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# A study on nearly recurrent generalized $(k, \mu)-$space forms

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**Abstract.** The object of the present paper is to study the notion of three dimensional locally nearly recurrent generalized $(k, \mu)-$space forms and nearly quasi-concircular $\phi-$recurrent generalized $(k, \mu)-$ space form $M(f_1, f_2, f_3, f_4, f_5, f_6)$.

## 1. Introduction

A generalized Sasakian space form was defined by Alegre et.al in 2004 [1] as an almost contact metric manifold $(M, \phi, \xi, \eta, g)$ whose curvature tensor $R$ is given by

$$R = f_1R_1 + f_2R_2 + f_3R_3, \tag{1}$$

where $f_1, f_2, f_3$ are some differentiable function on $M$ and

$$\left.\begin{array}{rl} R_1(X,Y)Z = & g(Y,Z)X - g(X,Z)Y, \\ R_2(X,Y)Z = & g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z, \\ R_3(X,Y)Z = & \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ & -g(Y,Z)\eta(X)\xi, \end{array}\right\} \tag{2}$$

for all $X, Y, Z \in TM$. The geometry of generalized Sasakian space forms have been developed by several authors as Shah [33], Hui and Chakraborty [14], Singh and Kishor [31] and many others.

Recently in 2013, Carriazo et.al [9] introduced a generalized $(k, \mu)-$space form as an almost metric manifold $(M, \phi, \xi, \eta, g)$ whose curvature tensor $R$ is given by

$$R = f_1R_1 + f_2R_2 + f_3R_3 + f_4R_4 + f_5R_5 + f_6R_6, \tag{3}$$

where $f_1, f_2, f_3, f_4, f_5, f_6$ are some differentiable function on $M$ and $R_1, R_{,2}, R_3$ are tensor defined by the equation (2) and $R_4, R_5$ and $R_6$ defined by the equation (4)

$$\left.\begin{array}{r} R_4(X,Y)Z = g(hY,Z)X - g(hX,Z)Y + g(\phi hX,Z)\phi hY - g(\phi hY,Z)\phi hX, \\ R_5(X,Y)Z = g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y, \\ R_6(X,Y)Z = \eta(X)\eta(Z)hY - \eta(Y)\eta(Z)hX + g(hX,Z)\eta(Y)\xi - g(hY,Z)\eta(X)\xi, \end{array}\right\} \tag{4}$$

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for all $X, Y, Z \in TM$ where $2h = L_\xi \phi$ and $L$ is the usual Lie derivative. This manifold was denoted by $M(f_1, f_2, f_3, f_4, f_5, f_6)$. Further such manifold have been studied by many authors such as Prakasha et.al [18], Premalatha and Nagaraja [23], Shanmukha et.al [27], Huchchappa et.al [13], Hui et.al [15], Biswas [6] . Very recently, Singh and Khatri [28] studied some interesting results on generalized $(k, \mu)$-space form such as a non-Sasakian generalized $(k, \mu)-$space form $(M^{2n+1}, g)$, is $\xi - Q-$flat if and only if $f_1 - f_3 = \frac{\nu}{2n}$ and $f_4 = f_6$, Angadi et. al [3] proved that in a weakly symmetric generalized $(k, \mu)-$space form with respect to semi-symmetric metric connection $\overline{\nabla}$, the sum of the 1-form $C_1 + C_3 + C_4$ vanishes .

Let $(M^n, g)$, $n > 3$ be a connected Riemannian manifold of $C^\infty$ and $D$ be its Riemannian connection. The quasi-concircular curvature tensor Prasad and Mourya [20] of $(M^{2n+1}, g)$ are defined as

$$\widetilde{V}(X, Y)Z = aR(X, Y)Z + \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)[g(Y, Z)X - g(X, Z)Y], \tag{5}$$

where $r$ is the scalar curvature tensor and if $a = 1$, $b = -\frac{1}{n-1}$, then the quasi-concircular curvature tensor $\widetilde{V}$ reduces to concircular curvature tensor Yano and Kon [35]. Quasi-concircular curvature tensor was studied by many authors such as Narain, Prakash and Prasad [10], Prasad and Yadav [22], Ahmad, Haseeb and Jun [2] etc.

Recurrence spaces have been of great importance and were studied by a large number of authors such as Ruse [25], Paatterson [24], Walker [34], Singh and Khan [29], [30] and Baishya and Chowdhury [8] etc.

In De and Guha [11] introduced and studied generalized recurrent manifold whose curvature tensor $R(X, Y)Z$ of type (1,3) satisfies the condition:

$$(D_U R)(X, Y)Z = A(U)R(X, Y)Z + B(U)[g(Y, Z)X - g(X, Z)Y], \tag{6}$$

where $A$ and $B$ are two non-zero 1-forms and $D$ denotes the operator of covariant differentiation with respect to metric tensor $g$. Such a space has been denoted by $GK_n$. In recent papers Bandyopadhyay, [7], Shaikh and Patra [32], Prakasha and Yildiz [19], Khan [17] and several authors explored various geometrical properties by using generalized recurrent manifold on Sasakain manifold and Lorentzian $\alpha$-Sasakian manifold.

Recently Prasad and Yadav [21] introduced a new type of non-flat recurrent Riemannian manifold whose curvature tensor $R(Y, Z)W$ of type (1,3) satisfies the condition

$$(D_X R)(Y, Z)W = [A(X) + B(X)]R(Y, Z)W + B(X)[g(Z, W)Y - g(Y, W)Z], \tag{7}$$

for all $X, Y, Z, W \in TM$ where $A$ and $B$ are two non-zero 1-forms and $\rho_1$ and $\rho_2$ are two vector fields such that

$$g(X, \rho_1) = A(X) \; and \; g(X, \rho_2) = B(X). \tag{8}$$

Such a manifold is called a nearly recurrent manifold and 1-forms $A$ and $B$ shall be called its associated 1-forms and n-dimensional recurrent manifold of this kind were denoted by them as $(NR)_n$.

If in particular $B = 0$ in (7), then the space reduced to a recurrent space according to Ruse [26] and Walker [34] which was denoted by $K_n$.

Moreover, in particular if $A = B = 0$ then (7) becomes $(D_X R)(Y, Z)W = 0$. That is , a Riemannian manifold is symmetric according to Kobayashi and Nomizu [16] and Desai and Amur [12]. The name nearly recurrent Riemannian manifold was chosen because if $B = 0$ in (7) then the manifold reduces to a recurrent manifold which is very close to recurrent space. This justifies the name " Nearly recurrent Riemannian manifold " for the manifold defined by (7) and the use of the symbol $(NR)_n$ for it.

## 2. Preliminaries

A (2n+1) dimensional Riemannian manifold $(M^{2n+1}, g)$ is said to be an almost contact metric manifold if admits a tensor $\phi$ of type (1,1), $\xi$ is a vector fields of type (0,1) and 1-form $\eta$ is a tensor of the type (1,0) satisfying (Blair, [4], [5])

$$\phi^2 = -I + \eta \otimes \xi, \; or \; \phi^2 X = -X + \eta(X)\xi, \; \eta(\xi) = 1, \; \phi\xi = 0, \; \eta(\phi X) = 0, \tag{9}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{10}$$

$$g(X, \phi Y) = -g(\phi X, Y), \;\; g(X, \xi) = \eta(X). \tag{11}$$

Such a manifold is said to be a contact metric manifold if

$$d\eta(X, Y) = F(X, Y),$$

where $F(X, Y) = g(X, \phi Y)$ is the fundamental 2-form of $M^{2n+1}$.

It is know that on a contact metric manifold $(M, \phi, \xi, \eta, g)$, the tensor $h$ is defined by $2h = L_\xi \phi$ which is symmetric and satisfies the following relations:

$$h\xi = 0, \;\; h\phi = -\phi h, \;\; trh = tr(\phi h) = 0, \;\; \eta(hX) = 0, \tag{12}$$

$$D_X \xi = -\phi X - \phi h X, \;\; (D_X \eta)(Y) = g(X + hX, \phi Y), \tag{13}$$

In a (2n+1) dimensional $(k, \mu)$ contact metric manifold, we have [5]

$$h^2 = (k - 1)\phi^2, \;\; k \leq 1, \tag{14}$$

$$(D_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{15}$$

$$(D_X h)(Y) = [(1 - k)g(X, \phi Y) + g(X, h\phi Y)]\xi + \eta(Y)h(\phi X + \phi h X) - \mu\eta(X)\phi h Y. \tag{16}$$

In a (2n+1)-dimensional $(k, \mu)$-space form, the following relation holds [23]

$$R(X, Y)\xi = (f_1 - f_3)[\eta(Y)X - \eta(X)Y] + (f_4 - f_6)[\eta(Y)hX - \eta(X)hY], \tag{17}$$

$$\eta(R(X, Y)Z) = (f_1 - f_3)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)] + (f_4 - f_6)[g(hY, Z)\eta(X) - g(hX, Z)\eta(Y)], \tag{18}$$

$$QX = (2nf_1 + 3f_2 - f_3)X + [(2n - 1)f_4 - f_6)]hX - [3f_2 + (2n - 1)f_3]\eta(X)\xi, \tag{19}$$

$$S(X, Y) = (2nf_1 + 3f_2 - f_3)g(X, Y) + [(2n - 1)f_4 - f_6)]g(hX, Y) - [3f_2 + (2n - 1)f_3]\eta(X)\eta(Y), \tag{20}$$

$$S(X, \xi) = 2n(f_1 - f_3)\eta(X), \tag{21}$$

$$r = 2n[(2n + 1)f_1 + 3f_2 - 2f_3], \tag{22}$$

$$S(\phi X, \phi Y) = S(X, Y) - 2n(f_1 - f_3)\eta(X)\eta(Y), \tag{23}$$

where $Q$ is the Ricci operator defined by $g(QX, Y) = S(X, Y)$, $S$ is the Ricci tensor and $r$ is the scalar curvature of $M(f_1, f_2, f_3, f_4, f_5, f_6)$.

A generalized $(k, \mu)-$ space form $M(f_1, f_2, f_3, f_4, f_5, f_6)$ is said to be Einstein manifold if its Ricci tensor $S$ of the form

$$S(X, Y) = \lambda g(X, Y), \tag{24}$$

where $\lambda$ is constant.

The present paper is organized as follows: After introduction and preliminaries in section 3, we obtained three dimensional nearly $\phi-$ recurrent generalized $(k, \mu)$ space form with $Q\phi = \phi Q$ is constant curvature. Finally, in last section, we shows that nearly quasi-concircular $\phi$-recurrent generalized $(k, \mu)-$ space form $M(f_1, f_2, f_3, f_4, f_5, f_6)$ is an Einstein manifold provided $f_1 \neq f_3, k \neq 0$.

## 3. Three dimensional locally nearly recurrent generalized $(k,\mu)-$ space form $M(f_1, f_2, f_3, f_4, f_5, f_6)$

**Definition** A generalized $(k,\mu)-$ space form $M(f_1, f_2, f_3, f_4, f_5, f_6)$ is called nearly $\phi-$ recurrent [21] if

$$(\phi^2 D_W R)(X,Y)Z = [A(W)+B(W)]R(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y], \tag{25}$$

for all $X, Y, Z, W \in TM$ where $A$ and $B$ are two non-zero 1-forms.

$$g(X,\rho_1) = A(X) \;\; and \;\; g(X,\rho_2) = B(X).$$

In three dimensional Riemannian manifold $(M^n, g)$, we have

$$\begin{aligned} R(X,Y)Z =& g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ &+ \frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{26}$$

where $Q$ is the Ricci operator that in $S(X,Y) = g(QX,Y)$ and $r$ is the scalar.

Now putting $Z = \xi$ in (26) and using (9), (16)and (21), we get

$$\begin{aligned} &(f_1 - f_3)[\eta(Y)X - \eta(X)Y] + (f_4 - f_6)[\eta(Y)hX - \eta(X)hY] = \eta(Y)QX - \\ &\eta(X)QY + 2(f_1 - f_3)[\eta(Y)X - \eta(X)Y] + \frac{r}{2}[\eta(Y)X - \eta(X)Y], \end{aligned}$$

which implies

$$\begin{aligned} &\left[(f_1 - f_3) - \frac{r}{2}\right][\eta(Y)X - \eta(X)Y] + \\ &(f_4 - f_6)[\eta(Y)hX - \eta(X)hY] + \eta(Y)QX - \eta(X)QY = 0. \end{aligned} \tag{27}$$

Putting $\xi$ for $Y$ in (27) and using (19), we get

$$QX = \left[\frac{r}{2} - (f_1 - f_3)\right][X - \eta(X)\xi] + 2(f_1 - f_3)\eta(X)\xi + (f_4 - f_6)hX. \tag{28}$$

Therefore, it follows from (28), that

$$\begin{aligned} S(X,Y) =& \left[\frac{r}{2} - (f_1 - f_3)\right][g(X,Y) - \eta(X)\eta(Y)] + \\ & 2(f_1 - f_3)\eta(X)\eta(Y) + (f_4 - f_6)g(hX,Y). \end{aligned} \tag{29}$$

Using (28), (29) in (26), we have

$$\begin{aligned} R(X,Y)Z =& g(Y,Z)\left[\left\{\frac{r}{2} - (f_1 - f_3)\right\}\{X - \eta(X)\xi\} + 2(f_1 - f_3)\eta(X)\xi + (f_4 - f_6)hX\right] \\ &- g(X,Z)\left[\left\{\frac{r}{2} - (f_1 - f_3)\right\}\{Y - \eta(Y)\xi\} + 2(f_1 - f_3)\eta(Y)\xi + (f_4 - f_6)hY\right] + \\ &\left[\left\{\frac{r}{2} - (f_1 - f_3)\right\}\{g(Y,Z) - \eta(Y)\eta(Z)\} + 2(f_1 - f_3)\eta(Y)\eta(Z) + (f_4 - f_6)g(hY,Z)\right]X \\ &- \left[\left\{\frac{r}{2} - (f_1 - f_3)\right\}\{g(X,Z) - \eta(X)\eta(Z)\} + 2(f_1 - f_3)\eta(X)\eta(Z) + (f_4 - f_6)g(hX,Z)\right]Y \\ &+ \frac{r}{2}[g(X,Z)Y - g(Y,Z)X]. \end{aligned} \tag{30}$$

From (22) and (30), we get

$$\begin{aligned} R(X,Y)Z =& (f_1 + 3f_2)[g(Y,Z)X - g(X,Z)Y] + (3f_2 + f_3). \\ & [\{g(X,Z)\eta(Y) - g(Y,Z)\eta(X)\}\xi + \{\eta(X)Y - \eta(Y)X\}\eta(Z)] \\ & + (f_4 - f_6)[g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y]. \end{aligned} \tag{31}$$

Taking covariant derivative on both side of equation (31), we have

$$\begin{aligned}(D_WR)(X,Y)Z =& d(f_1+3f_2)(W)[g(Y,Z)X-g(X,Z)Y]+d(3f_2+f_3)(W)[\{g(X,Z)\eta(Y)-g(Y,Z)\eta(X)\}\xi\\ &+\{\eta(X)Y-\eta(Y)X\}\eta(Z)]++(3f_2+f_3)[\{g(X,Z)(D_W\eta)(Y)-g(Y,Z)(D_W\eta)(X)\}\xi+\\ &\{g(X,Z)\eta(Y)-g(Y,Z)\eta(X)\}(D_W\xi)+\{(D_W\eta)(X)Y-(D_W\eta)(Y)X\}\eta(Z)+\\ &\{\eta(X)Y-\eta(Y)X\}(D_W\eta)(Z)]+d(f_4-f_6)(W)[g(Y,Z)hX-g(X,Z)hY+\\ &g(hY,Z)X-g(hX,Z)Y]+(f_4-f_6)[g(Y,Z)(D_Wh)X-g(X,Z)(D_Wh)Y\\ &+g((D_Wh)Y,Z)X-g((D_Wh)X,Z)Y].\end{aligned} \tag{32}$$

From (12), (13), (14) and (32), we have

$$\begin{aligned}(D_WR)(X,Y)Z =& d(f_1+3f_2)(W)[g(Y,Z)X-g(X,Z)Y]+d(3f_2+f_3)(W)[\{g(X,Z)\eta(Y)-g(Y,Z)\eta(X)\}\xi\\ &+\{\eta(X)Y-\eta(Y)X\}\eta(Z)]+(3f_2+f_3)[\Big\{g(X,Z)g(W+hW,\phi Y)-g(Y,Z)g(W+hW,\phi X)\Big\}\xi\\ &+\{g(Y,Z)\eta(X)-g(X,Z)\eta(Y)\}(\phi W+\phi hW)+\Big\{g(W+hW,\phi X)Y-g(W+hW,\phi Y)X\Big\}\eta(Z)+\\ &\{\eta(X)Y-\eta(Y)X\}g(W+hW,\phi Z)]+d(f_4-f_6)(W)[g(Y,Z)hX-g(X,Z)hY+g(hY,Z)X-g(hX,Z)Y]\\ &+(f_4-f_6)[\Big\{(1-k)g(W,\phi X)+g(W,h\phi X)\Big\}g(Y,Z)\xi-\Big\{(1-k)g(W,\phi Y)+g(W,h\phi Y)\Big\}g(X,Z)\xi\\ &-\Big\{(1-k)g(W,\phi X)+g(W,h\phi X)\Big\}\eta(Z)Y+\Big\{(1-k)g(W,\phi Y)+g(W,h\phi Y)\Big\}\eta(Z)X+\\ &\{g(X,Z)\eta(Y)-g(Y,Z)\eta(X)\}\Big\{(1-k)\phi W+\phi hW\Big\}\\ &-\mu\Big\{g(Y,Z)\phi hX-g(X,Z)\phi hY+g(\phi hY,Z)X-g(\phi hX,Z)Y\Big\}\eta(W)\\ &+\{\eta(Y)X-\eta(X)Y\}\Big\{(k-1)g(\phi W,Z)+g(hW,\phi Z)\Big\}].\end{aligned} \tag{33}$$

Applying $\phi^2$ on both side of (33) and using (9), we get

$$\begin{aligned}\phi^2((D_WR)(X,Y)Z) =& d(f_1+3f_2)(W)[g(Y,Z)\phi^2X-g(X,Z)\phi^2Y]+d(3f_2+f_3)(W).\\ &[\Big\{\eta(X)\phi^2Y-\eta(Y)\phi^2X\Big\}\eta(Z)]+\\ &(3f_2+f_3)[-\{g(X,Z)\eta(Y)-g(Y,Z)\eta(X)\}(\phi^3W+\phi^3hW)+\\ &\Big\{g(W+hW,\phi X)\phi^2Y-g(W+hW,\phi Y)\phi^2X\Big\}\eta(Z)+\\ &\Big\{\eta(X)\phi^2Y-\eta(Y)\phi^2X\Big\}g(W+hW,\phi Z)]+d(f_4-f_6)(W).\\ &[g(Y,Z)\phi^2hX-g(X,Z)\phi^2hY+g(hY,Z)\phi^2X-g(hX,Z)\phi^2Y]\\ &+(f_4-f_6)[-\Big\{(1-k)g(W,\phi X)+g(W,h\phi X)\Big\}\eta(Z)\phi^2Y+\\ &\Big\{(1-k)g(W,\phi Y)+g(W,h\phi Y)\Big\}\eta(Z)\phi^2X+\\ &\{g(X,Z)\eta(Y)-g(Y,Z)\eta(X)\}\Big\{(1-k)\phi^3W+\phi^3hW\Big\}-\\ &\mu\Big\{g(Y,Z)\phi^3hX-g(X,Z)\phi^3hY+g(\phi hY,Z)\phi^2X-g(\phi hX,Z)\phi^2Y\Big\}\eta(W)\\ &+\Big\{\eta(Y)\phi^2X-\eta(X)\phi^2Y\Big\}g(h(\phi W+\phi hW),Z)].\end{aligned} \tag{34}$$

If all vector field $X, Y, Z, W$ are orthogonal to $\xi$, then eq$^n$ (34) will be reduces to

$$\begin{aligned}\phi^2((D_WR)(X,Y)Z) =& -d(f_1+3f_2)(W)[g(Y,Z)X-g(X,Z)Y]\\ &-d(f_4-f_6)(W)[g(Y,Z)hX-g(X,Z)hY\\ &+g(hY,Z)X-g(hX,Z)Y].\end{aligned} \tag{35}$$

From (25) and (5.10), we get

$$R(X,Y)Z = -\left[\frac{B(W) + d(f_1 + 3f_2)(W)}{A(W) + B(W)}\right][g(Y,Z)X - g(X,Z)Y] - \left[\frac{d(f_4 - f_6)(W)}{A(W) + B(W)}\right][g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y]. \tag{36}$$

Putting $W = e_i, i = 1,2,3$ is an orthonormal basis of the tangent space at any point of the manifold and taking summation over $i, 1 \leq i \leq 3$, we have

$$R(X,Y)Z = -\left[\frac{B(e_i) + d(f_1 + 3f_2)(e_i)}{A(e_i) + B(e_i)}\right][g(Y,Z)X - g(X,Z)Y] - \left[\frac{d(f_4 - f_6)(e_i)}{A(e_i) + B(e_i)}\right][g(Y,Z)hX - g(X,Z)hY + g(hY,Z)X - g(hX,Z)Y]. \tag{37}$$

If $M^3(f_1, f_2, ..., f_6)$ is a contact metric generalized $(k,\mu)-$ space form then $Q\phi = \phi Q$ is true if and only if $f_4 - f_6 = 0$ [9] where $Q$ denotes the Ricci operator on $M$.
Thus eq$^n$ (37) will be

$$R(X,Y)Z = -\left[\frac{B(e_i) + d(f_1 + 3f_2)(e_i)}{A(e_i) + B(e_i)}\right][g(Y,Z)X - g(X,Z)Y],$$

which implies that

$$R(X,Y)Z = \lambda[g(Y,Z)X - g(X,Z)Y],$$

where

$$\lambda = -\left[\frac{B(e_i) + d(f_1 + 3f_2)(e_i)}{A(e_i) + B(e_i)}\right],$$

where $A$ and $B$ are non-zero 1-forms.
Hence we have the following theorem:

**Theorem 3.1.** *A three dimensional locally nearly $\phi-$recurrent contact metric generalized $(k,\mu)-$ space form $M(f_1, f_2, f_3, f_4, f_5, f_6)$ with $Q\phi = \phi Q$ is a constant curvature.*

## 4. Nearly quasi-concircular $\phi-$recurrent generalized $(k,\mu)-$ space form $M(f_1, f_2, f_3, f_4, f_5, f_6)$

**Definition 4.1** A Riemannian manifold is called nearly quasi-concircular $\phi-$recurrent if its quasi-concircular curvature tensor $\widetilde{V}$ [20] satisfying the following condition:

$$\phi^2\left((D_W\widetilde{V})(X,Y)Z\right) = [A(W) + B(W)]\widetilde{V}(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y]. \tag{38}$$

From (9) and (38), we get

$$-(D_W\widetilde{V})(X,Y)Z + \eta\left((D_W\widetilde{V})(X,Y)Z\right)\xi = [A(W) + B(W)]\widetilde{V}(X,Y)Z + B(W)[g(Y,Z)X - g(X,Z)Y]. \tag{39}$$

Taking covariant derivative of (5), we get

$$(D_W\widetilde{V})(X,Y)Z = a(D_WR)(X,Y)Z + \frac{Wr}{2n+1}\left(\frac{a}{2n}+2b\right)[g(Y,Z)X - g(X,Z)Y], \tag{40}$$

From (39) and (40), we get

$$\begin{aligned}&a\left[-(D_WR)(X,Y)Z + \eta\left((D_WR)(X,Y)Z\right)\xi\right] - \frac{Wr}{2n+1}\left(\frac{a}{2n}+2b\right).\\&[g(Y,Z)X - g(X,Z)Y - g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi] = [A(W)+B(W)].\\&\left[aR(X,Y)Z + \frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\{g(Y,Z)X - g(X,Z)Y\}\right] +\\&B(W)[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{41}$$

From (18) and (41), we get

$$\begin{aligned}&a[-(D_WR)(X,Y)Z + (f_1 - f_3)\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}\xi -\\&(f_4 - f_6)\{g(hY,Z)\eta(X) - g(hX,Z)\eta(Y)\}\xi] - \frac{Wr}{2n+1}\left(\frac{a}{2n}+2b\right).\\&[g(Y,Z)X - g(X,Z)Y - g(Y,Z)\eta(X)\xi + g(X,Z)\eta(Y)\xi] = [A(W)+B(W)].\\&\left[aR(X,Y)Z + \frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\{g(Y,Z)X - g(X,Z)Y\}\right] +\\&B(W)[g(Y,Z)X - g(X,Z)Y].\end{aligned} \tag{42}$$

Contracting (42) w.r.t. $X$, we have

$$\begin{aligned}&a[-(D_WS)(Y,Z) + (f_1 - f_3)\{g(Y,Z) - \eta(Y)\eta(Z)\} - (f_4 - f_6)g(hY,Z)]\\&- \frac{Wr}{2n+1}\left(\frac{a}{2n}+2b\right)[(2n-1)g(Y,Z) + \eta(Y)\eta(Z)] = [A(W)+B(W)].\\&\left[aS(Y,Z) + \frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)2ng(Y,Z)\right] + 2nB(W)g(Y,Z).\end{aligned} \tag{43}$$

Putting $Z = \xi$ in (43) and using (9), (12) and (22), we get

$$\begin{aligned}&- a(D_WS)(Y,\xi) - \left(\frac{4n^2}{2n+1}\right)d\{(2n+1)f_1 + 3f_2 - 2f_3\}(W)\left[\frac{a}{2n}+2b\right]\eta(Y) =\\&[A(W)+B(W)]\left[aS(Y,\xi) + \frac{2nr}{2n+1}\left(\frac{a}{2n}+2b\right)\eta(Y)\right] + 2nB(W)\eta(Y).\end{aligned} \tag{44}$$

Now we have

$$(D_WS)(Y,\xi) = D_WS(Y,\xi) - S(D_WY,\xi) - S(Y,D_W\xi).$$

Using (13) and (21) in the above relation, it follows that

$$\begin{aligned}(D_WS)(Y,\xi) =\,&2nd(f_1 - f_3)(W)\eta(Y) + 2n(f_1 - f_3)[g(W,\phi Y) - g(hW,\phi Y)]\\&+ S(Y,\phi W) + S(Y,\phi hW).\end{aligned} \tag{45}$$

From (44) and (45), we get

$$\begin{aligned}&- a[2nd(f_1 - f_3)(W)\eta(Y) + 2n(f_1 - f_3)[g(W,\phi Y) - g(hW,\phi Y)] + S(Y,\phi W)+\\&S(Y,\phi hW)] - \left(\frac{4n^2}{2n+1}\right)d\{(2n+1)f_1 + 3f_2 - 2f_3\}(W)\left[\frac{a}{2n}+2b\right]\eta(Y) =\\&[A(W)+B(W)]\left[aS(Y,\xi) + \frac{2nr}{2n+1}\left(\frac{a}{2n}+2b\right)\eta(Y)\right] + 2nB(W)\eta(Y).\end{aligned} \tag{46}$$

Substituting $Y$ by $\phi Y$ in (46) and using (9) and (14), we get

$$-a.2n(f_1 - f_3)\left[g\left(W, \phi^2 Y\right) + g\left(hW, \phi^2 Y\right)\right] + S(\phi Y, \phi W) + S(\phi Y, \phi hW) = 0,$$

which implies that

$$S(Y, W) + S(Y, hW) = -2na(f_1 - f_3)[g(Y, W) + g(Y, hW)]. \tag{47}$$

Again replacing $W$ by $hW$ in (47) and then using (9) and (18), we get

$$\begin{aligned} S(Y, hW) + S(Y, hW) - kS(Y, W) = &- 2na(f_1 - f_3)[g(Y, hW)+ \\ &g(Y, W) - kg(Y, W)]. \end{aligned} \tag{48}$$

Subtracting (47) and (48), we get

$$k[S(Y, W) + a.2n(f_1 - f_3)g(Y, W)] = 0, \tag{49}$$

either $k = 0$ or $S(Y, W) + a.2n(f_1 - f_3)g(Y, W) = 0$.
When $k \neq 0$ then $S(Y, W) = -a.2n(f_1 - f_3)g(Y, W)$. If $-a.2n(f_1 - f_3) = \lambda_1(constant)$, then

$$S(Y, W) = \lambda_1 g(Y, W).$$

In (49), when $k = 0$, then (47) will be

$$S(Y, W) + S(Y, hW) = -2na(f_1 - f_3)[g(Y, W) + g(Y, hW)]. \tag{50}$$

Hence we have the following theorem:

**Theorem 4.1.** *In a nearly quasi-concircular $\phi-$recurrent generalized $(k, \mu)-$ space form $M(f_1, f_2, f_3, f_4, f_5, f_6)$, $f_1 \neq f_3$ holds, then*

*(i) the manifold M is Einstein manifold if $k \neq 0$,*

*(ii) the Ricc tensor are related by (50) if $k = 0$.*

**Corollary 4.2.** *A nearly quasi-concircular $\phi-$recurrent generalized $(k, \mu)-$ space form $M(f_1, f_2, f_3, f_4, f_5, f_6)$ is nearly recurrent generalized $(k, \mu)-$ space form $M(f_1, f_2, f_3, f_4, f_5, f_6)$ if $a = 1$.*

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