y_m as follows:

$$y_m = \begin{cases} x_{m_k}, & m = m_k \\ l, & \text{otherwise} \end{cases}.$$

It is evident that $\delta_{p,q}(\{k : x_m \neq y_m\})=0$, and $y_m \to l$ as $m \to \infty$. Due to the regularity of the matrix $A=(a_{mk})$ and the regularity of statistical deferred convergence, $y_m \to l\left(D_{p,q}S(A)\right)$ holds. Hence, the proof is concluded. □

Corollary 2.13. *If $x_k \to l(S)$, then there exists a sequence (y_k) with $\delta_{0,n}(\{k : x_k \neq y_k\})=0$ such that $y_k \to l(S(A))$ holds.*

Question 2.14. *Can we find a regular matrix $\widetilde{A}$ that makes a deferred statistically A-convergent sequence deferred $\widetilde{A}$-convergent?*

Theorem 2.15. *Assume that p and q are sequences that satisfy the deferred property, and let $A=(a_{mk})$ be an infinite regular matrix. Then $x_k \to l\left(D_{p,q}S(A)\right)$ if and only if there exists an index set $K=\{m_n : n \in \mathbb{N}\}$ with $\delta_{p,q}(K)=1$ such that x_k converges to l in the matrix $\widetilde{A}=(a_{m_n k})$ formed by selecting rows from A based on the set K.*

Proof. To prove this result, we will utilize the idea of Cantor's nested intervals theorem. A similar proof can be found in the studies [17] and [21]. Here, we adapt the method defined in (*) and follow similar steps. Let's assume that $x_k \to l\left(D_{p,q}S(A)\right)$. This means that for any $\varepsilon > 0$, the following condition holds as n approaches infinity:

$$\frac{1}{q_n - p_n} \left|\left\{p_n+1 \leq m \leq q_n : \left|(Ax)_m - l\right| \geq \varepsilon\right\}\right| \to 0.$$

Now, we define the set

$$K_j(A) := \left\{m \in \mathbb{N} : \left|(Ax)_m - l\right| < \frac{1}{j}\right\}$$

for any $j \in \mathbb{N}$. It can be observed that the sequence $\left\{K_j(A)\right\}_{j=1}^{\infty}$ is monotonically decreasing with respect to set inclusion, and the above equation implies that $\delta_{p,q}\left(K_j(A)\right)=1$ for all $j \in \mathbb{N}$. Let $n_1 \in K_1(A)$. Hence, there exists $n_2 \in K_2(A)$ such that for every $n \geq n_2$, the following inequality must hold:

$$\frac{1}{q_n - p_n} \left|\left\{p_n+1 \leq m \leq q_n : \left|(Ax)_m - l\right| < \frac{1}{2}\right\}\right| \geq \frac{1}{2}.$$

Similarly, there exists $n_3 \in K_3(A)$ such that for all $n \geq n_3$, the following inequality holds:

$$\frac{1}{q_n-p_n}\left|\left\{p_n+1\leq m\leq q_n: \left|(Ax)_m-l\right|<\frac{1}{3}\right\}\right|\geq\frac{2}{3}.$$

We can continue this process consecutively, and for each j, there exists an $n_j \in K_j(A)$ such that for all $n \geq n_j$, we have:

$$\frac{1}{q_n-p_n}\left|\left\{p_n+1\leq m\leq q_n: \left|(Ax)_m-l\right|<\frac{1}{j}\right\}\right|\geq\frac{j-1}{j}$$

By defining the sequence $\mathbb{N}=\bigcup_{j=1}^{\infty}[n_j, n_{j+1}$, we can take $K:=\bigcup_{j=1}^{\infty}\left([n_j, n_{j+1})\cap K_j(A)\right)$. For any arbitrary $n \in \mathbb{N}$, there exists $n_j \leq n < n_{j+1}$ and the above inequality gives us:

$$\delta_{p,q}(K)\geq\frac{1}{q_n-p_n}\left|\left\{p_n+1\leq m\leq q_n: \left|(Ax)_m-l\right|<\frac{1}{j}\right\}\right|\geq\frac{j-1}{j}\to 1$$

as j approaches infinity. Let's define the set K as $K=\{m_n: n \in \mathbb{N}\}$, where m_n is a monotonically increasing sequence in $\mathbb{N}$. We select the rows of matrix A based on the monotonically increasing sequence m_n and denote it as matrix $\widetilde{A}=(\widetilde{a}_{m_n k})$. It is evident from Theorem 1 that the following conditions are satisfied:

1. $\lim_{n\to\infty} a_{m_n k}=0$, for all $k\in\mathbb{N}$,
2. $\sup_n \sum_{k=1}^{\infty}\left|a_{m_n k}\right|<\infty$,
3. $\lim_{n\to\infty}\sum_{k=1}^{\infty} a_{m_n k}=1$.

For simplicity, let's denote the sequence $\left(\widetilde{A}x\right)_n$ as $(y_n):=(\sum_{k=1}^{\infty} a_{m_n k}x_k)$ .Based on the construction of set K, it can be easily concluded that the sequence (y_n) converges to l. Thus, the proof is complete. □

Definition 2.16. *A sequence* (x_k) *is said to be deferred statistically A-Cauchy if there exists a natural number* $m(\varepsilon)\in\mathbb{N}$ *such that*

$$\lim_{n\to\infty}\frac{1}{q_n-p_n}|\{p_n< m\leq q_n: \left|(Ax)_m-(Ax)_{m(\varepsilon)}\right|\geq\varepsilon\}|=0,$$

holds for every $\varepsilon>0$.

It can be observed that the value of $m(\varepsilon)$ in Definition 2 is not unique. To see this, consider a natural number $m_1> m(\varepsilon)$. Then, the following inclusion holds:

$$\{p_n< m\leq q_n: \left|(Ax)_m-(Ax)_{m_1}\right|\geq\varepsilon\}\subseteq\{p_n< m\leq q_n: \left|(Ax)_m-(Ax)_{m(\varepsilon)}\right|\geq\frac{\varepsilon}{2}\}\cup\{p_n< m\leq q_n: \left|(Ax)_{m(\varepsilon)}-(Ax)_{m_1}\right|\geq\frac{\varepsilon}{2}\}$$

for any $\varepsilon>0$. Therefore, this inclusion implies that

$$\delta_{p,q}(\{p_n< m\leq q_n: \left|(Ax)_m-(Ax)_{m_1}\right|\geq\varepsilon\})=0.$$

Theorem 2.17. *A deferred statistically A-convergent sequence with respect to two sequences p and q satisfying the deferred property, and a regular matrix* $A=(a_{mk})$*is deferred statistically A-Cauchy.*

Proof. Let $x_n\to l\left(D_{p,q}S(A)\right)$ as $n\to\infty$, and consider an arbitrary sufficiently large element $n(\varepsilon)$ in the set $\left\{p(n)<m\leq q(n): \left|(Ax)_m-l\right|\geq\varepsilon\right\}$ for an arbitrary ε. For this $n(\varepsilon)$, we have the inclusion:

$$\left\{p(n)<m\leq q(n): \left|(Ax)_m-(Ax)_n(\varepsilon)\right|\geq\varepsilon\right\}\subseteq\left\{p(n)<m\leq q(n): \left|(Ax)_m-l\right|\geq\varepsilon\right\}\cup\left\{p(n)<m\leq q(n): \left|(Ax)_n(\varepsilon)-l\right|\geq\varepsilon\right\}.$$

Therefore, based on the assumption of Theorem, this inclusion implies that (x_n) is a deferred statistically A-Cauchy sequence. □

For the converse of Theorem 11, we provide the following result.

Theorem 2.18. *A deferred statistically A-Cauchy sequence with sequences p and q satisfying the deferred property, and a regular matrix* $A=(a_{mk})$ *is deferred statistically A-convergent.*

Proof. Suppose x is a deferred statistically A-Cauchy sequence. Then, there exists $n(\varepsilon)\in\mathbb{N}$ such that $\delta_{p,q}(B)=0$ for every ε, where $B=\{n:\left|(Ax)_n-(Ax)_{n(\varepsilon)}\right|\geq\varepsilon\}$. It can be observed that the sets:

$$Y=\{y\in\mathbb{R}:\delta_{p,q}\{n:(Ax)_n<y\}=1\}$$

and

$$Z=\{z\in\mathbb{R}:\delta_{p,q}\{n:(Ax)_n>z\}=1\}$$

are non-empty because the inclusions

$$B^c\subseteq\{n:(Ax)_{n(\varepsilon)}-\varepsilon<(Ax)_n\}$$

and

$$B^c\subseteq\{n:(Ax)_n<(Ax)_{n(\varepsilon)}+\varepsilon\}.$$

hold. We claim that for some $y\in Y$ and $z\in Z$, $z<y$. Suppose $z_0\geq y_0$ for some z_0 and y_0. Thus, we obtain the following:

$$\{n:(Ax)_n>z_0\}\subseteq\{n:(Ax)_n>y_0\},$$

which implies $\delta_{p,q}(\{n:(Ax)_n\leq y_0\})=0$. This contradicts $y_0\in Y$. Therefore, we conclude that $z<y$ for all $y\in Y$ and $z\in Z$, which yields

$$(Ax)_{n(\varepsilon)}-\varepsilon\leq\sup Z\leq\inf Y\leq(Ax)_{n(\varepsilon)}+\varepsilon.$$

This implies that $\sup Z=\inf Y$ since ε is arbitrary. Let $l=\sup Z$ and $s=\inf Y$, and let $\mu>0$. Then, there exist $y_\mu\in Y$ and $z_\mu\in Z$ such that $l-\mu<z_\mu<y_\mu<l+\mu$. This implies that $\delta_{p,q}(\{n:(Ax)_n<l+\mu\})=1$ and $\delta_{p,q}(\{n:(Ax)_n>l-\mu\})=1$. Consider

$$\{n:\left|(Ax)_n-l\right|<\mu\}=\{n:(Ax)_n>l-\mu\}\cap\{n:(Ax)_n<l+\mu\}.$$

From this, we deduce that

$$\delta_{p,q}\left(\{n:\left|(Ax)_n-l\right|\geq\mu\}\right)=0.$$

Therefore, we have obtained the deferred statistically A-convergence of x to l as desired. □

Definition 2.19. *If the following limit exists*

$$\lim_{n\to\infty}\frac{1}{q_n-p_n}|\{p_n<m\leq q_n:\left|(Ax)_m\right|\geq M\}|=0$$

for some positive scalar $M>0$*, then the sequence* $x=(x_k)$ *is called deferred statistically A-bounded.*

Theorem 2.20.

1. *Every deferred statistically A-convergent sequence with a regular matrix A and arbitrary sequences p and q satisfying the deferred property is deferred statistically A-bounded.*
2. *A deferred statistically A-Cauchy sequence with a regular matrix A and arbitrary sequences p and q satisfying the deferred property is deferred statistically A-bounded.*

Proof. The proof for these statements can be easily established by utilizing the definitions of deferred statistical A-convergence, deferred statistical A-Cauchiness, and deferred statistically A-boundedness. However, for brevity, we omit the detailed proof here. □

Definition 2.21. *Let* $\mathrm{A} = (\mathrm{a}_{\mathrm{mk}})$ *be a regular matrix and* $\mathrm{x} = (\mathrm{x}_{\mathrm{k}})$ *be a sequence. Then,* x *is called strongly deferred* A*-convergent to* l *if*

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \sum_{m=p_n+1}^{q_n} \left|(Ax)_m - l\right| = 0$$

holds for some p *and* q *satisfying the deferred property sequences.*

In this paper, we denote $SD_{p,q}(A)$ as the set of all strongly deferred A-convergent sequences.

Theorem 2.22. *Let p and q be sequences satisfying the deferred property, and let* $\mathrm{A} = (\mathrm{a}_{\mathrm{mk}})$ *be a regular matrix. If the sequence* $(Ax)_m$ *converges strongly to l, then it also converges deferred statistically to l.*

Proof. Assuming the deferred property holds, we have:

$$\lim_{n\to\infty} \frac{1}{q_n - p_n} \sum_{m=p_n+1}^{q_n} \left|(Ax)_m - l\right| = 0$$

for arbitrary p and q satisfying the deferred property. For any $\varepsilon > 0$, we define the set $B(\varepsilon)$ as follows:

$$B(\varepsilon) = \{p(n) + 1 \le m \le q(n) : \left|(Ax)_m - l\right| \ge \varepsilon\}.$$

Thus, the following inequality holds:

$$\sum_{m=p_n+1}^{q_n} \left|(Ax)_m - l\right| = \Big(\sum_{m\in B(\varepsilon)} + \sum_{m\notin B(\varepsilon)}\Big) \left|(Ax)_m - l\right| \ge \varepsilon\, |B(\varepsilon)|.$$

Dividing both sides of the inequality by $q_n - p_n$and taking the limit as n approaches infinity, we obtain the desired result. Therefore, deferred strong A-convergence implies deferred statistical A-convergence. □

In the following example, we demonstrate that Theorem 14 does not hold conversely in general.

Example 2.23. *Consider the sequence* $\mathrm{x} := (\mathrm{x}_{\mathrm{k}})$ *defined as follows:*

$$x_k =: \begin{cases} \frac{1}{a_{mk}} k^2, & \left[\left|\sqrt{q_n}\right|\right] \le k \le q_n, n \in \mathbb{N}, \\ 0, & \text{otherwise}, \end{cases}$$

where $\mathrm{A} = (\mathrm{a}_{\mathrm{mk}})$*is a regular matrix defined as:*

$$a_{mk} =: \begin{cases} \frac{1}{q_n - \left[\left|\sqrt{q_n}\right|\right]}, & \left[\left|\sqrt{q_n}\right|\right] \le k \le q_n, n \in \mathbb{N}, \\ 0, & \text{otherwise}, \end{cases}.$$

It is evident that

$$(Ax)_m = \begin{cases} k^2, & \left[\left|\sqrt{q_n}\right|\right] \le k \le q_n, \\ 0, & \text{otherwise}. \end{cases},$$

and thus, we observe that $(\mathrm{x}_{\mathrm{n}})$ *is a deferred statistically A-null sequence for any* $p_n \le \left[\left|\sqrt{q_n}\right|\right]$*, but it is not strongly deferred A-convergent [8].*

We denote the set $\ell_\infty(A) := \{x = (x_k) : (Ax)_m \in \ell_\infty\}$. Therefore, we establish the following theorem.

Theorem 2.24. *Let p and q be sequences satisfying the deferred property, and* $A = (a_{mk})$ *be a regular matrix. If* $x \in \ell_\infty(A)$ *and x converges deferred statistically to l, then x converges to l in the strongly deferred A-summable sense, denoted as* $x_k \to l\left(D_{p,q}S(A)\right)$.

Proof. Assume that x is an A-bounded sequence and converges deferred statistically to l. Let $M := \|(Ax)\|_\infty + |l|$ be a finite constant satisfying the inequality

$$\left|(Ax)_k - l\right| \leq M$$

for every $n \in \mathbb{N}$. For any $\varepsilon > 0$, we fix an N_ε such that:

$$\frac{1}{q_n - p_n}\left|\{p_n < k \leq q_n : \left|(Ax)_k - l\right| \geq \frac{\varepsilon}{2}\}\right| < \frac{\varepsilon}{2M}$$

holds for every $n > N_\varepsilon$. Let $B_n := \{p_n < k \leq q_n : \left|(Ax)_k - l\right| \geq \frac{\varepsilon}{2}\}$ for each n. Then, for all $n > N_\varepsilon$, we have:

$$\frac{1}{q_n - p_n}\sum_{k=p_n+1}^{q_n}\left|(Ax)_k - l\right| = \frac{1}{q_n - p_n}\Big(\sum_{\substack{k=p_n+1\\ k\in B_n}}^{q_n}\left|(Ax)_k - l\right| + \sum_{\substack{k=p_n+1\\ k\in B_n^c}}^{q_n}\left|(Ax)_k - l\right|\Big)$$

$$\leq \frac{1}{q_n - p_n}\Big(\sum_{\substack{k=p_n+1\\ k\in B_n}}^{q_n} M + \sum_{\substack{k=p_n+1,\\ k\in B_n^c}}^{q_n}\frac{\varepsilon}{2}\Big) \leq \frac{1}{q_n - p_n}\left(M|B_n| + \frac{\varepsilon}{2}(q_n - p_n)\right) \leq M\frac{\varepsilon}{2M} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, we obtain:

$$\lim_{n\to\infty}\frac{1}{q_n - p_n}\sum_{k=p_n+1}^{q_n}\left|(Ax)_k - l\right| = 0.$$

Thus, the sequence (x_k) converges strongly deferred A-summable to l. □

Corollary 2.25. *Let p and q be sequences satisfying the deferred property, and* $A = (a_{mk})$ *be a regular matrix. Then, we have* $\ell_\infty \cap \left(D_{p,q}S(A)\right) = \ell_\infty \cap \left(SD_{p,q}(A)\right)$.

Definition 2.26. *Two sequences* $x = (x_n)$ *and* $y = (y_n)$ *are called deferred statistically A-equivalent, denoted as* $x \sim y$*, if the sequence* $(x_n - y_n)$ *converges deferred statistically to zero.*

It can be demonstrated that the relation "$\sim$" is an equivalence relation on $D_{p,q}S(A)$. Let us denote the quotient space by $D_{p,q}S(A)/_\sim$, defined as:

$$D_{p,q}S(A)/_\sim := \{[x] : x \in D_{p,q}S(A)\},$$

where $[x] := \{y \in \omega : x_n - y_n \to 0\left(D_{p,q}S(A)\right)\}$.

Theorem 2.27. *Let* $A = (a_{mk})$ *be a regular matrix, and let p and q be two sequences satisfying the deferred property. If a sequence* $y \in [x]$ *is deferred statistical A-convergent to l, then the sequence* $x = (x_k)$ *being deferred statistically A-convergent to l implies the deferred statistical A-convergence of y.*

Proof. Consider a deferred statistically A-convergent sequence $x = (x_k)$ to l, and let $y = (y_k) \in [x]$. This implies that for any $\varepsilon > 0$:

$$\frac{1}{q(n)-p(n)}|\{p(n)+1\leq m\leq q(n):\left|(Ax)_m - l\right|\geq\varepsilon\}|\to 0,$$

and

$$\frac{1}{q(n)-p(n)}|\{p(n)+1\leq m\leq q(n):\left|A(x_k - y_k)\right|\geq\varepsilon\}|\to 0,$$

as n approaches infinity. Fix $\varepsilon > 0$. Then, we can establish the following inclusion:

$$\{p(n)+1\leq m\leq q(n):\left|(Ay)_m - l\right|\geq\varepsilon\}\subseteq\left\{p(n)+1\leq m\leq q(n):\left|(Ay - Ax)_m\right|\geq\frac{\varepsilon}{2}\right\}$$

$$\cup\,\{p(n)+1\leq m\leq q(n):\left|(Ax)_m - l\right|\geq\varepsilon/2\}.$$

By completing the necessary steps in the proof, it follows from the above equations that the deferred density of the right-hand side of the last inclusion is zero, which concludes the proof. □

Corollary 2.28. *For every* $\widetilde{y} = (y_k) \in [x]$, *if* $\widetilde{x} = (x_k)$ *is not deferred statistically A-convergent, then* $\widetilde{y}$ *is also not deferred statistically A-convergent.*

Consider the regular matrices $A = (a_{mk})$ and $B = (b_{mk})$ such that:

$$\limsup_{m\to\infty}\sum_{k=1}^{\infty}|a_{mk} - b_{mk}| = 0(**)$$

We observe the following theorem.

Theorem 2.29. *Let p and q be sequences satisfying the deferred property, and A and B be two arbitrary regular matrices satisfying condition (**). If* $x = (x_k) \in \ell_\infty(A)$ *,then the deferred statistically A-convergence of* (x_k) *implies the deferred statistically B-convergence of* (x_k) *with the same limit, and vice versa.*

Proof. Let $(x_k) \in \ell_\infty(A)$ be deferred statistically A-convergent to l. The proof is straightforward when $x = (0)_{n=0}^{\infty}$. Suppose $x \neq (0)_{n=0}^{\infty}$. We observe that for each $\varepsilon > 0$, the inequality:

$$\begin{aligned}|\{p_n < m\leq q_n:\left|(Bx)_m - (Ax)_m\right|\geq\frac{\varepsilon}{2}\}| &= |\{p_n < m\leq q_n: |\sum_{k=1}^{\infty}b_{mk}x_k - \sum_{m=1}^{\infty}a_{mk}x_k|\geq\frac{\varepsilon}{2}\}| \\ &= \left|\left\{p_n < m\leq q_n \left|\sum_{m=1}^{\infty}(b_{mk} - a_{mk})x_m\right|\geq\frac{\varepsilon}{2}\right\}\right| \\ &\leq \left|\left\{p_n < m\leq q_n: \sum_{k=1}^{\infty}|b_{mk} - a_{mk}|\ |x_k|\geq\frac{\varepsilon}{2}\right\}\right| \\ &\leq |\{p_n < m\leq q_n: \sum_{k=1}^{\infty}|b_{mk} - a_{mk}|\geq\frac{\varepsilon}{2\|x\|_\infty\}}|\end{aligned}$$

holds. By multiplying the above inequality by $\frac{1}{q_n - p_n}$ and taking the limit as $n \to \infty$, it follows that:

$$\lim_{n\to\infty}\frac{1}{q_n - p_n}|\{p_n < m\leq q_n:\left|(Bx)_m - (Ax)_m\right|\geq\frac{\varepsilon}{2}\}| = 0$$

because $\limsup_{m\to\infty}\sum_{k=1}^{\infty}|b_{mk}-a_{mk}|=0$. Moreover, the following inequality:

$$|\{p_n< m\leq q_n: \left|(Bx)_m-\ell\right|\geq\varepsilon\}| = |\{p_n< m\leq q_n: |\sum_{k=1}^{\infty} b_{mk}x_k-l|\geq\varepsilon\}|$$

$$\leq \left|\left\{p_n< m\leq q_n \left|\sum_{k=1}^{\infty} a_{mk}x_k-l\right|\geq\frac{\varepsilon}{2}\right\}\right| + |\{p_n< m\leq q_n \sum_{k=1}^{\infty} b_{mk}x_k-\sum_{k=1}^{\infty} a_{mk}x_k|\geq\frac{\varepsilon}{2}\}|$$

is also true for all $\varepsilon > 0$. Multiplying each side by $\frac{1}{q_n-p_n}$ and taking the limit as n tends to infinity, we obtain the desired result. □

3. INCLUSION RESULTS FOR $C_\lambda S(A)$ and $D_\lambda S(A)$

In this section, we explore a special case of deferred statistically A-convergence for a strictly increasing sequence $\lambda=(\lambda_n)$ with $\lambda_0=0$. We define two types of convergence, namely $C_\lambda S(A)$ and $D_\lambda S(A)$ convergence, for a sequence $x=(x_k)$ with respect to a scalar $l\in\mathbb{R}$ and an arbitrary $\varepsilon > 0$.

Definition 3.1. *A sequence* $x=(x_k)$ *is said to be* $C_\lambda S(A)$*-convergent and* $D_\lambda S(A)$*-convergent, denoted as* $x_k\to l\,(C_\lambda S(A))$ *and* $x_k\to l\,(D_\lambda S(A))$*, respectively, if the following conditions hold for any* $\varepsilon > 0$*:*

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\left\{0\leq m\leq\lambda_n: \left|(Ax)_m-l\right|\geq\varepsilon\right\}\right|=0$$

and

$$\lim_{n\to\infty}\frac{1}{\lambda_n-\lambda_{n-1}}\left|\left\{\lambda_{n-1}\leq m\leq\lambda_n: \left|(Ax)_m-l\right|\geq\varepsilon\right\}\right|=0.$$

Theorem 3.2. *Let* $\lambda:=(\lambda_n)$ *be a strictly increasing sequence with* $\lambda_0=0$ *and* $A=(a_{nk})$ *be a regular matrix. If* $x_k\to l\,(D_\lambda S(A))$*, then it implies* $x_k\to l\,(C_\lambda S(A))$*.*

Proof. Let us assume that $x_k\to l\,(D_\lambda S(A))$ holds. For an arbitrary $\varepsilon > 0$, we have:

$$\left\{\lambda_0\leq m\leq\lambda_n: \left|(Ax)_m-l\right|\geq\varepsilon\right\}=\bigcup_{k=1}^{k=n}\left\{\lambda_{k-1}\leq m\leq\lambda_k: \left|(Ax)_m-l\right|\geq\varepsilon\right\}.$$

This implies that:

$$\left|\left\{\lambda_0\leq m\leq\lambda_n: \left|(Ax)_m-l\right|\geq\varepsilon\right\}\right|=\sum_{k=1}^{k=n}\left|\left\{\lambda_{k-1}\leq m\leq\lambda_k: \left|(Ax)_m-l\right|\geq\varepsilon\right\}\right|.$$

By multiplying the equation above by the appropriate expressions, it can be rewritten as:

$$\frac{1}{\lambda_n}\left|\left\{\lambda_0\leq m\leq\lambda_n: \left|(Ax)_m-l\right|\geq\varepsilon\right\}\right|=\sum_{k=1}^{k=n}\frac{\lambda_k-\lambda_{k-1}}{\lambda_n}\frac{1}{\lambda_k-\lambda_{k-1}}\left|\left\{\lambda_{k-1}\leq m\leq\lambda_k: \left|(Ax)_m-l\right|\geq\varepsilon\right\}\right|.$$

Hence, the last equality shows that $C_\lambda S(A)$ convergence is a linear combination of $D_\lambda S(A)$ convergence for the sequence (x_k). Let us consider a matrix $T=(t_{nk})$ defined by:

$$t_{nk}=\begin{cases}\frac{\lambda_k-\lambda_{k-1}}{\lambda_n}, & k\leq n,\\ 0, & k>n\end{cases}.$$

Therefore, we have:

$$\frac{1}{\lambda_n}\left|\left\{\lambda_0\leq m\leq\lambda_n : \left|(Ax)_m - l\right|\geq\varepsilon\right\}\right| = \sum_{k=1}^{k=n} t_{nk}\frac{1}{\lambda_k-\lambda_{k-1}}\left|\left\{\lambda_{k-1}\leq m\leq\lambda_k : \left|(Ax)_m - l\right|\geq\varepsilon\right\}\right|.$$

Since the matrix $T=(t_{nk})$ is regular (satisfying the Silverman-Toeplitz Theorem), we conclude that (x_k) is $C_\lambda S(A)$convergent to l. □

Theorem 3.3. *Let* $\lambda:=(\lambda_n)$ *be a strictly increasing sequence with* $\lambda_0=0$ *and* $A=(a_{nk})$ *be a regular matrix. Then,* $x_k\to l\,(C_\lambda S(A))$ *implies that* $x_k\to l(D_\lambda S(A)$ *if and only if* $\liminf_{n\to\infty}\frac{\lambda_n}{\lambda_{n-1}}>1$.

Proof. Let us assume that $x_k\to l\,(C_\lambda S(A))$ holds. For any $\varepsilon>0$, we have:

$$\left\{\lambda_{n-1}\leq m<\lambda_n : \left|(Ax)_m-l\right|\geq\varepsilon\right\}=\left\{1\leq m<\lambda_n : \left|(Ax)_m-l\right|\geq\varepsilon\right\}\setminus\left\{1\leq m\leq\lambda_{n-1} : \left|(Ax)_m-l\right|\geq\varepsilon\right\}.$$

This implies the following equality:

$$\left|\left\{\lambda_{n-1}\leq m<\lambda_n : \left|(Ax)_m-l\right|\geq\varepsilon\right\}\right|=\left|\left\{1\leq m<\lambda_n : \left|(Ax)_m-l\right|\geq\varepsilon\right\}\right|-\left|\left\{1\leq m\leq\lambda_{n-1} : \left|(Ax)_m-l\right|\geq\varepsilon\right\}\right|.$$

By multiplying with the appropriate coefficients, the expression above can be transformed as follows:

$$\begin{aligned}\frac{1}{\lambda_n-\lambda_{n-1}}\left|\left\{\lambda_{n-1}\leq m<\lambda_n : \left|(Ax)_m-l\right|\geq\varepsilon\right\}\right| &= \frac{\lambda_n}{\lambda_n-\lambda_{n-1}}\frac{1}{\lambda_n}\left|\left\{1\leq m<\lambda_n : \left|(Ax)_m-l\right|\geq\varepsilon\right\}\right| \\ &\quad -\frac{1}{\lambda_n-\lambda_{n-1}}\left|\left\{1\leq m<\lambda_{n-1} : \left|(Ax)_m-l\right|\geq\varepsilon\right\}\right|.\end{aligned}$$

If we consider a matrix $T:=(t_{nk})$ defined as follows: $t_{nk}=\frac{\lambda_n}{\lambda_n-\lambda_{n-1}}$ when $k=n$; $t_{nk}=\frac{\lambda_{n-1}}{\lambda_n-\lambda_{n-1}}$ when $k=n-1$ and $t_{nk}=0$ otherwise, then the $D_\lambda S(A)$ transformation of the sequence (x_k) is the $T=(t_{nk})$ transform of the $C_\lambda S(A)$ transformation of the sequence. □

4. DEFERRED STATISTICAL *A*-TYPE KOROVKIN THEOREMS

In 1960, P. P. Korovkin introduced one of the most significant results in approximation theory, known as Korovkin's theorem. The theorem focuses on real-valued vector spaces defined as follows:

$$C[a,b]:=\{f : f \text{ is a real}-\text{valued continuous function on } [a,b]\}$$

and

$$B[a,b]:=\{f : f \text{ is a real}-\text{valued bounded function on } [a,b]\}$$

Let $T_n : C[a,b]\to B[a,b]$ be a positive and linear operator, which means (T_n) is linear and $T_n(f)\geq 0$ for every $f(x)\geq 0$, where $x\in[a,b]$. Korovkin's theorem provides conditions for the approximation of $T_n(g)$ to a continuous function $g\in C[a,b]$. The original Korovkin theorem is given in the follow theorem.

Theorem 4.1. *(P. P. Korovkin): Consider the sequence* (T_n) *of positive and linear operators* $T_n : C[a,b]\to B[a,b]$. *The following conditions are equivalent:*

1. *For each function* $f_i(x)=x^i$, *where* $i=0,1,2$, *etc., we have* $\lim_{n\to\infty}\|T_n(f_i)-f_i\|_\infty=0$,
2. *For each continuous function* f *in* $C[a,b]$, *we have* $\lim_{n\to\infty}\|T_n(f)-f\|_\infty=0$.

A generalization of Korovkin's theorem was initially explored by Gadjiev-Orhan [20], introducing the concept of asymptotic density. Since then, numerous generalizations and applications of Korovkin's theorem have been developed, considering different types of densities on natural numbers (see for example [31]). For our specific case, let's introduce some notations before discussing the conclusions. Consider the operator $\widetilde{T}_n$: $C[a,b] \to B[a,b]$ defined as follows:

$$\widetilde{T}_n\colon C[a,b] \to B[a,b]$$

where $T(f,.) : C[a,b] \to B[a,b]$ is a positive and linear operator. The operator $\widetilde{T}_n(f,.)$ satisfies the following properties:

$$\widetilde{T}_n(f+g,.) = \sum_{k=1}^{\infty} a_{n,k} T_k(f+g,.) = \sum_{k=1}^{\infty} a_{n,k} T_k(f,.) + \sum_{k=1}^{\infty} a_{n,k} T_k(g,.) = \widetilde{T}_n(f,.) + \widetilde{T}_n(g,.)$$

$\widetilde{T}_n(f,x) \geq 0$, whenever $f(x) \geq 0$ for all $x \in [a,b]$. In the above expressions, $\|\cdot\|_\infty$ represents the uniform norm on $C[a,b]$.

Korovkin's Theorems have been extended and applied to various summability methods, including Cesàro summation (see for example [37]), Abel summation and Borel summation (see for example [33]). These extensions further demonstrate the versatility and importance of Korovkin's Theorems in the context of approximation theory and its connections to other areas of mathematical analysis (see for example [3, 11]), including summability methods.

Theorem 4.2. *(Korovkin Theorem Generalization) Let* $A = (a_{nk})$ *be a regular matrix with non-negative elements, and let p and q be sequences of natural numbers satisfying the deferred property. The following conditions are equivalent:*

1. *For the functions* $f_i(x) = x^i$, *where* $i = 0, 1, 2$, *etc., we have* $\left\|\widetilde{T}_n(f_i) - f_i\right\|_\infty \to 0 \left(D_{p,q}S(A)\right)$,
2. *For every* $f \in C[a, b]$, *we have* $\left\|\widetilde{T}_n(f) - f\right\|_\infty \to 0 \left(D_{p,q}S(A)\right)$.

Proof. $(i) \Longrightarrow (ii)$ Since $f_i(x) = x^i \in C[a,b]$ for each $i = 0, 1, 2$, etc., we have $\lim_{n\to\infty} \left\|\widetilde{T}_n(f_i) - f_i\right\|_\infty = 0$, we have $\lim_{n\to\infty} \left\|\widetilde{T}_n(f) - f\right\|_\infty = 0$ for every $f \in C[a, b]$. Hence, we only need to focus on the other part of the theorem.

$(ii) \Longrightarrow (i)$ If a function $f \in C[a, b]$ is continuous, there exists a positive scalar $M > 0$ such that $|f(t) - f(x)| \leq 2M$ for all $x, t \in [a, b]$. Furthermore, by the continuity of $f \in C[a, b]$, for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $|f(t) - f(x)| < \varepsilon$ holds for all $x, t \in [a, b]$ satisfying $|x - t| < \delta$. Consider $\Phi(x) := (x - t)^2$. For all $x, t \in [a, b]$ satisfying $|x - t| < \delta$, we observe the following inequality:

$$|f(t) - f(x)| < \varepsilon + \frac{2M}{\delta^2}\Phi.$$

Now, we have $\widetilde{T}_n(f,x) - f(x) = \widetilde{T}_n(f(t) - f(x), x) + f(x)\left(\widetilde{T}_n(1,x) - 1\right)$. From this equality, we obtain the following inequality:

$$\left\|\widetilde{T}_n(f) - f\right\| \leq \left(\varepsilon + M + \frac{2M}{\delta^2}\right)\left\|\widetilde{T}_n(1) - 1\right\| + \frac{4Mb}{\delta^2}\left\|\widetilde{T}_n(t) - x\right\| + \frac{2M}{\delta^2}\left\|\widetilde{T}_n\left(t^2\right) - x^2\right\| \leq M_1\left(\sum_{i=0}^{i=2}\left\|\widetilde{T}_n\left(t^i\right) - x^i\right\|\right)$$

where $M_1 := \max\left\{\varepsilon + M + \frac{2M}{\delta^2}, \frac{4Mb}{\delta^2}, \frac{2M}{\delta^2}\right\}$. The above inequality shows that for any $\varepsilon_1 > 0$, we have:

$$\left\{k \leq n : \left\|\widetilde{T}_n(f) - f\right\| \geq \varepsilon_1\right\} \subseteq \{k \leq n : \sum_{i=0}^{i=2}\left\|\widetilde{T}_n\left(t^i\right) - x^i\right\| \geq \frac{\varepsilon_1}{M_1}\}.$$

This implies that:

$$\frac{1}{q_n - p_n}\left|\left\{k \le n : \left\|\widetilde{T}_n(f) - f\right\| \ge \varepsilon_1\right\}\right| \le \frac{1}{q_n - p_n}\left|\left\{k \le n : \left\|\widetilde{T}_n(1) - 1\right\| \ge \frac{\varepsilon_1}{M_1}\right\}\right|$$
$$+ \frac{1}{q_n - p_n}\left|\left\{k \le n : \left\|\widetilde{T}_n(t) - x\right\| \ge \frac{\varepsilon_1}{M_1}\right\}\right| + \frac{1}{q_n - p_n}\left|\left\{k \le n : \left\|\widetilde{T}_n\left(t^2\right) - x^2\right\| \ge \frac{\varepsilon_1}{M_1}\right\}\right|.$$

By the assumptions of the theorem, the functions 1, x, and x^2 are deferred statistically A-convergent to the functions 1, x, and x^2, respectively. This implies that the deferred density of the three clusters on the right side of the above inequality is zero. Thus, we have established the desired proof. □

Corollary 4.3. *When $q(n) = n$ and $p(n) = 0$, Theorem 23 coincides with Theorem 1 given by Gadjiev-Orhan and Theorem 4 given by Alotaibi, corresponding to the unit matrix and Cesàro matrix, respectively.*

We will now present a sequence of positive linear operators that satisfies the assumptions of Theorem 23 but does not satisfy both the classical and statistical cases of the Korovkin theorem. Consider the Bernstein polynomial $B_n(f,x) : C[0,1] \to B[0,1]$ given by

$$B_n(f,x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k},$$

where $B_n(1,x) = 1$ converges to 1, $B_n(t,x) = x$ converges to x, and $B_n\left(t^2, x\right) = x^2 + \frac{x - x^2}{n}$ converges to x^2 as n approaches infinity. Let $A = (a_{nk})$ be a matrix and $x = (x_k)$ be a sequence defined as follows:

$$a_{nk} := \begin{cases} \frac{1}{2}, & n \neq m^2, k = n^2 - 2, k = n^2 - 1, \\ 1, & n \neq m^2, k = n^2, \\ 0 & \text{otherwise}, \end{cases}$$

and

$$x_k := \begin{cases} 1, & \text{if } k \text{ is odd}, \\ k, & \text{if } k \text{ is even square}, \\ 0 & \text{if } k \text{ is nonsquare and even}. \end{cases}$$

Then, the transformed sequence

$$(Ax)_n := \begin{cases} \frac{1}{2}, & \text{if } n \text{ is nonsquare}, \\ k, & \text{if } n \text{ is even square}, \\ 0, & \text{otherwise}, \end{cases}$$

is deferred statistically A-convergent to $1/2$ for any p_n and q_n satisfying the deferred property. Now, let us define the modified form $\widetilde{B}_n(f,x) : C[0,1] \to B[0,1]$ of the Bernstein polynomial as:

$$\widetilde{B}_n(f,x) := (1 + x_n) B_n(f,x).$$

It is evident that $\widetilde{B}_n(f_i, x)$ does not converge to $f_i = x^i, i = 0,1,2$ in the usual case (or in the statistical case). However, it is deferred statistically A-convergent to $f_i = x^i, i = 0,1,2$.

Theorem 4.4. *Let p and q be a sequence of natural numbers satisfying the deferred property, and $A = (a_{nk})$ be a regular matrix with non-negative elements. If $\left\|\widetilde{T}_n(f_i) - f_i\right\|_\infty \to 0 \left(D_{p,q} S(A)\right)$ holds for functions defined by $f_i(x) = x^i$, i = 0, 1, 2, then there exists a set $K \subset \mathbb{N}$ with $\delta_{p,q}(K) = 1$ such that $\left\|T_n^*(f) - f\right\|_\infty \to 0 \left(D_{p,q}(A)\right)$ holds for all $f \in C[a,b]$, where T_n^* is a positive and linear operator with respect to K.*

Proof. Assuming that $\left\|\widetilde{T}_n(f_i)-f_i\right\|_\infty \to 0\left(D_{p,q}S(A)\right)$ holds for functions defined by $f_i(x)=x^i$, we can obtain sets $K_i, i=0,1,2$, such that $\delta_{p,q}(K_i)=1$. Let $K=\bigcap_{i=0}^2 K_i$ such that $\delta_{p,q}(K)=1$. Denote the set K as $K:=\{m_n: n\in\mathbb{N}\}$, and define the corresponding matrix $A^*=(a_{m_n,k})$ and the related operator as

$$T_n^*(f):=\sum_{k=1}^{\infty} a_{m_n,k}.T_k(f).$$

In can be observed that

$$\left\|T_n^*(f_i)-f_i\right\| \to 0\left(D_{p,q}(A^*)\right)$$

holds for $i=0,\ 1,\ 2$. Therefore, we have

$$\left\|T_n^*(f)-f\right\| \leq M_1\left(\sum_{i=0}^{2}\left\|T_n^*(f_i)-f_i\right\|\right).$$

Thus, we get $\left\|T_n^*(f)-f\right\| \to 0\left(D_{p,q}(A^*)\right)$. Hence, the proof is completed. □

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