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Weighted set sharing and related uniqueness problems for L-function

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Abstract. In this paper, we investigate the uniqueness problem of L-functions belonging to the (extended) Selberg class by considering the weighted set sharing with an arbitrary meromorphic function having finitely many poles. The result obtained in this article significantly extends and also improves in some sense a result of the present authors [11] and a recent result due to Banerjee-Kundu [1].

1. Introduction, Definitions and Results

The Riemann zeta function, together with the Riemann hypothesis, has been considered as one of the most significant topics in certain branches of mathematics. L-functions which are defined by taking the Riemann zeta function as a prototype are naturally important, especially in analytic number theory. Recently, value distribution of L-functions have been studied extensively by many mathematicians (see [4], [6], [9]-[10], [15]). The value distribution of an L-function and also the related notion of sharing some values or sets are defined as that of the meromorphic functions.

Let us denote by $\mathcal{M}(\mathbb{C})$ the field of meromorphic functions over $\mathbb{C}$, where and in what follows, $\mathbb{C}$ denotes the usual complex plane. Two nonconstant functions f, g in $\mathcal{M}(\mathbb{C})$ are said to share a value $v \in \mathbb{C} \cup \{\infty\}$ IM (ignoring multiplicity) if $f^{-1}(v) = g^{-1}(v)$ as two sets in $\mathbb{C}$, where $f^{-1}(v) := \{s \in \mathbb{C} : f(s) = v\}$. We say, f and g share v CM (counting multiplicity) if the roots of $f(s) = v$ and the associated multiplicities are exactly same with those of $g(s) = v$. For some $S \subset \mathbb{C} \cup \{\infty\}$, suppose that $E(S, f) := \cup_{v \in S}\{s \in \mathbb{C} : f(s) = v\}$, where a root of multiplicity p is counted p times in the set $E(S, f)$. If we ignore the multiplicities, the notation $E(S, f)$ is replaced by $\overline{E}(S, f)$. If $E(S, f) = E(S, g)$ (resp. $\overline{E}(S, f) = \overline{E}(S, g)$) for some $f, g \in \mathcal{M}(\mathbb{C})$, then f and g are said to share the set S CM (resp. IM). Clearly, set sharing coincides with value sharing in case of singleton set.

The L-functions which are discussed in this paper are such that these can be analytically continued to a meromorphic function in the whole complex place $\mathbb{C}$ and also have some other crucial properties as that of the Riemann zeta function. To be specific, by an L-function we mean a function in the (extended) Selberg class. The Selberg class $\mathcal{S}$ of L-functions is defined as the set of all those Dirichlet series $\mathcal{L}(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$ of a complex variable s, that satisfy the following axioms (see [13, 14]):

(i) *Ramanujan hypothesis:* $a(n) \ll n^{\varepsilon}$ for each $\varepsilon > 0$;

(ii) *Analytic continuation:* There is a nonnegative integer k such that $(s-1)^k \mathcal{L}(s)$ is an entire function of

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finite order;

(iii) *Functional equation:* $\mathcal{L}$ satisfies a functional equation of the type

$$\Lambda_{\mathcal{L}}(s) = \omega\overline{\Lambda_{\mathcal{L}}(1-\overline{s})},$$

where

$$\Lambda_{\mathcal{L}}(s) = \mathcal{L}(s)Q^s \prod_{j=1}^{K} \Gamma(\lambda_j s + \nu_j)$$

with positive real numbers Q, λ_j, positive integer K, and complex numbers ν_j, ω with $\mathrm{Re}(\nu_j) \geq 0$ and $|\omega| = 1$.

(iv) *Euler product hypothesis:* $\mathcal{L}$ can be written over primes:

$$\mathcal{L}(s) = \prod_p exp\left(\sum_{k=1}^{\infty} \frac{b(p^k)}{p^{ks}}\right)$$

with suitable coefficients $b(p^k)$ such that $b(p^k) \ll p^{k\theta}$ for some $\theta < \frac{1}{2}$, where the product is taken over all prime numbers p.

$\mathcal{L}$ is said to be in the extended Selberg class $\mathcal{S}^\sharp$ if it satisfies axioms (i)-(iii). It is worth mentioning that considering $\mathcal{L}$ in $\mathcal{S}^\sharp$ (which we adopt throughout the paper) means considering a wider class of functions than $\mathcal{S}$. Therefore, the results proved in this article are also true for L-functions in $\mathcal{S}$. The degree $d_{\mathcal{L}}$ of an L-function $\mathcal{L}$ is given by

$$d_{\mathcal{L}} := 2\sum_{j=1}^{K} \lambda_j,$$

where λ_j and K are respectively the positive real number and the positive integer defined in axiom (iii).

From Nevanlinna's five-value theorem it is known that two nonconstant functions in $\mathcal{M}(\mathbb{C})$ are identically equal if they share five distinct values in the extended complex plane ($\mathbb{C} \cup \{\infty\}$). Also, the number "five" is the best possible as shown by Nevanlinna (see [5, 17]). In terms of shared values, in 2007, Steuding ([14], p. 152) proved that two L-functions $\mathcal{L}_1$, $\mathcal{L}_2$ with $a(1) = 1$ are identical when they share a complex value c ($\neq \infty$) CM. Later, it was shown by Hu-Li [6] that the theorem does not hold when $c = 1$.

Due to having the property of meromorphic continuation with L-functions, it becomes interesting to investigate how an L-function and an arbitrary meromorphic function are uniquely determined by their shared values. In this direction, Li [9] proved the following uniqueness result.

Theorem 1.1. *([9]) Let f be a meromorphic function in $\mathbb{C}$ with finitely many poles and let a, $b \in \mathbb{C}$ be any two distinct values. If f and a nonconstant L-function $\mathcal{L}$ share a CM and b IM, then $\mathcal{L} \equiv f$.*

In 2001, Lahiri [7, 8] introduced the concept of weighted sharing of values which can measure any sharing by assigning certain nonnegative integer as its weight including ∞. To proceed further, we must recall the following definition.

Definition 1.1. *Let $v \in \mathbb{C} \cup \{\infty\}$ and l be nonnegative integer or ∞. We denote by $E_l(v; f)$ the set of all v-points of f where a v-point of multiplicity p is counted p times if $p \leq l$ and $l+1$ times if $p > l$. If $E_l(v; f) = E_l(v; g)$ we say that f, g share the value v with weight l.*

We write f, g share (v, l) to mean that f, g share some value v with weight l. Clearly if f, g share (v, l), then f, g share (v, l_1) for any integer $0 \leq l_1 < l$. Also we note that $l = \infty$ and $l = 0$ refers to CM sharing and IM sharing respectively.

Let S be any subset of $\mathbb{C} \cup \{\infty\}$ and l be nonnegative integer or ∞. Denote by $E_l(S; f)$ the set $\cup_{v \in S} E_l(v; f)$, where $E_l(v; f)$ is defined as in Definition 1.1. Obviously, $E_\infty(S; f) = E(S; f)$ and $E_0(S; f) = \overline{E}(S; f)$. If $E_l(S; f) = E_l(S; g)$, then we say f, g share (S, l), i.e., f, g share the set S with weight l.

In 2018, using the notion of weighted sharing of values Hao-Chen [4] established the following theorem which replaces CM and IM by certain weights in Theorem 1.1.

Theorem 1.2. *([4]) Let f be a meromorphic function in $\mathbb{C}$ with finitely many poles, and let a_1, $a_2 \in \mathbb{C}$ be any two distinct values. If f and a nonconstant L-function $\mathcal{L}$ share (a_1, m_1) and (a_2, m_2) for two positive integers m_1 and m_2 with $m_1m_2 > 1$, then $\mathcal{L} \equiv f$.*

Earlier to this, in 2015, Wu-Hu [15] considered weighted sharing for two shared values and obtained the same conclusion as above with the same weights of sharing for two L-functions $\mathcal{L}_1$ and $\mathcal{L}_2$. The result certainly extends Steuding's theorem ([14], p. 152). Let us denote by $\#(S)$ the cardinality of any set S. If a shared value is regarded as a set of cardinality 1, then naturally the set sharing problems between some f and $\mathcal{L}$ involving sets of more cardinalities becomes a matter of utmost interest to the researchers. In 2016, Lin-Lin [10] established the following result in this direction.

Theorem 1.3. *([10]) Let f be a meromorphic function in $\mathbb{C}$ with finitely many poles, and $S_1, S_2 \subset \mathbb{C}$ be two sets such that $S_1 \cap S_2 = \emptyset$ and $\#(S_i) \leq 2$; $i = 1, 2$. Suppose that for a finite set $S = \{\alpha_i | i = 1, 2, \ldots, n\}$, $C(S)$ is defined by $C(S) = \frac{1}{n}\sum_{i=1}^{n} \alpha_i$. If f and a nonconstant L-function $\mathcal{L}$ share (S_1, ∞) and $(S_2, 0)$, then $\mathcal{L} \equiv f$ when $C(S_1) \neq C(S_2)$.*

Moreover, either $\mathcal{L} \equiv f$ or $\mathcal{L} + f \equiv 2C(S_1)$ holds when $C(S_1) = C(S_2)$.

In [10], the authors also posed the following question.

Question 1.1. *(Ques. 1.17, [10]) What can be said about the conclusion of Theorem 1.3 if max $\{\#(S_1), \#(S_2)\} \geq 3$?*

Considering this question, in 2019, the present authors [11] proved the following uniqueness result.

Theorem 1.4. *([11]) Let f be a nonconstant meromorphic function in $\mathbb{C}$ with finitely many poles. Suppose that $S_1 = \{a_1, a_2, \ldots, a_k\}$, $(k \geq 3)$ and $S_2 = \{b_1, b_2\}$ are two subsets of $\mathbb{C}$ such that $S_1 \cap S_2 = \emptyset$ and $\prod_{i=1}^{k}(b_1 - a_i)^2 \neq \prod_{j=1}^{k}(b_2 - a_j)^2$. If f and a nonconstant L-function $\mathcal{L}$ share (S_1, ∞) and $(S_2, 0)$, then $\mathcal{L} \equiv f$.*

A close inspection into the statement of Theorem 1.3 reveals that in order to get the only conclusion $\mathcal{L} \equiv f$ for $\#(S_2) = 2$, the shared-set elements satisfy the relations as follows: $(b_1 - a_1) \neq -(b_2 - a_2)$, $(b_1 - a_2) \neq -(b_2 - a_1)$ when $S_1 = \{a_1, a_2\}$, $S_2 = \{b_1, b_2\}$; $(b_1 - a_1) \neq -(b_2 - a_1)$ when $S_1 = \{a_1\}$, $S_2 = \{b_1, b_2\}$. Also it is seen that $(b_1 - a_1) \neq -(b_1 - a_2)$, $(b_1 - a_2) \neq -(b_1 - a_1)$ when $S_1 = \{a_1, a_2\}$, $S_2 = \{b_1\}$ and $(b_1 - a_1) \neq -(b_1 - a_1)$ when $S_1 = \{a_1\}$, $S_2 = \{b_1\}$. Moreover, Theorem 1.4 includes the condition $\prod_{i=1}^{k}(b_1 - a_i) \neq -\prod_{j=1}^{k}(b_2 - a_j)$ and $\prod_{i=1}^{k}(b_1 - a_i) \neq \prod_{j=1}^{k}(b_2 - a_j)$. However, it becomes a challenge to obtain the conclusion of Theorem 1.4 by changing the CM sharing of the set S_1 into IM sharing keeping the same type of restrictions on the elements of the sets as that of Theorem 1.4. In this direction in [3], Chen-Qiu obtained a result with the condition $\prod_{i=1}^{k}(b_1 - a_i)^2 \neq \prod_{j=1}^{k}(b_2 - a_j)^2$, but this involves three set sharing. Recently, Banerjee-Kundu [1] obtained the following result, which replaces CM sharing of S_1 by IM.

Theorem 1.5. *([1]) Let f be a nonconstant meromorphic function in $\mathbb{C}$ with finitely many poles. Suppose that $S_1 = \{a_1, a_2, \ldots, a_k\}$ $(k \geq 2)$ and $S_2 = \{b_1, b_2\}$, where $a_1, a_2, \ldots, a_k, b_1, b_2$ are $k + 2$ distinct finite complex numbers satisfying $(b_1 - a_i) \neq -(b_2 - a_j)$ for $1 \leq i, j \leq k$. If f and a nonconstant L-function $\mathcal{L}$ share $(S_1, 0)$ and (S_2, ∞), then $\mathcal{L} \equiv f$.*

Note 1.1. *In [1], Banerjee-Kundu proved that if in Theorem 1.3, (S_1, ∞) and $(S_2, 0)$ are replaced by (S_1, m_1) and (S_2, m_2) with $m_1m_2 \geq 2$, then conclusions of Theorem 1.3 hold (see Thm. 1.22, [1]). Although it can be seen that the conclusions mostly hold, there is a gap in the argument in the proof (see p. 3774, [1]). In the penultimate paragraph of the proof the authors claimed that after having $\Sigma \equiv 0$, proceeding in the same manner as done in the last part of Proposition 2.6 [10] one would get $f \equiv \mathcal{L}$. Hence they obtained a contradiction to the initial assumption $f \not\equiv \mathcal{L}$, and using this they completed the rest part of the proof. Firstly, we note that without using the assumptions $\chi_0 \not\equiv 0$, $\chi_1 \not\equiv 0$ and proceeding in the aforementioned way (after obtaining $\Sigma \equiv 0$) we precisely obtain that (i) $f \equiv \mathcal{L}$ when $\alpha_1 + \alpha_2 \neq \beta_1 + \beta_2$, and (ii) either $f \equiv \mathcal{L}$ or $f + \mathcal{L} \equiv \alpha_1 + \alpha_2$ when $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$ (see p. 3805, [10]). Since it is already assumed in the proof that $\chi_0 \not\equiv 0$, $\chi_1 \not\equiv 0$, we observe that neither $f \equiv \mathcal{L}$ nor $f + \mathcal{L} \equiv \alpha_1 + \alpha_2$ holds, as otherwise we have $\chi_0 \equiv 0$, $\chi_1 \equiv 0$. Thus we see that the authors' argument is not true. Also it is clear from (i) and (ii) that without using the said assumptions we can not conclude that only $f \equiv \mathcal{L}$ holds. For example, let $\mathcal{L} = \zeta$, $f = -\zeta$, where ζ denotes the Riemann zeta function, and let $\alpha_1 = -1$, $\alpha_2 = 1$, $\beta_1 = -3$, $\beta_2 = 3$. Then $\mathcal{L}$ and f share*

S_1, S_2 with weight ≥ 2, and $\frac{e^{\nu}}{H} \equiv \frac{(f-\alpha_1)(f-\alpha_2)}{(\mathcal{L}-\alpha_1)(\mathcal{L}-\alpha_2)} \equiv \frac{(-\zeta+1)(-\zeta-1)}{(\zeta+1)(\zeta-1)} \equiv 1$, i.e., $\Sigma \equiv 0$ but $f \not\equiv \mathcal{L}$. Secondly, after claiming to obtain $f \equiv \mathcal{L}$ (due to $\Sigma \equiv 0$), the authors concluded that any one of χ_0, χ_1 must be zero, as otherwise it violates the assumption $f \not\equiv \mathcal{L}$. This means that in case $f \not\equiv \mathcal{L}$ due to $\Sigma \equiv 0$, the conclusion that at least one of χ_0, χ_1 must be zero can not be drawn. This is not true in view of the above example, where $f \not\equiv \mathcal{L}$ but $\chi_0 \equiv 0$. Therefore, due to assumption $f \not\equiv \mathcal{L}$ in the beginning, the proof remains incorrect. However, by avoiding this assumption in the beginning, the proof can reach to a position where any one of $\Sigma \equiv 0$, $\chi_0 \equiv 0$ and $\chi_1 \equiv 0$ must hold, whence the conclusions similar to Theorem 1.3 can be obtained.

Furthermore, as the authors [1] used the last part of proof of Proposition 2.6 [10], from this proof (see Case 1, p. 3805, [10]) they surely obtained in a particular situation that $(\beta_1-\alpha_1)^2(\beta_1-\alpha_2)^2 = (\beta_2-\alpha_1)^2(\beta_2-\alpha_2)^2$ implies either $\beta_1 = \beta_2$ or $\alpha_1+\alpha_2 = \beta_1+\beta_2$. If we consider $S_1 = \{\alpha_1, \alpha_2\}$, $S_2 = \{\beta_1, \beta_2\}$ with $\alpha_1 = \frac{1}{3}, \alpha_2 = \frac{41}{22}, \beta_1 = 2, \beta_2 = \frac{1}{2}$, then we have $\beta_1 \neq \beta_2$, $\alpha_1+\alpha_2 \neq \beta_1+\beta_2$ but $(\beta_1-\alpha_1)^2(\beta_1-\alpha_2)^2 = (\beta_2-\alpha_1)^2(\beta_2-\alpha_2)^2$. The existence of such shared sets for some suitable f and $\mathcal{L}$ (and so conclusion of Thm. 1.22 [1] in this case) is although not established, one can proceed as in Subcase 1.1 of this paper whenever possibly $(\beta_1-\alpha_1)(\beta_1-\alpha_2) = -(\beta_2-\alpha_1)(\beta_2-\alpha_2)$ holds, as shown in the third section.

Regarding Theorem 1.4 and Theorem 1.5 it is quite natural to ask the following question:

Question 1.2. *How far can we minimize the weights of sharing to obtain the same conclusion as of Theorems 1.4 and 1.5 so that "CM" is removed from both the sets?*

In this paper, taking into account the above question we have been able to remove the CM sharing from both the sets without adding any extra condition as compared to Theorems 1.4 and 1.5. The main result of this paper is now given by means of the following theorem.

Theorem 1.6. *Let f be a nonconstant meromorphic function in $\mathbb{C}$ with finitely many poles. Suppose that $S_1 = \{a_1, a_2, \ldots, a_k\}$ $(k \geq 2)$ and $S_2 = \{b_1, b_2\}$, where $a_1, a_2, \ldots, a_k, b_1, b_2$ are $k+2$ distinct finite complex numbers satisfying $(b_1-a_i) \neq -(b_2-a_j)$ for $1 \leq i, j \leq k$. If f and a nonconstant L-function $\mathcal{L}$ share (S_1, μ_1) and (S_2, μ_2), where $\mu_1\mu_2 > 1$, then $\mathcal{L} \equiv f$.*

A sufficient number of examples are available to show that removal of the condition $(b_1-a_i) \neq -(b_2-a_j)$ for $1 \leq i, j \leq k$ may not imply that $\mathcal{L} \equiv f$. We mention the following one in this regard.

Example 1.1. *Let $\mathcal{L} = \zeta$, $f = -\zeta$, where ζ denotes the Riemann zeta function, and let $S_1 = \{i, -i, c, -c\}$, $S_2 = \{d, -d\}$ be two disjoint sets, where $c, d \in \mathbb{C}$ and $i^2 = -1$. Then f, $\mathcal{L}$ share S_1 and S_2 with weight both greater than 1, and $(b_1-a_i) = -(b_2-a_j)$ for some i, j in $1 \leq i, j \leq 4$ but $\mathcal{L} \not\equiv f$.*

Since we will employ Nevanlinna theory to prove our main result, we refer the reader to [5, 16, 17], where standard notations and results of this theory can be found in details. For the sake of convenience, we mention that $N(r, f)$ (also written as $N(r, \infty; f)$) is used to denote the counting function of poles, ($\overline{N}(r, f)$ the corresponding reduced counting function) and $T(r, f)$ to denote the characteristic function. We recall that $m(r, f) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$ and $N(r, a; f) = N(r, \frac{1}{f-a})$ for any $a \in \mathbb{C}$. For $f \in \mathcal{M}(\mathbb{C})$, the order $\lambda(f)$ of f is defined as

$$\lambda(f) := \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}.$$

The symbol $S(r, f)$ will mean any quantity satisfying $S(r, f) = O(\log(rT(r, f)))$ $(r \to \infty)$ for all r possibly outside a set of finite linear measure. If $\lambda(f)$ is finite, then $S(r, f) = O(\log r)$ for all r. We also need the following definition for future use in the paper.

Definition 1.2. *[7, 8] For any $p \in \mathbb{N}$, we denote by $\overline{N}(r, a; f| \geq p)$ the counting function of those a-points of f whose multiplicities are greater than or equal to p, where each a-point is counted only once.*

2. Lemmas

We present some lemmas that will be useful in the next section.

Lemma 2.1. *([17, Theorem 1.14]) Let nonconstant $f(z)$, $g(z) \in \mathcal{M}(\mathbb{C})$. If $\lambda(f)$ and $\lambda(g)$ are the orders of f and g respectively, then*

$$\lambda(f.g) \leq max\{\lambda(f), \lambda(g)\},$$
$$\lambda(f+g) \leq max\{\lambda(f), \lambda(g)\}.$$

Lemma 2.2. *Let f be a meromorphic function in $\mathbb{C}$ having finitely many poles in the complex plane and let $S_1 = \{a_1, a_2, \ldots, a_m\}$, $S_2 = \{b_1, b_2\}$, where a_i $(i = 1, 2, \ldots, m)$, b_1, b_2 are $m+2$ distinct complex values. If f and a nonconstant L-function $\mathcal{L}$ share the set $(S_1, 0)$ and $(S_2, 0)$, then $\lambda(f) = \lambda(\mathcal{L})$=1.*

Proof. The proof is omitted as the same can be found in the proof of Lemma 3 [11]. □

Lemma 2.3. *([17, Theorem 1.42]) Let a nonconstant $f(z) \in \mathcal{M}(\mathbb{C})$. If 0 and ∞ are two Picard exceptional values of f, then $f(z) = e^{h(z)}$ for some entire function $h(z)$.*

3. Proof of the Theorem

Proof. [**Proof of Theorem 1.6**] First of all, we note that f is with finitely many poles and $\mathcal{L}$ has at most one pole at $z = 1$. Therefore,

$$\overline{N}(r, f) = O(\log r),\ \overline{N}(r, \mathcal{L}) = O(\log r). \tag{3.1}$$

If $d_{\mathcal{L}}$ denotes the degree of $\mathcal{L}$, then from a result due to Steuding (see [14], p. 150), we have

$$T(r, \mathcal{L}) = \frac{d_{\mathcal{L}}}{\pi} r \log r + O(r). \tag{3.2}$$

From the assumption of set sharing between f and $\mathcal{L}$, and Lemma 2.2, we deduce that $\lambda(f) = \lambda(\mathcal{L}) = 1$ and so $S(r, f) = S(r, \mathcal{L}) = O(\log r)$. Let us now consider the following auxiliary functions:

$$\Delta_1 = \frac{L_a'}{L_a} - \frac{F_a'}{F_a}, \tag{3.3}$$

where $L_a = (\mathcal{L} - a_1)(\mathcal{L} - a_2)\ldots(\mathcal{L} - a_k)$, $F_a = (f - a_1)(f - a_2)\ldots(f - a_k)$;

$$\Delta_2 = \frac{L_b'}{L_b} - \frac{F_b'}{F_b}, \tag{3.4}$$

where $L_b = (\mathcal{L} - b_1)(\mathcal{L} - b_2)$, $F_b = (f - b_1)(f - b_2)$. Clearly, F_a and L_a share $(0, \mu_1)$ as f and $\mathcal{L}$ share (S_1, μ_1), and F_b and L_b share $(0, \mu_2)$ as f and $\mathcal{L}$ share (S_2, μ_2).

Assume that $\Delta_1 \not\equiv 0$ and $\Delta_2 \not\equiv 0$. By the lemma of logarithmic derivative (see [17], Lemma 1.4') it is obvious that

$$m(r, \Delta_1) = O(\log r) = m(r, \Delta_2). \tag{3.5}$$

Since f and $\mathcal{L}$ share (S_2, μ_2), any zero of L_b is a common zero of $\mathcal{L} - b_i$ and $f - b_j$ for some $i, j \in \{1, 2\}$. Moreover, any zero of $(\mathcal{L} - b_1)(\mathcal{L} - b_2)$ with multiplicity $\geq \mu_2 + 1$ is a zero of Δ_1 with multiplicity at least μ_2. Again, due to $(0, \mu_1)$ sharing between L_a and F_a it is evident that the possible poles of Δ_1 that come from a zero of L_a must be of multiplicity at least $\mu_1 + 1$. Therefore, from (3.1), (3.3), (3.5) and the first fundamental theorem we obtain

$$\begin{aligned} & \mu_2\{\overline{N}(r, b_1; \mathcal{L}| \geq \mu_2 + 1) + \overline{N}(r, b_2; \mathcal{L}| \geq \mu_2 + 1)\} \\ \leq\ & N(r, 0; \Delta_1) \\ \leq\ & T(r, \Delta_1) + O(1) \end{aligned}$$

$$\begin{aligned}
&\leq N(r,\Delta_1)+O(\log r)\\
&\leq \overline{N}(r,0;L_a|\geq \mu_1+1)+\overline{N}(r,\mathcal{L})+\overline{N}(r,f)+O(\log r)\\
&\leq \sum_{i=1}^{k}\overline{N}(r,a_i;\mathcal{L}|\geq \mu_1+1)+O(\log r). \qquad (3.6)
\end{aligned}$$

Similarly, considering the sharing between L_a and F_a for a zero of Δ_2, and the sharing of L_b and F_b for a pole of Δ_2 we have from (3.1), (3.4) and (3.5) that

$$\begin{aligned}
&\mu_1\sum_{i=1}^{k}\overline{N}(r,a_i;\mathcal{L}|\geq \mu_1+1)\\
&\leq N(r,0;\Delta_2)\\
&\leq T(r,\Delta_2)+O(1)\\
&\leq N(r,\Delta_2)+O(\log r)\\
&\leq \overline{N}(r,0;L_b|\geq \mu_2+1)+\overline{N}(r,\mathcal{L})+\overline{N}(r,f)+O(\log r)\\
&\leq \overline{N}(r,b_1;\mathcal{L}|\geq \mu_2+1)+\overline{N}(r,b_2;\mathcal{L}|\geq \mu_2+1)+O(\log r). \qquad (3.7)
\end{aligned}$$

From (3.6) and (3.7) we have

$$\left(1-\frac{1}{\mu_1\mu_2}\right)\{\overline{N}(r,b_1;\mathcal{L}|\geq \mu_2+1)+\overline{N}(r,b_2;\mathcal{L}|\geq \mu_2+1)\}\leq O(\log r),$$

which implies in view of the condition $\mu_1\mu_2>1$ that

$$\overline{N}(r,b_1;\mathcal{L}|\geq \mu_2+1)+\overline{N}(r,b_2;\mathcal{L}|\geq \mu_2+1)\leq O(\log r). \qquad (3.8)$$

Similarly,

$$\sum_{i=1}^{k}\overline{N}(r,a_i;\mathcal{L}|\geq \mu_1+1)\leq O(\log r). \qquad (3.9)$$

Therefore, the sharing assumption of L_b, F_b and that of L_a, F_a together with (3.8) and (3.9) respectively yields that

$$\overline{N}(r,b_1;f|\geq \mu_2+1)+\overline{N}(r,b_2;f|\geq \mu_2+1)\leq O(\log r). \qquad (3.10)$$

and

$$\sum_{i=1}^{k}\overline{N}(r,a_i;f|\geq \mu_1+1)\leq O(\log r). \qquad (3.11)$$

We now consider the function $\frac{L_a}{F_a}$ for its possible zeros and poles. From (3.1), (3.9) and (3.11) we have

$$\begin{aligned}
\overline{N}\left(r,\frac{L_a}{F_a}\right) &\leq \overline{N}(r,\mathcal{L})+\overline{N}(r,0;F_a|\geq \mu_1+1)\\
&\leq \sum_{i=1}^{k}\overline{N}(r,a_i;f|\geq \mu_1+1)+O(\log r)\\
&\leq O(\log r), \qquad (3.12)
\end{aligned}$$

and

$$\overline{N}\left(r,0;\frac{L_a}{F_a}\right) \leq \overline{N}(r,f)+\overline{N}(r,0;L_a|\geq \mu_1+1)$$

$$\begin{aligned} &\leq \sum_{i=1}^{k} \overline{N}(r, a_i; \mathcal{L}| \geq \mu_1 + 1) + O(\log r) \\ &\leq O(\log r). \end{aligned} \tag{3.13}$$

From (3.12) and (3.13) it is clear that there exists a rational function $\mathcal{J}$ such that the function defined as

$$\Gamma := \frac{\mathcal{J}(\mathcal{L} - a_1)(\mathcal{L} - a_2)\ldots(\mathcal{L} - a_k)}{(f - a_1)(f - a_2)\ldots(f - a_k)} \tag{3.14}$$

is a non-vanishing entire function. Since we get from Lemmas 2.1 and 2.2 that $\lambda(\Gamma) \leq 1$, by Lemma 2.3 or Hadamard Factorization Theorem (see p. 484, [2]) we can write

$$\Gamma = \frac{\mathcal{J}(\mathcal{L} - a_1)(\mathcal{L} - a_2)\ldots(\mathcal{L} - a_k)}{(f - a_1)(f - a_2)\ldots(f - a_k)} = e^{\varphi}, \tag{3.15}$$

where $\varphi(z)$ is a polynomial of degree at most 1. Let us set

$$\Lambda = \left(\frac{e^{\varphi}}{\mathcal{J}} - 1\right)\left(\frac{e^{\varphi}}{\mathcal{J}} - \gamma\right)\left(\frac{e^{\varphi}}{\mathcal{J}} - \frac{1}{\gamma}\right),$$

where $\gamma = \frac{(b_2-a_1)(b_2-a_2)\ldots(b_2-a_k)}{(b_1-a_1)(b_1-a_2)\ldots(b_1-a_k)} \neq 0$. Then it is easy to see that each zero of L_b is a zero of Λ in view of sharing of the set S_2. Assuming $\Lambda \not\equiv 0$, by Nevanlinna's first and second fundamental theorem we deduce that

$$\begin{aligned} T(r, \mathcal{L}) &\leq \overline{N}(r, \mathcal{L}) + \overline{N}(r, b_1; \mathcal{L}) + \overline{N}(r, b_2; \mathcal{L}) + O(\log r) \\ &\leq \overline{N}(r, 0; \Lambda) + O(\log r) \\ &\leq 3T\left(r, \frac{e^{\varphi}}{\mathcal{J}}\right) + O(\log r) \\ &\leq O(r), \end{aligned}$$

which clearly contradicts (3.2) as $\mathcal{L}$ is nonconstant. This contradiction arises due to the assumption $\Lambda \not\equiv 0$ together with $\Delta_1 \not\equiv 0$ and $\Delta_2 \not\equiv 0$. Therefore one of the following must hold: (i) $\Lambda \equiv 0$, (ii) $\Delta_1 \equiv 0$, (iii) $\Delta_2 \equiv 0$. To discuss the possibilities, we distinguish the following three cases:

Case 1. Suppose that (i) holds. Then $\frac{e^{\varphi}}{\mathcal{J}} \equiv \delta$, where δ is equal to either 1 or γ or $\frac{1}{\gamma}$. If the given condition

$$(b_1 - a_i) \neq -(b_2 - a_j) \quad \forall\ 1 \leq i, j \leq k \tag{3.16}$$

implies that $\prod_{i=1}^{k}(b_1 - a_i)^2 \neq \prod_{j=1}^{k}(b_2 - a_j)^2$, then we will proceed in the same manner as described in Case 1.1, Case 1.2 and Case 1.3 of the proof of Theorem 1.4 (see p. 607, [11]). Arguing as in Case 1.1 of proof of Theorem 1.4 [11] we therefore obtain that f and $\mathcal{L}$ share b_1, b_2 CM when $\frac{e^{\varphi}}{\mathcal{J}} \equiv 1$ and $\prod_{i=1}^{k}(b_1 - a_i)^2 \neq \prod_{j=1}^{k}(b_2 - a_j)^2$. Hence in this case, by Theorem 1.1 we get $\mathcal{L} \equiv f$. Furthermore, if $\frac{e^{\varphi}}{\mathcal{J}} \equiv \gamma$ or $\frac{e^{\varphi}}{\mathcal{J}} \equiv \frac{1}{\gamma}$, then likewise the above mentioned Case 1.2 and Case 1.3 we will get contradiction to the assumption. Thus, in case $\prod_{i=1}^{k}(b_1 - a_i)^2 \neq \prod_{j=1}^{k}(b_2 - a_j)^2$, we have $\mathcal{L} \equiv f$.

If (3.16) does not imply the above inequality, then it implies that either

$$\prod_{i=1}^{k}(b_1 - a_i) = -\prod_{j=1}^{k}(b_2 - a_j) \tag{3.17}$$

or

$$\prod_{i=1}^{k}(b_1 - a_i) = \prod_{j=1}^{k}(b_2 - a_j). \tag{3.18}$$

Subcase 1.1. Let us first suppose that (3.17) holds, i.e., $\gamma = \frac{1}{\gamma} = -1$. Then either $\frac{e^{\varphi}}{\mathcal{J}} \equiv -1$ as (3.17) holds or $\frac{e^{\varphi}}{\mathcal{J}} \equiv 1$, i.e., $\frac{e^{\varphi}}{\mathcal{J}} \equiv \delta$, where $\delta \in \{-1, 1\}$.

Subcase 1.1.(i). Let $\frac{e^{\varphi}}{\mathcal{J}} \equiv -1$. In view of sharing of S_2 and Lemma 2.3 [10], $\frac{e^{\varphi}}{\mathcal{J}} \equiv -1$ can occur only when $f - b_i$ and $\mathcal{L} - b_j$ share 0 for $i \neq j$; $i, j = 1, 2$. For, otherwise in case $f - b_i$ and $\mathcal{L} - b_i$ share 0, then from $\frac{e^{\varphi}}{\mathcal{J}} \equiv -1$ and (3.15) we get $1 = -1$, a contradiction. We now suppose that z_0 is a common zero of $f - b_i$ and $\mathcal{L} - b_j$ $(i \neq j)$ with multiplicity p_0 and q_0 respectively. Then setting $P(z) = (z - a_1)(z - a_2) \dots (z - a_k)$, we see that $P(f) - P(b_i)$ and $P(\mathcal{L}) - P(b_j)$ have zeros at z_0 with multiplicity p_0 and q_0 respectively. From $\frac{e^{\varphi}}{\mathcal{J}} \equiv -1$ and (3.15) we have $P(f) = -P(\mathcal{L})$. Hence by (3.17) we have $P(f) - P(b_i) = -(P(\mathcal{L}) - P(b_j))$ for $i \neq j$, which implies that $p_0 = q_0$. Therefore, in this case $f - b_i$ and $\mathcal{L} - b_j$ $(i \neq j)$ share 0 CM for $i, j = 1, 2$. Hence we deduce that f and $\mathcal{L}$ share the set S_2 CM. Thus due to the condition (3.16), we obtain from Theorem 1.5 that $\mathcal{L} \equiv f$.

Subcase 1.1.(ii). Let $\frac{e^{\varphi}}{\mathcal{J}} \equiv 1$. From the sharing of S_2, we see that if $f - b_i$ and $\mathcal{L} - b_j$ $(i \neq j)$ share 0, then at a common zero of $f - b_i$ and $\mathcal{L} - b_j$ the value of $\frac{e^{\varphi}}{\mathcal{J}}$ is either γ or $\frac{1}{\gamma}$, i.e., the value is -1 due to (3.17). On the other hand, at a common zero of $f - b_i$ and $\mathcal{L} - b_i$ the value of $\frac{e^{\varphi}}{\mathcal{J}}$ is 1. Therefore, $\frac{e^{\varphi}}{\mathcal{J}} \equiv 1$ can occur only when $f - b_i$ and $\mathcal{L} - b_i$ share 0 for $i = 1, 2$. If z_1 is a common zero of $f - b_i$ and $\mathcal{L} - b_i$ with multiplicity p_1 and q_1 respectively, then similarly as above, from $\frac{e^{\varphi}}{\mathcal{J}} \equiv 1$ and (3.15) we have $P(f) - P(b_i) = P(\mathcal{L}) - P(b_i)$ for $i = 1, 2$. Hence $p_1 = q_1$ and so $f - b_i$, $\mathcal{L} - b_i$ share 0 CM for $i = 1, 2$, i.e., f and $\mathcal{L}$ share S_2 CM. Using the condition (3.16), by Theorem 1.5 we obtain that $\mathcal{L} \equiv f$.

Subcase 1.2. Let us now suppose that (3.18) holds, i.e., $\gamma = \frac{1}{\gamma} = 1$. Then $\delta = 1$ i.e., $\frac{e^{\varphi}}{\mathcal{J}} \equiv 1$. More precisely, we note that as (i) holds, the possibility $\frac{e^{\varphi}}{\mathcal{J}} \equiv 1$ may hold without the aid of (3.18), or else it may hold due to (3.18). It is observed that if $f - b_i$ and $\mathcal{L} - b_i$ share 0, then obviously $\frac{e^{\varphi}}{\mathcal{J}} = 1$ holds at their common zeros. Also $\frac{e^{\varphi}}{\mathcal{J}} = 1$ holds at the common zeros of $f - b_i$ and $\mathcal{L} - b_j$ $(i \neq j)$ due to (3.18). Therefore if z_1 is a common zero of $f - b_i$ and $\mathcal{L} - b_i$ with multiplicity p_1 and q_1 respectively, then with the similar argument as in *Subcase 1.1.(ii)* we get that $P(f) - P(b_i) = P(\mathcal{L}) - P(b_i)$, which implies that $p_1 = q_1$. Therefore, $f - b_i$ and $\mathcal{L} - b_i$ share 0 CM for $i = 1, 2$, i.e., f and $\mathcal{L}$ share the set S_2 CM. Similarly, if z_0 is a common zero of $f - b_i$ and $\mathcal{L} - b_j$ $(i \neq j)$, then $P(f) - P(b_i) = P(\mathcal{L}) - P(b_j)$ and so $p_0 = q_0$. Therefore, $f - b_i$ and $\mathcal{L} - b_j$ share 0 CM for $i \neq j$; $i, j = 1, 2$, which means that f and $\mathcal{L}$ share S_2 CM. Since f and $\mathcal{L}$ share S_2 CM, by Theorem 1.5 we obtain that $\mathcal{L} \equiv f$.

Case 2. Suppose, (ii) holds. Integrating (3.3) we get $L_a = CF_a$, where $C(\neq 0)$ is a constant. Therefore L_a and F_a share 0 CM. In other words, f and $\mathcal{L}$ share S_1 CM. Then proceeding similarly as in the proof of Theorem 1.4 (see [11]), we certainly obtain that either $\frac{e^{\varphi}}{\mathcal{J}} \equiv 1$ or $\frac{e^{\varphi}}{\mathcal{J}} \equiv \gamma$ or $\frac{e^{\varphi}}{\mathcal{J}} \equiv \frac{1}{\gamma}$, where $\frac{e^{\varphi}}{\mathcal{J}} = \frac{\prod_{i=1}^{k}(\mathcal{L}-a_i)}{\prod_{j=1}^{k}(f-a_j)}$. Then similarly as in Case 1 above we get the conclusion.

Case 3. Suppose that (iii) holds. Then from (3.4), on integration we deduce that L_b and F_b share 0 CM, i.e., f and $\mathcal{L}$ share S_2 CM. In this case, from Theorem 1.5 we have that $\mathcal{L} \equiv f$ because of the assumption $(b_1 - a_i) \neq -(b_2 - a_j)$ for $1 \leq i, j \leq k$.

This completes the proof of Theorem 1.6. □

4. Further Remarks

It is reasonable to consider Subcase 1.1 and Subcase 1.2 in the proof of Theorem 1.6, since the existence of such S_1 and S_2 is possible, as seen from the following examples.

Example 4.1. *Let* $S_1 = \{-\frac{2}{5}, -\frac{145}{36}\}$, $S_2 = \{\frac{1}{4}, -\frac{3}{2}\}$. *Then* $(b_1 - a_i) \neq -(b_2 - a_j)$ *for* $1 \leq i, j \leq 2$, $\prod_{i=1}^{2}(b_1 - a_i) = -\prod_{j=1}^{2}(b_2 - a_j)$ *and* $\prod_{i=1}^{2}(b_1 - a_i)^2 = \prod_{j=1}^{2}(b_2 - a_j)^2$.

Example 4.2. *Let* $S_1 = \{-\frac{1}{2}, \frac{1}{3}, \frac{1}{6}\}$, $S_2 = \{1, -\frac{1}{2} + \frac{\sqrt{5}}{3}i\}$. *Then* $(b_1 - a_i) \neq -(b_2 - a_j)$ *for* $1 \leq i, j \leq 3$, $\prod_{i=1}^{3}(b_1 - a_i) = \prod_{j=1}^{3}(b_2 - a_j)$ *and* $\prod_{i=1}^{3}(b_1 - a_i)^2 = \prod_{j=1}^{3}(b_2 - a_j)^2$.

From Theorem 1.6 it is clear that the conclusion $\mathcal{L} \equiv f$ is obtained by taking weights μ_1 and μ_2 in such a way that $\mu_1\mu_2 \geq 2$. Naturally, the case $\mu_1 = 0$, $\mu_2 = 0$ or at least $\mu_1 = 0$, $\mu_2 = 1$ or 2 is left unsolved. Therefore, the following question is raised in the direction of further investigation.

Question 4.1. *What happens to Theorem 1.6 if the sharing weight of one shared set is taken as* 0*, keeping the weight of other shared set close to* 0*?*

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References

[1] A. Banerjee and A. Kundu, Uniqueness of L-function and certain class of meromorphic function under two weighted shared sets of least cardinalities, Filomat, 36 (2022), 3755-3774.
[2] C.A. Berenstein and R. Gay, Complex Variables, Springer-Verlag, New York, 1991.
[3] J.F. Chen and C. Qiu, Uniqueness theorems of L-function in the extended Selberg class, Acta Math. Sci., 40B(4) (2020), 970-980.
[4] W.J. Hao and J.F. Chen, Uniqueness theorems for L-functions in the extended Selberg class, Open Math., 16 (2018), 1291-1299.
[5] W.K. Hayman, Meromorphic Functions, The Clarendon Press, Oxford (1964).
[6] P.C. Hu and B.Q. Li, A simple proof and strengthening of a uniqueness theorem for L-functions, Canad. Math. Bull., 59 (2016), 119-122.
[7] I. Lahiri, Weighted sharing and uniqueness of meromorphic functions, Nagoya Math. J., 161 (2001), 193-206.
[8] I. Lahiri, Weighted value sharing and uniqueness of meromorphic functions, Complex Var. Theory Appl., 46 (2001), 241-253.
[9] B.Q. Li, A result on value distribution of L-functions, Proc. Amer. Math. Soc., 138 (2010), 2071-2077.
[10] P. Lin and W. Lin, Value distribution of L-functions concerning sharing sets, Filomat, 30(2016), 3795-3806.
[11] P. Sahoo and S. Halder, Some results on L-functions related to sharing two finite sets, Comput. Methods Funct. Theo., 19 (2019), 601-612.
[12] P. Sahoo and S. Halder, Correction to : Some results on L-functions related to sharing two finite sets, Comput. Methods Funct. Theo., 22 (2022), 191-196.
[13] A. Selberg, Old and new conjectures and results about a class of Dirichlet series, in: Proccedings of the Amalfi Conference on Analytic Number Theory (Maiori, 1989), 367-385, Univ. Salerno, Salerno, 1992.
[14] J. Steuding, Value distribution of L-functions, Lect. Notes Math., Vol. 1877, Springer-Verlag, Berlin, 2007.
[15] A.D. Wu and P.C. Hu, Uniqueness theorems for Dirichlet series, Bull. Aust. Math. Soc., 91 (2015), 389-399.
[16] L. Yang, Value Distribution Theory, Springer-Verlag, Berlin, Heidelberg, 1993.
[17] C.C. Yang and H.X. Yi, Uniqueness Theory of Meromorphic Functions, Kluwer Academic Publishers, Dordrecht, Boston, London, 2003.