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Duality results in terms of convexifactors for a bilevel multiobjective optimization problem

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Abstract. In this work, we have formulated a Wolfe type dual for a multiobjective bilevel programming problem, which involves vector-valued objective function in the upper level and single objective function in the lower level. Under generalized convexity assumptions, on the functions involved weak and strong duality theorems are derived. Some examples are given to illustrate the applicability of the obtained results.

1. Introduction

Bilevel programming problems are hierarchical optimization problems consisting of a second optimization problem as a part of the constraint of the first optimization problem. H. V. Stackelberg [15] formulated the first bilevel programming problem on the market economy. During past few years, many researchers [1–4, 9, 10, 13, 16, 17], has studied bilevel programming. Bard [2] established the first-order necessary optimality conditions for the linear bilevel programming problem. Later, in [17], under nonsmooth constraint qualifications, Ye developed Karush-Kuhn-Tucker type necessary optimality conditions. Dempe [4] derived necessary and sufficient conditions for the case where the solution set of the lower level problem is a singleton. Babahadda and Gadhi [1] used the concept of convexifactors to derive the necessary optimality conditions for bilevel optimization problems in terms of Lagrange-Kuhn-Tucker multipliers. Lafhim et al. [13] gave Karush-Kuhn-Tucker type necessary optimality conditions of a multiobjective bilevel optimization problem with the help of Ψ-reformulation. Later in [10], Gadhi et al. established the sufficient optimality conditions and also formulated the Mond-Weir dual and gave duality results. Recently, Chuong [3] derived the necessary optimality conditions for the nonsmooth multiobjective bilevel optimization problem concerning the vector-valued objective functions in both levels of the program.

The bilevel programming problem is reformulated as a single level mathematical programming problem with the help of Ψ-reformulation. The concept of Ψ-reformulation is depended on a scalarization function proposed in optimization by Hiriart-Urruty [11] and is utilized by Gadhi and Dempe [9], Lafhim et al. [13], Gadhi et al. [10], etc.

We have formulated Wolfe type dual of bilevel multiobjective problem by using generalized convexity introduced by Dutta and Chandra [7]. Demyanov [5] introduced the concept of convexifactor as a generalization of the notion of lower concave and upper convex approximations and was then examined in [6] and [12]. A new notion of ∂^*-pseudoconvex function in terms of convexifactors was introduced by Dutta

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and Chandra [7, 8]. For a nonsmooth optimization problem involving locally Lipschitz functions, Li and Zhang [14] obtained the necessary optimality conditions, under certain constraint qualification, by using the concept of upper and lower convexifactors. Suneja and Kohli [16] used convexifactors to give sufficient optimality conditions and derived duality results for the Wolfe dual and the Mond-Weir dual.

This paper is organized as follows. In section 2, we consider the bilevel multiobjective programming problem, recall some notations, basic definitions of convexifactors and some preliminary results. In section 3, considered bilevel problem is reformulated as a single level programming problem with the help of Ψ-reformulation and an example is given to show the applicability of the model and notations. In section 4, we associate the Wolfe type dual to the bilevel problem, derive weak and strong duality results under generalized convexity and an example is provided to exhibit the applicability of the attained outcomes. Concluding remarks are given in section 5.

2. Notations and Preliminaries

We consider the following bilevel multiobjective programming problem in this paper:

$$(P) \qquad \mathbb{R}^n_+ - \operatorname*{Minimize}_{x,y} F(x,\ y) = \left\{F_1(x,\ y), \dots, F_n(x,\ y)\right\}$$
$$s.t. \quad G_j(x,\ y) \leq 0,\ \forall j \in J$$
$$y \in \Lambda(x),$$

where for each $x \in \mathbb{R}^{n_1}$, $\Lambda(x)$ is the set of optimal solutions of the following parametric optimization problem

$$(P_x) \qquad \operatorname*{Minimize}_{y} f(x,\ y)$$
$$s.t. \quad g_s(x,\ y) \leq 0,\ \forall s \in S.$$

Here, $f : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $g_s : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $s \in S = \{1, \dots, q\}$, $G_j : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $j \in J = \{1, \dots, p\}$, $F_i : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$, $i \in I = \{1, \dots, n\}$ are given locally Lipschitz continuous functions and $n_1,\ n_2,\ p,\ q,\ n \geq 1$ are integers.

A pair $(\bar{x},\ \bar{y})$ is said to be a weak efficient solution if there exists no other $(x,\ y) \in E$ such that

$$F(\bar{x},\ \bar{y}) - F(x,\ y) \in \text{ int } \mathbb{R}^n_+$$

where

$$E = \{(x,\ y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} :\ G_j(x,\ y) \leq 0,\ \forall j \in J \text{ and } y \in \Lambda(x)\}$$

is the feasible set of (P).
Let the closure, the convex hull and the negative polar cone of a subset C of $\mathbb{R}^n$ be denoted by $cl\, C$, $conv\, C$ and C^o, respectively. Let us consider a function $\Delta_C : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\Delta_C(x) = \begin{cases} -d(x,\ \mathbb{R}^n \setminus C) & \text{if } x \in C, \\ d(x,\ C) & \text{if } x \in \mathbb{R}^n \setminus C, \end{cases}$$

where

$$d(x,\ C) = \inf\left\{\|x - y\|,\ y \in C\right\}.$$

Proposition 2.1. [11] *Let $C \subset \mathbb{R}^n$ $(C \neq \mathbb{R}^n)$ be a closed and convex cone with nonempty interior. The function Δ_C is convex, positively homogenous, 1-Lipschitzian and decreasing on $\mathbb{R}^n$ with respect to the order introduced by C. Moreover $(\mathbb{R}^n \setminus C) = \{x \in \mathbb{R}^n :\ \Delta_C(x) > 0\}$, $int(C) = \{x \in \mathbb{R}^n :\ \Delta_C(x) < 0\}$ and the boundary of C is the set $bd(C) = \{x \in \mathbb{R}^n :\ \Delta_C(x) = 0\}$.*

Proposition 2.2. [10] *Let $C \subset \mathbb{R}^n$ $(C \neq \mathbb{R}^n)$ be a nonempty, closed and convex cone with nonempty interior. Then for all $x \in \mathbb{R}^n$*

$$0 \notin \partial\Delta_C(x).$$

Where, the set $\partial f(x)$ represents the subdifferential of convex analysis of f at x.

Let $F : \mathbb{R}^n \to \mathbb{R} \bigcup \{\pm\infty\}$ be an extended real-valued function and let $x \in \mathbb{R}^n$ where $F(x)$ is finite. The lower and upper Dini directional derivatives of F at x in the direction v are defined, respectively, by

$$F_d^-(x,\ v) = \liminf_{t \to 0^+} \frac{F(x + tv) - F(x)}{t}$$

and

$$F_d^+(x,\ v) = \limsup_{t \to 0^+} \frac{F(x + tv) - F(x)}{t}$$

Definition 2.3. [8, 12] *Let $F : \mathbb{R}^n \to \mathbb{R} \bigcup \{\pm\infty\}$ be an extended real-valued function*

- *The function F is said to admit an upper convexifactor (UCF) $\partial^* F(x)$ at x if $\partial^* F(x) \subset \mathbb{R}^n$ is closed and for each $v \in \mathbb{R}^n$,*

$$F_d^-(x,\ v) \leq \sup_{x^* \in \partial^* F(x)} \langle x^*,\ v \rangle.$$

- *The function F is said to admit a lower convexifactor (LCF) $\partial_* F(x)$ at x if $\partial_* F(x) \subset \mathbb{R}^n$ is closed and for each $v \in \mathbb{R}^n$,*

$$F_d^+(x,\ v) \geq \inf_{x^* \in \partial_* F(x)} \langle x^*,\ v \rangle.$$

- *F is said to admit a convexifactor $\partial^* F(x)$ at x if $\partial^* F(x)$ is both an upper and lower convexifactor of F at x.*

- *F is said to have an upper semi-regular convexifactor (USRCF) $\partial^* F(x)$ at x if $\partial^* F(x)$ is an upper convexifactor at x and for each $v \in \mathbb{R}^n$,*

$$F_d^+(x,\ v) \leq \sup_{x^* \in \partial^* F(x)} \langle x^*,\ v \rangle.$$

Definition 2.4. [16] *Let $F : \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}$ and $(u,\ v) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Assume that F admits convexifactor $\partial^* F(u,\ v)$, F is said to be ∂^*-convex at $(u,\ v)$ iff for all $(x,\ y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$:*

$$\langle \rho, (x,\ y) - (u,\ v) \rangle \leq F(x,\ y) - F(u,\ v), \qquad \forall \rho \in \partial^* F(u,\ v).$$

If the strict inequality holds in above definition for $(x,\ y) \neq (u,\ v)$, then F is said to be strict ∂^*-convex at $(u,\ v)$.

Definition 2.5. [13] *We say that the nonsmooth Abadie constraint qualification holds at $\bar{u} = (\bar{x},\ \bar{y}) \in E$ with respect to (UCFs) $\partial^* G_j(\bar{u})$; $j \in J_0(\bar{u})$, $\partial^* g_s(\bar{u})$; $s \in S_0(\bar{u})$ and $\partial^* \Psi(\bar{u})$ if*

$$[C(\bar{u})]^o \subseteq K(E,\ \bar{u}),$$

where

$$J_0(\bar{u}) = \{j \in J :\ G_j(\bar{u}) = 0\},\ S_0(\bar{u}) = \{s \in S :\ g_s(\bar{u}) = 0\}$$

and

$$C(\bar{u}) = \left(\bigcup_{j \in J_0(\bar{u})} \partial^* G_j(\bar{u}) \right) \cup \left(\bigcup_{s \in S_0(\bar{u})} \partial^* g_s(\bar{u}) \right) \cup \partial^* \Psi(\bar{u}).$$

Here, $K(E,\ \bar{u})$ denotes the contingent cone to E at $\bar{u}$.

3. Ψ- reformulation of the (P)

In this section, according to [9], (P) can be reformulated as a single level programming problem (RP) given as follows:
Let $x \in \mathbb{R}^{n_1}$ and let

$$Y(x) = \{y \in \mathbb{R}^{n_2} : \; g_s(x, \, y) \leq 0, \;\; \forall s \in S\}$$

be the feasible region of the lower level problem (P_x).

$$(RP) \qquad \mathbb{R}^n_+ - \underset{x,y}{\text{Minimize}}\; F(x, \, y) = \Big\{F_1(x, \, y), \ldots, F_n(x, \, y)\Big\}$$

$$\text{subject to } : \begin{cases} G_j(x, \, y) \leq 0 & j \in J, \\ g_s(x, \, y) \leq 0 & s \in S, \\ \Psi(x, \, y) \leq 0, \\ (x, \, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}, \end{cases}$$

where

$$\Psi(x, \, y) = \max_{z \in Y(x)} \psi(x, \, y, \, z)$$

and

$$\psi(x, \, y, \, z) = \min\Big\{f(x, \, y) - f(x, \, z), -\Delta_{(-\mathbb{R}^q_+)}\Big(g_1(x, \, z), \ldots, g_q(x, \, z)\Big)\Big\}.$$

Taking $x \in \mathbb{R}^{n_1}$, we assume that $Y(x)$ is closed and bounded. The optimal value function of the lower level problem (P_x) is defined by

$$V(x) = \inf_y \{f(x, y) : g_s(x, y) \leq 0, \;\; \forall s \in S\}.$$

Lemma 3.1. [10] $\left\{ \begin{array}{c} (x, \, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \\ y \in Y(x) \textit{ and } f(x, \, y) - V(x) < 0 \end{array} \right\} = \left\{ \begin{array}{c} (x, \, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \\ y \in Y(x) \textit{ and } \Psi(x, \, y) < 0 \end{array} \right\} = \emptyset.$

Lemma 3.2. [10] $\left\{ \begin{array}{c} (x, \, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \\ y \in Y(x) \textit{ and } f(x, \, y) - V(x) = 0 \end{array} \right\} = \left\{ \begin{array}{c} (x, \, y) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} : \\ y \in Y(x) \textit{ and } \Psi(x, \, y) = 0 \end{array} \right\}.$

Theorem 3.3. [10] *Let $\bar{u} = (\bar{x}, \, \bar{y}) \in E$ be a local weak efficient solution of (P). Assume that F_i; $i \in I$ admit bounded* (USRCF) *$\partial^* F_i(\bar{u})$ at $\bar{u}$, G_j; $j \in J$, g_s; $s \in S$, admit* (UCFs) *$\partial^* G_j(\bar{u})$, $\partial^* g_s(\bar{u})$, respectively at $\bar{u}$. If the nonsmooth Abadie constraint qualification holds at $\bar{u}$, then there exists $\bar{\pi}^* = (\lambda^*, \, \mu^*, \, \nu^*, \, \eta^*) \in \mathbb{R}^{\bar{n}}_+$, $\lambda^* \neq 0_{\mathbb{R}^n}$,*

$$\begin{gathered} 0 \in \sum_{i=1}^{n} \lambda_i^* \partial^* F_i(\bar{u}) \; + \; \sum_{j=1}^{p} \mu_j^* \partial^* G_j(\bar{u}) \; + \; \sum_{s=1}^{q} \nu_s^* \partial^* g_s(\bar{u}) \; + \; \eta^* \partial^* \Psi(\bar{u}) \\ \mu_j^* G_j(\bar{u}) = 0, \; \nu_s^* g_s(\bar{u}) = 0, \qquad j \in J, \; s \in S. \end{gathered} \tag{1}$$

Theorem 3.4. [10] *Let $\bar{u} \in E$ be a feasible solution of (P). Suppose that F_i, $i \in I$, G_j, $j \in J_0(\bar{u})$, g_s, $s \in S_0(\bar{u})$, and Ψ are ∂^*-convex at $\bar{u}$ and that there exists $\bar{\pi}^* = (\lambda^*, \, \mu^*, \, \nu^*, \, \eta^*) \in \mathbb{R}^{\bar{n}}_+$, $\lambda^* \neq 0_{\mathbb{R}^n}$, satisfying* (1). *Then, $\bar{u}$ is a weak efficient solution of (P).*

Example 3.5. *Consider the following multiobjective bilevel optimization problem:*

$$(Q) : \begin{cases} \mathbb{R}^2_+ - \underset{x,y}{min}\; F(x, \, y) = \Big\{x^2 - y, \; x + y + 1\Big\} \\ subject\ to : \begin{cases} G_1(x, \, y) = -x - 2y \leq 0, \\ G_2(x, \, y) = -y \leq 0, \\ y \in \Lambda(x), \end{cases} \end{cases}$$

where for each $x \in \mathbb{R}$, $\Lambda(x)$ is the set of optimal solutions of the following parametric optimization problem

$$(Q_x) : \begin{cases} \underset{y}{min}\; f(x, \, y) = |x| + y^2 \\ subject\ to : g_1(x, \, y) = y \leq 0. \end{cases}$$

We have,

$$F_1(x, \, y) = x^2 - y \textit{ and } F_2(x, \, y) = x + y + 1.$$

We claim that $\bar{u} = (0,\ 0)$ *is a feasible point of* (Q) *with*

$$\Lambda(x) = \{0\},\ E = R^{+} \times \{0\},\ J_0(\bar{u}) = \{1,2\},\ S_0(\bar{u}) = \{1\}$$

and

$$\Psi(x,\ y) = min\{y^2,\ 0\}.$$

- *The sets* $\partial^* F_1(\bar{u}) = \{(0,-1)\}$ *and* $\partial^* F_2(\bar{u}) = \{(1,1)\}$ *are bounded upper semi-regular convexifactors of* F_1 *and* F_2. *Moreover,* F_1 *and* F_2 *are* ∂^**-convex functions at* $\bar{u}$.
- *The sets* $\partial^* G_1(\bar{u}) = \{(-1,-2)\}$, $\partial^* G_2(\bar{u}) = \{(0,-1)\}$, $\partial^* g_1(\bar{u}) = \{(0,1)\}$ *and* $\partial^* \Psi(\bar{u}) = \{(0,0)\}$ *are upper convexifactors of* $G_1,\ G_2,\ g_1$ *and* Ψ. *Moreover,* $G_1,\ G_2,\ g_1$ *and* Ψ *are* ∂^**-convex functions at* $\bar{u}$.
- *The nonsmooth Abadie constraint qualification holds at* $\bar{u}$. *Indeed, since*

  $$C(\bar{u}) = \{(-1,-2),\ (0,-1),\ (0,1),\ (0,0)\} \text{ and } E = \mathbb{R}^{+} \times \{0\}$$

  we obtain,

  $$[C(\bar{u})]^{\circ} = \mathbb{R}^{+} \times \{0\} = K(E,\ \bar{u}).$$

- *For* $\pi^* = \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{3}{2},\ 1\right)$, *the condition (1) is satisfied. Indeed,*

  $$\frac{1}{2}(0,-1) + \frac{1}{2}(1,1) + \frac{1}{2}(-1,-2) + \frac{1}{2}(0,-1) + \frac{3}{2}(0,1) + 1(0,0) = (0,0)$$

  by Theorem 3.4, we conclude that $\bar{u}$ *is a weak efficient solution of* (Q).

4. Duality

Let $\bar{n} = n + p + q + 1$ and suppose that $F_i;\ i \in I$ admit bounded (USRCF) $\partial^* F_i(v)$ at $v \in \mathbb{R}^{n_1+n_2}$ and $G_j,\ j \in J,\ g_s,\ s \in S$, admit (UCFs) $\partial^* G_j(t)$ and $\partial^* g_s(t)$ at v. We formulate the Wolfe dual problem (D) and establish duality theorems for (P) and (D). Consider the Wolfe dual problem (D) of (P):

$$(D) \qquad \mathbb{R}^n_+ - \underset{v,\pi^*}{\text{Maximize}}\ \phi(v,\pi^*) = \left\{\phi_1(v,\pi^*),\dots,\phi_n(v,\pi^*)\right\}$$

$$\text{subject to} \quad 0 \in \sum_{i=1}^{n} \lambda_i^* \partial^* F_i(v) \ +\ \sum_{j=1}^{p} \mu_j^* \partial^* G_j(v) \ +\ \sum_{s=1}^{q} \nu_s^* \partial^* g_s(v) \ +\ \eta^* \partial^* \Psi(v),$$

$$\pi^* = (\lambda_1^*,\dots,\lambda_n^*,\ \mu_1^*,\dots,\mu_p^*,\ \nu_1^*,\dots,\ \nu_q^*,\ \eta^*) \geq 0,$$

$$(\lambda_1^*,\dots,\lambda_n^*) \neq (0,\dots,0),\ \sum_{i=1}^{n} \lambda_i^* = 1,$$

where,

$$\phi_i(v,\pi^*) = F_i(v) \ +\ \sum_{j=1}^{p} \mu_j^* G_j(v) \ +\ \sum_{s=1}^{q} \nu_s^* g_s(v) \ +\ \eta^* \Psi(v).$$

The set $\tilde{E}$ of all feasible points of (D) is defined as:

$$\tilde{E} = \Big\{(v,\pi^*) \in \mathbb{R}^{n_1+n_2} \times \mathbb{R}^{\bar{n}} :\ 0 \in \sum_{i=1}^{n} \lambda_i^* \partial^* F_i(v) \ +\ \sum_{j=1}^{p} \mu_j^* \partial^* G_j(v) \ +\ \sum_{s=1}^{q} \nu_s^* \partial^* g_s(v) \ +\ \eta^* \partial^* \Psi(v),$$

$$\pi^* \geq 0,\ (\lambda_1^*,\dots,\lambda_n^*) \neq (0,\dots,0),\ \sum_{i=1}^{n} \lambda_i^* = 1\Big\}.$$

Remark 4.1. *Under the hypotheses of Theorem 3.3, the set $\tilde{E}$ of all feasible points of (D) is nonempty.*

Theorem 4.2 (Weak Duality). *Let u be feasible for (P) and for any feasible point $(v,\ \pi^*)$ of (D), such that $F_i,\ i \in I,\ G_j,\ j \in J_0(\bar{u}),\ g_s,\ s \in S_0(\bar{u})$ and Ψ are ∂^*-convex at v, then $F(u) \nleq \phi(v,\ \pi^*)$.*

Proof Suppose, to the contrary that $F(u) \leq \phi(v,\ \pi^*)$
that is,

$$\Big\{F_1(u),\ldots,F_n(u)\Big\} \leq \Big\{\phi_1(v,\ \pi^*),\ldots,\phi_n(v,\ \pi^*)\Big\}$$

Since $(\lambda_1^*,\ \ldots,\ \lambda_n^*) \geq 0$ and $(\lambda_1^*,\ \ldots,\ \lambda_n^*) \neq 0$ we obtain,

$$\sum_{i=1}^{n} \lambda_i^* F_i(u) \leq \sum_{i=1}^{n} \lambda_i^* \phi_i(v,\ \pi^*). \tag{2}$$

Since v is feasible for dual so,

$$0 \in \sum_{i=1}^{n} \lambda_i^* \partial^* F_i(v)\ +\ \sum_{j=1}^{p} \mu_j^* \partial^* G_j(v)\ +\ \sum_{s=1}^{q} \nu_s^* \partial^* g_s(v)\ +\ \eta^* \partial^* \Psi(v).$$

Therefore there exist $\xi_i \in \partial^* F_i(v),\ \eta_j \in \partial^* G_j(v),\ \theta_s \in \partial^* g_s(v)$ and $\chi \in \partial^* \Psi(v)$, such that,

$$0 = \sum_{i=1}^{n} \lambda_i^* \xi_i\ +\ \sum_{j=1}^{p} \mu_j^* \eta_j\ +\ \sum_{s=1}^{q} \nu_s^* \theta_s\ +\ \eta^* \chi, \tag{3}$$

Since $F_i,\ i \in I$ is ∂^*-convex at v, we have,
$\langle\, \xi_i,\ u - v\rangle \leq F_i(u) - F_i(v),\quad \forall i \in I$
for $\lambda_i^* \geq 0$ and $\lambda_i^* \neq 0$, we have, $\lambda_i^*(F_i(u) - F_i(v)) \geq \langle\, \lambda_i^* \xi_i,\ u - v\rangle,\quad \forall i \in I.$

$$\sum_{i=1}^{n} \lambda_i^* F_i(u) \geq \sum_{i=1}^{n} \lambda_i^* F_i(v) + \Big\langle\ \sum_{i=1}^{n} \lambda_i^* \xi_i,\ u - v\Big\rangle.$$

From (3)

$$\sum_{i=1}^{n} \lambda_i^* F_i(u) \geq \sum_{i=1}^{n} \lambda_i^* F_i(v) + \Big\langle\ -\Big(\sum_{j=1}^{p} \mu_j^* \eta_j\ +\ \sum_{s=1}^{q} \nu_s^* \theta_s\ +\ \eta^* \chi\Big),\ u - v\Big\rangle,$$

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i^* F_i(u) \geq \sum_{i=1}^{n} \lambda_i^* F_i(v) &- \Big\langle\ \sum_{j=1}^{p} \mu_j^* \eta_j,\ u - v\Big\rangle \\ &-\ \Big\langle\ \sum_{s=1}^{q} \nu_s^* \theta_s,\ u - v\Big\rangle\ -\ \langle\, \eta^* \chi,\ u - v\rangle.\end{aligned} \tag{4}$$

Since $G_j;\ j \in J_0(\bar{u}),\ g_s;\ s \in S_0(\bar{u})$ and Ψ are ∂^*-convex at v and $\mu_j^* \geq 0,\ \nu_s^* \geq 0$ and $\eta^* \geq 0$, we have,

$$\begin{aligned}&\sum_{j=1}^{p} \mu_j^* \Big(G_j(u) - G_j(v)\Big) \geq \Big\langle\ \sum_{j=1}^{p} \mu_j^* \eta_j,\ u - v\Big\rangle,\quad \forall j \in J_0(\bar{u}),\\ &\sum_{s=1}^{q} \nu_s^* \Big(g_s(u) - g_s(v)\Big) \geq \Big\langle\ \sum_{s=1}^{q} \nu_s^* \theta_s,\ u - v\Big\rangle,\quad \forall s \in S_0(\bar{u}),\\ &\eta^* \Big(\Psi(u) - \Psi(v)\Big) \geq \langle\, \eta^* \chi,\ u - v\rangle.\end{aligned} \tag{5}$$

Putting (5) in (4)

$$\sum_{i=1}^{n} \lambda_i^* F(u) \geq \sum_{i=1}^{n} \lambda_i^* F(v) + \sum_{j=1}^{p} \mu_j^* \Big(G_j(v) - G_j(u) \Big) + \sum_{i=1}^{n} \nu_s^* \Big(g_s(v) - g_s(u) \Big) + \eta^* \Big(\Psi(v) - \Psi(u) \Big)$$

since u is feasible for (P) therefore, by Lemma 3.1 and Lemma 3.2 it is feasible for (RP), we have,

$$\sum_{i=1}^{n} \lambda_i^* F_i(u) \geq \sum_{i=1}^{n} \lambda_i^* F_i(v) + \sum_{j=1}^{p} \mu_j^* G_j(v) + \sum_{i=1}^{n} \nu_s^* g_s(v) + \eta^* \Psi(v)$$

Since $\sum_{i=1}^{n} \lambda_i^* = 1,\ \lambda_i^* \geq 0$ and $\lambda_i^* \neq 0$, we obtain,

$$\sum_{i=1}^{n} \lambda_i^* F_i(u) \geq \sum_{i=1}^{n} \lambda_i^* \phi_i(v,\ \pi^*)$$

which is contradiction to (2). Hence the result. □

Theorem 4.3 (Strong Duality). *Let $\bar{u}$ be a weak efficient solution of (P) where the nonsmooth Abadie constraint qualification holds. Then, there exists $\bar{\pi}^* = (\lambda^*,\ \mu^*,\ \nu^*,\ \eta^*) \in \mathbb{R}_+^{\bar{n}},\ \lambda^* \neq 0_{\mathbb{R}^{\bar{n}}}$, such that $(\bar{u},\ \bar{\pi}^*)$ is a feasible point of (D) and optimal values of objective functions are equal. Moreover, if $F_i,\ i \in I,\ G_j,\ j \in J_0(\bar{u}),\ g_s,\ s \in S_0(\bar{u})$, and Ψ are ∂^*-convex at $\bar{u}$, then $(\bar{u},\ \bar{\pi}^*)$ is a weak efficient solution of (D).*

Proof Let $\bar{u}$ be a weak efficient solution of (P) where the nonsmooth Abadie constraint qualification holds.

- By Theorem 3.3 one can find $\lambda^* \in (-\mathbb{R}_+^n)^o \setminus \{0\}$ and $(\mu^*,\ \nu^*,\ \eta^*) \in \mathbb{R}_+^{p+q+1}$ such that

$$0 \in \sum_{i=1}^{n} \lambda_i^* \partial^* F_i(\bar{u}) \ +\ \sum_{j=1}^{p} \mu_j^* \partial^* G_j(\bar{u}) \ +\ \sum_{s=1}^{q} \nu_s^* \partial^* g_s(\bar{u}) \ +\ \eta^* \partial^* \Psi(\bar{u})$$
$$\mu_j^* G_j(\bar{u}) = 0, \quad \nu_s^* g_s(\bar{u}) = 0, \quad j \in J,\ s \in S.$$

  Since $\Psi(\bar{u}) = 0$, concludes that $(\bar{u},\ \bar{\pi}^*)$ is a feasible point of (D).

- Let us prove that $(\bar{u},\ \bar{\pi}^*)$ is a weak efficient solution of (D) by contrary, suppose that there exists a point $(\bar{v}, \bar{\pi}_1{}^*) \in \tilde{E}$ such that

$$\phi(\bar{v}, \bar{\pi}_1{}^*) - F(\bar{u}) \in \operatorname{int}(\mathbb{R}_+^n) \tag{6}$$

  Since $(\bar{u},\ \bar{\pi}^*)$ is a feasible point of (D) and since $\bar{u}$ is a feasible solution of (P), by Theorem 4.2 one get a contradiction to (6). Hence the result. □

Example 4.4. *The Wolfe dual of the problem (Q) considered in Example 3.5 is*

$$(D): \begin{cases} \mathbb{R}_+^2 - \underset{\hat{x},\hat{y},\pi^*}{max} \Big\{ \phi_1(\hat{x},\ \hat{y},\ \pi^*),\ \phi_2(\hat{x},\ \hat{y},\ \pi^*) \Big\} \\ \textit{subject to}: \\ 0 \in \lambda_1^* \partial^* F_1(\hat{x},\ \hat{y}) + \lambda_2^* \partial^* F_2(\hat{x},\ \hat{y}) + \mu_1^* \partial^* G_1(\hat{x},\ \hat{y}) \\ \quad + \mu_2^* \partial^* G_2(\hat{x},\ \hat{y}) + \nu_1^* \partial^* g_1(\hat{x},\ \hat{y}) + \eta^* \partial^* \Psi(\hat{x},\ \hat{y}), \\ \pi^* = (\lambda_1^*,\ \lambda_2^*,\ \mu_1^*,\ \mu_2^*,\ \nu_1^*,\ \eta^*) \geq 0, \\ (\lambda_1^*,\ \lambda_2^*) \neq (0,\ 0),\ \lambda_1^* + \lambda_2^* = 1 \\ (\hat{x},\ \hat{y}) \in \mathbb{R}^2. \end{cases}$$

We claim that $v = (-1, 0)$ *is a feasible point of* (D). *Indeed, for*
$\pi^* = \left(\frac{1}{4},\ \frac{3}{4},\ \frac{1}{4},\ \frac{1}{2},\ \frac{1}{4},\ 1\right)$*, one get*

$$\frac{1}{4}(-2,-1) + \frac{3}{4}(1,1) + \frac{1}{4}(-1,-1) + \frac{1}{2}(0,-1) + \frac{1}{4}(0,1) + 1(0,0) = (0,0).$$

Which implies

$$\begin{aligned} 0 \in\ & \lambda_1^* \partial^* F_1(-1,0) + \lambda_2^* \partial^* F_2(-1,0) + \mu_1^* \partial^* G_1(-1,0) \\ & + \mu_2^* \partial^* G_2(-1,0) + \nu_1^* \partial^* g_1(-1,0) + \eta^* \partial^* \Psi(-1,0). \end{aligned}$$

Here, the sets $\partial^* F_1(-1,0) = \{(-2,-1)\}$ *and* $\partial^* F_2(-1,0) = \{(1,1)\}$ *are bounded upper semi-regular convexifactors of* F_1 *and* F_2 *at* $(-1,0)$ *and the sets*
$\partial^* G_1(-1,0) = \{(-1,-2)\}$, $\partial^* G_2(-1,0) = \{(0,-1)\}$, $\partial^* g_1(-1,0) = \{(0,1)\}$ *and* $\partial^* \Psi(-1,0) = \{(0,0)\}$ *are upper convexifactors of* $G_1,\ G_2,\ g_1$ *and* Ψ *at* $(-1,0)$. *Moreover,* $F_1,\ F_2,\ G_1,\ G_2,\ g_1$ *and* Ψ *are* ∂^**-convex at* $(-1,0)$.
Since $E = \mathbb{R}^+ \times \{0\}$, *for any feasible solution* $(x,\ y) \in E$ *of* (Q) *and any feasible solution* $(\hat{x},\ \hat{y},\ \pi^*) \in \tilde{E}$ *of* (D), *we have*

$$\phi(\hat{x},\ \hat{y},\ \pi^*) - F(x,\ y) \notin \mathbb{R}^2_+/\{0\}.$$

Hence, the Theorem 4.2 holds for (P) *and* (D).

5. Conclusion

In this paper, we have formulated a Wolfe type dual for a multiobjective bilevel programming problem. Using the single level programming problem we obtained weak and strong duality assertions under ∂^*-convexity assumption. Some examples are constructed to demonstrate the applicability of results. For future research, same approach could be used to obtain duality results for nonsmooth bilevel optimization problems, where both the lower and the upper level problems are vector optimization problems.

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