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Non-linear mixed Jordan bi-skew Lie triple derivations on $*$-algebras

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Abstract. In this article, we investigate the behaviour of a non-linear map Θ on $*$-algebra $\mathfrak{A}$, which satisfies $\Theta([X \circ Y, Z]_\bullet) = [\Theta(X) \circ Y, Z]_\bullet + [X \circ \Theta(Y), Z]_\bullet + [X \circ Y, \Theta(Z)]_\bullet$, where $X \circ Y = XY + YX$ and $[X, Y]_\bullet = XY^* - YX^*$ (namely, Jordan and bi-skew Lie product, respectively), for all $X, Y, Z \in \mathfrak{A}$. Furthermore, we apply the above mentioned result to several distinct algebras.

1. Introduction

Let $\mathfrak{A}$ be an associative $*$-algebra over the field of complex numbers $\mathbb{C}$. The products defined by $X \circ Y = XY + YX$, $X * Y = XY + YX^*$, $[X, Y] = XY - YX$ and $[X, Y]_* = XY - YX^*$ are called Jordan product, $*$-Jordan product, Lie product and $*$-Lie product of $X, Y \in \mathfrak{A}$ respectively. In recent years, several authors investigated the structure of derivations concerning these products see ([6, 9, 10, 14, 18, 19, 23]). A linear map $\Theta : \mathfrak{A} \to \mathfrak{A}$ is called a derivation if $\Theta(XY) = \Theta(X)Y + X\Theta(Y)$ for all $X, Y \in \mathfrak{A}$. Further, if Θ satisfies $\Theta(X^*) = \Theta(X)^*$ for all $X \in \mathfrak{A}$, then Θ is called a $*$-derivation. Obviously, every $*$-derivation is a derivation. Without assuming the linearity assumption if a map $\Theta : \mathfrak{A} \to \mathfrak{A}$ satisfies

$$\Theta(X \circ Y) = \Theta(X) \circ Y + X \circ \Theta(Y)$$

or

$$\Theta(X \circ Y \circ Z) = \Theta(X) \circ Y \circ Z + X \circ \Theta(Y) \circ Z + X \circ Y \circ \Theta(Z)$$

for all $X, Y, Z \in \mathfrak{A}$, then Θ is called a non-linear Jordan derivation or a non-linear Jordan triple derivation respectively. By considering Lie (or Lie triple) product, a non-linear Lie (or Lie triple) derivation is defined analogously. Very recently, Kong and Zhang [13] introduced a new product $[X, Y]_\bullet = XY^* - YX^*$, called as bi-skew Lie product of $X, Y \in \mathfrak{A}$ and they proved that every non-linear bi-skew Lie derivation on a factor von Neumann algebra $\mathfrak{A}$ (with $dim(\mathfrak{A}) \geq 2$) is an additive $*$-derivation. The third author further extended this result to multiplicative bi-skew Lie triple derivation [1]. Recall that a map $\Theta : \mathfrak{A} \to \mathfrak{A}$ is called a non-linear bi-skew Lie or Lie triple derivation if Θ satisfies

$$\Theta([X, Y]_\bullet) = [\Theta(X), Y]_\bullet + [X, \Theta(Y)]_\bullet$$

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or

$$\Theta([[X,Y]_\bullet,Z]_\bullet) = [[\Theta(X),Y]_\bullet,Z]_\bullet + [[X,\Theta(Y)]_\bullet,Z]_\bullet + [[X,Y]_\bullet,\Theta(Z)]_\bullet$$

for all $X,Y,Z \in \mathfrak{A}$. Researchers have been studying the additivity or characterization of maps preserving various kinds of products. Many mathematicians studied alternative rings as a more general class of rings that preserve these products (see [3, 4, 11, 16]).

In recent years, several scholars considered mixed products constituting Jordan ($*$-Jordan) and Lie ($*$-Lie) products and characterize the structure of derivations preserving these products ([5, 8, 12, 20–22]). For instance, in [22] Zhou et.al. proved that every non-linear mixed Lie triple derivation on prime $*$-algebras, is an additive $*$-derivation. In [5], Li and Zhang investigated the structure of non-linear mixed Jordan triple $*$-derivation on $*$-algebras. Ferreira and Wei [2] proved that every mixed $*$-Jordan (i.e., the mixed product of $X \circ Y = XY + YX$ and $X \bullet Y = X^*Y + Y^*X$) type derivation on a $*$-algebra, is an additive $*$-derivation. The authors of [17] studied non-linear mixed $*$-Jordan type derivations preserving the mixed product of $X \circ Y = XY + YX^*$ and $X \bullet Y = XY - YX^*$, on alternative $*$-algebras and proved that they are additive $*$-derivations. Let $\mathfrak{A}$ be a $*$-algebra. Consider a map (not necessarily linear) $\Theta : \mathfrak{A} \to \mathfrak{A}$ satisfying

$$\Theta([X \circ Y, Z]_\bullet) = [\Theta(X) \circ Y, Z]_\bullet + [X \circ \Theta(Y), Z]_\bullet + [X \circ Y, \Theta(Z)]_\bullet$$

for all $X,Y,Z \in \mathfrak{A}$, then Θ is called a non-linear mixed Jordan bi-skew Lie triple derivation on $\mathfrak{A}$.

Motivated by the aforementioned works, our primary focus will be on the mixed product constructed by Jordan and bi-skew Lie product and we try to give the description of non-linear mixed Jordan bi-skew Lie triple derivations on $*$-algebras.

2. Preliminaries and Main Result

Throughout the article unless otherwise stated, $\mathfrak{A}$ represents a $*$-algebra over $\mathbb{C}$, the field of complex numbers. Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An idempotent operator $\mathcal{P} \in \mathcal{B}(\mathcal{H})$ is called a projection if it is self-adjoint (i.e., $\mathcal{P}^2 = \mathcal{P}$ and $\mathcal{P}^* = \mathcal{P}$). Any operator $X \in \mathcal{B}(\mathcal{H})$ can be expressed as $X = RX + iImX$, where $i \in \mathbb{C}$ i.e., $i^2 = -1$, $RX = \frac{X+X^*}{2}$ and $ImX = \frac{X-X^*}{2i}$. It is evident that both RX and ImX are self-adjoint.

Denote by $\mathcal{P}_1 = \mathcal{P}$ and $\mathcal{P}_2 = \mathfrak{I} - \mathcal{P}$ be two non-trivial projections in $\mathfrak{A}$. Then our main theorem reads as follows.

Main Theorem. Let $\mathfrak{A}$ be a unital $*$-algebra containing non-trivial projections $\mathcal{P}_1, \mathcal{P}_2$ and satisfies

$$X\mathfrak{A}\mathcal{P}_k = 0 \ \ implies \ \ X = 0, k = 1,2. \tag{1}$$

Then a map $\Theta : \mathfrak{A} \to \mathfrak{A}$ satisfies

$$\Theta([X \circ Y, Z]_\bullet) = [\Theta(X) \circ Y, Z]_\bullet + [X \circ \Theta(Y), Z]_\bullet + [X \circ Y, \Theta(Z)]_\bullet$$

for all $X,Y,Z \in \mathfrak{A}$ if and only if Θ is an additive $*$-derivation.

Let $\mathcal{P}_1 = \mathcal{P}$ and $\mathcal{P}_2 = \mathfrak{I} - \mathcal{P}$ be two non-trivial projections in $\mathfrak{A}$. Write $\mathfrak{A}_{ij} = \mathcal{P}_i\mathfrak{A}\mathcal{P}_j$. Then $\mathfrak{A} = \mathfrak{A}_{11} + \mathfrak{A}_{12} + \mathfrak{A}_{21} + \mathfrak{A}_{22}$. Let $\mathcal{L} = \{X \in \mathfrak{A} | X^* = X\}$ and $\mathcal{K} = \{X \in \mathfrak{A} | X^* = -X\}$, $\mathcal{K}_{12} = \{\mathcal{P}_1K\mathcal{P}_2 + \mathcal{P}_2K\mathcal{P}_1 | K \in \mathcal{K}\}$ and $\mathcal{K}_{ii} = \mathcal{P}_i\mathcal{K}\mathcal{P}_i$ ($i = 1,2$). Thus, for every $K \in \mathcal{K}, K = K_{11} + K_{12} + K_{22}$, for every $K_{12} \in \mathcal{K}_{12}$ and $K_{ii} \in \mathcal{K}_{ii}$ ($i = 1,2$).

Only the necessity must be established. The proof of the main theorem is done by proving a series of lemmas which are as follows.

Lemma 2.1. $\Theta(0) = 0$.

Proof. By the hypothesis, we have

$$\Theta(0) = \Theta([0 \circ 0, 0]_\bullet) = [\Theta(0) \circ 0, 0]_\bullet + [0 \circ \Theta(0), 0]_\bullet + [0 \circ 0, \Theta(0)]_\bullet = 0.$$

□

Lemma 2.2. $\Theta(K)^* = -\Theta(K)$ *for every* $K \in \mathcal{K}$.

Proof. Observe, for any $K \in \mathcal{K}$ that $K = [\frac{1}{2}\mathfrak{I} \circ K, \frac{1}{2}\mathfrak{I}]_\bullet$. Thus

$$\begin{aligned} \Theta(K) &= \Theta([\frac{1}{2}\mathfrak{I} \circ K, \frac{1}{2}\mathfrak{I}]_\bullet) \\ &= [\Theta(\frac{1}{2}\mathfrak{I}) \circ K, \frac{1}{2}\mathfrak{I}]_\bullet + [\frac{1}{2}\mathfrak{I} \circ \Theta(K), \frac{1}{2}\mathfrak{I}]_\bullet \\ &+ [\frac{1}{2}\mathfrak{I} \circ K, \Theta(\frac{1}{2}\mathfrak{I})]_\bullet \\ &= \frac{3}{2}\Big(\Theta(\frac{1}{2}\mathfrak{I})K + K\Theta(\frac{1}{2}\mathfrak{I})^*\Big) + \frac{1}{2}\Big(\Theta(\frac{1}{2}\mathfrak{I})^*K + K\Theta(\frac{1}{2}\mathfrak{I})\Big) \\ &+ \frac{1}{2}\Big(\Theta(K) - \Theta(K)^*\Big). \end{aligned} \tag{2}$$

This gives

$$\begin{aligned} \Theta(K) &= 3\Big(\Theta(\frac{1}{2}\mathfrak{I})K + K\Theta(\frac{1}{2}\mathfrak{I})^*\Big) \\ &+ \Theta(\frac{1}{2}\mathfrak{I})^*K + K\Theta(\frac{1}{2}\mathfrak{I}) - \Theta(K)^*. \end{aligned} \tag{3}$$

Accordingly

$$\begin{aligned} \Theta(K)^* &= -3\Big(\Theta(\frac{1}{2}\mathfrak{I})K + K\Theta(\frac{1}{2}\mathfrak{I})^*\Big) \\ &- K\Theta(\frac{1}{2}\mathfrak{I}) - \Theta(\frac{1}{2}\mathfrak{I})^*K - \Theta(K). \end{aligned} \tag{4}$$

Adding Equations (3) and (4), we obtain $\Theta(K)^* = -\Theta(K)$. □

Lemma 2.3. *For any* $X_{11} \in \mathcal{K}_{11}, Y_{12} \in \mathcal{K}_{12}$ *and* $Z_{22} \in \mathcal{K}_{22}$*, we have*

(*i*) $\Theta(X_{11} + Y_{12}) = \Theta(X_{11}) + \Theta(Y_{12})$;

(*ii*) $\Theta(Y_{12} + Z_{22}) = \Theta(Y_{12}) + \Theta(Z_{22})$.

Proof. (*i*) Let $\Omega = \Theta(X_{11} + Y_{12}) - \Theta(X_{11}) - \Theta(Y_{12})$. It is evident from Lemma 2.2 that $\Omega \in \mathcal{K}$, i.e., $\Omega^* = -\Omega$. It is sufficient to show that $\Omega = \Omega_{11} + \Omega_{12} + \Omega_{22} = 0$. We have

$$\begin{aligned} & \Theta([\mathcal{P}_2 \circ (X_{11} + Y_{12}), \mathcal{P}_2]_\bullet) = \Theta([\mathcal{P}_2 \circ X_{11}, \mathcal{P}_2]_\bullet) + \Theta([\mathcal{P}_2 \circ Y_{12}, \mathcal{P}_2]_\bullet) \\ =\ & [\Theta(\mathcal{P}_2) \circ X_{11}, \mathcal{P}_2]_\bullet + [\mathcal{P}_2 \circ \Theta(X_{11}), \mathcal{P}_2]_\bullet \\ +\ & [\mathcal{P}_2 \circ X_{11}, \Theta(\mathcal{P}_2)]_\bullet + [\Theta(\mathcal{P}_2) \circ Y_{12}, \mathcal{P}_2]_\bullet \\ +\ & [\mathcal{P}_2 \circ \Theta(Y_{12}), \mathcal{P}_2]_\bullet + [\mathcal{P}_2 \circ Y_{12}, \Theta(\mathcal{P}_2)]_\bullet \\ =\ & [\Theta(\mathcal{P}_2) \circ (X_{11} + Y_{12}), \mathcal{P}_2]_\bullet + [\mathcal{P}_2 \circ (\Theta(X_{11}) + \Theta(Y_{12})), \mathcal{P}_2]_\bullet \\ +\ & [\mathcal{P}_2 \circ (X_{11} + Y_{12}), \Theta(\mathcal{P}_2)]_\bullet. \end{aligned}$$

Alternatively, we have

$$\begin{aligned} & \Theta([\mathcal{P}_2 \circ (X_{11} + Y_{12}), \mathcal{P}_2]_\bullet) = [\Theta(\mathcal{P}_2) \circ (X_{11} + Y_{12}), \mathcal{P}_2]_\bullet \\ +\ & [\mathcal{P}_2 \circ \Theta(X_{11} + Y_{12}), \mathcal{P}_2]_\bullet + [\mathcal{P}_2 \circ (X_{11} + Y_{12}), \Theta(\mathcal{P}_2)]_\bullet. \end{aligned}$$

From the last two relations we obtain $[\mathcal{P}_2 \circ \Omega, \mathcal{P}_2]_\bullet = 0$. This gives $\Omega_{12} = \Omega_{22} = 0$. It remains to show that $\Omega_{11} = 0$. Since $[(\mathcal{P}_2 - \mathcal{P}_1) \circ Y_{12}, \frac{1}{2}\mathfrak{I}]_\bullet = 0$, then we have

$$\begin{aligned}
& [\Theta(\mathcal{P}_2 - \mathcal{P}_1) \circ (X_{11} + Y_{12}), \frac{1}{2}\mathfrak{I}]_\bullet + [(\mathcal{P}_2 - \mathcal{P}_1) \circ \Theta(X_{11} + Y_{12}), \frac{1}{2}\mathfrak{I}]_\bullet \\
+ \quad & [(\mathcal{P}_2 - \mathcal{P}_1) \circ (X_{11} + Y_{12}), \Theta(\frac{1}{2}\mathfrak{I})]_\bullet \\
= \quad & \Theta([(\mathcal{P}_2 - \mathcal{P}_1) \circ (X_{11} + Y_{12}), \frac{1}{2}\mathfrak{I}]_\bullet) \\
= \quad & \Theta([(\mathcal{P}_2 - \mathcal{P}_1) \circ X_{11}, \frac{1}{2}\mathfrak{I}]_\bullet) + \Theta([(\mathcal{P}_2 - \mathcal{P}_1) \circ Y_{12}, \frac{1}{2}\mathfrak{I}]_\bullet) \\
= \quad & [\Theta(\mathcal{P}_2 - \mathcal{P}_1) \circ X_{11}, \frac{1}{2}\mathfrak{I}]_\bullet + [(\mathcal{P}_2 - \mathcal{P}_1) \circ \Theta(X_{11}), \frac{1}{2}\mathfrak{I}]_\bullet + [(\mathcal{P}_2 - \mathcal{P}_1) \circ X_{11}, \Theta(\frac{1}{2}\mathfrak{I})]_\bullet \\
+ \quad & [\Theta(\mathcal{P}_2 - \mathcal{P}_1) \circ Y_{12}, \frac{1}{2}\mathfrak{I}]_\bullet) + [(\mathcal{P}_2 - \mathcal{P}_1) \circ \Theta(Y_{12}), \frac{1}{2}\mathfrak{I}]_\bullet + [(\mathcal{P}_2 - \mathcal{P}_1) \circ Y_{12}, \Theta(\frac{1}{2}\mathfrak{I})]_\bullet \\
= \quad & [\Theta(\mathcal{P}_2 - \mathcal{P}_1) \circ (X_{11} + Y_{12}), \frac{1}{2}\mathfrak{I}]_\bullet + [(\mathcal{P}_2 - \mathcal{P}_1) \circ (\Theta(X_{11}) + \Theta(Y_{12})), \frac{1}{2}\mathfrak{I}]_\bullet \\
+ \quad & [(\mathcal{P}_2 - \mathcal{P}_1) \circ (X_{11} + Y_{12}), \Theta(\frac{1}{2}\mathfrak{I})]_\bullet.
\end{aligned}$$

Thus, we have $[(\mathcal{P}_2 - \mathcal{P}_1) \circ \Omega, \frac{1}{2}\mathfrak{I}]_\bullet = 0$. This together with the fact $\Omega^* = -\Omega$ gives $\Omega_{11} = 0$. Therefore $\Omega = 0$ i.e.,

$$\Theta(X_{11} + Y_{12}) = \Theta(X_{11}) + \Theta(Y_{12}).$$

Following the same procedure we can establish (*ii*). This completes the proof. □

Lemma 2.4. *For any $X_{11} \in \mathcal{K}_{11}, Y_{12} \in \mathcal{K}_{12}$ and $Z_{22} \in \mathcal{K}_{22}$, we have*

$$\Theta(X_{11} + Y_{12} + Z_{22}) = \Theta(X_{11}) + \Theta(Y_{12}) + \Theta(Z_{22}).$$

Proof. Assume that $\Omega = \Theta(X_{11} + Y_{12} + Z_{22}) - \Theta(X_{11}) - \Theta(Y_{12}) - \Theta(Z_{22})$. We will show that $\Omega = 0$. From Lemma 2.3 and $[\mathcal{P}_1 \circ Z_{22}, \mathcal{P}_1]_\bullet = 0$, we can write

$$\begin{aligned}
& \Theta([\mathcal{P}_1 \circ (X_{11} + Y_{12} + Z_{22}), \mathcal{P}_1]_\bullet) \\
= \quad & \Theta([\mathcal{P}_1 \circ (X_{11} + Y_{12}), \mathcal{P}_1]_\bullet) + \Theta([\mathcal{P}_1 \circ Z_{22}, \mathcal{P}_1]_\bullet) \\
= \quad & [\Theta(\mathcal{P}_1), (X_{11} + Y_{12}), \mathcal{P}_1]_\bullet + [\mathcal{P}_1 \circ \Theta(X_{11} + Y_{12}), \mathcal{P}_1]_\bullet \\
+ \quad & [\mathcal{P}_1 \circ (X_{11} + Y_{12}), \Theta(\mathcal{P}_1)]_\bullet + [\Theta(\mathcal{P}_1) \circ Z_{22}, \mathcal{P}_1]_\bullet \\
+ \quad & [\mathcal{P}_1 \circ \Theta(Z_{22}), \mathcal{P}_1]_\bullet + [\mathcal{P}_1 \circ Z_{22}, \Theta(\mathcal{P}_1)]_\bullet \\
= \quad & [\Theta(\mathcal{P}_1) \circ (X_{11} + Y_{12}), \mathcal{P}_1]_\bullet + [\mathcal{P}_1 \circ (\Theta(X_{11}) + \Theta(Y_{12})), \mathcal{P}_1]_\bullet \\
+ \quad & [\mathcal{P}_1 \circ (X_{11} + Y_{12}), \Theta(\mathcal{P}_1)]_\bullet + [\Theta(\mathcal{P}_1) \circ Z_{22}, \mathcal{P}_1]_\bullet \\
+ \quad & [\mathcal{P}_1 \circ \Theta(Z_{22}), \mathcal{P}_1]_\bullet + [\mathcal{P}_1 \circ Z_{22}, \Theta(\mathcal{P}_1)]_\bullet \\
= \quad & [\Theta(\mathcal{P}_1) \circ (X_{11} + Y_{12} + Z_{22}), \mathcal{P}_1]_\bullet + [\mathcal{P}_1 \circ (\Theta(X_{11}) + \Theta(Y_{12}) + \Theta(Z_{22})), \mathcal{P}_1]_\bullet \\
+ \quad & [\mathcal{P}_1 \circ (X_{11} + Y_{12} + Z_{22}), \Theta(\mathcal{P}_1)]_\bullet.
\end{aligned}$$

Also, we have

$$\begin{aligned}
& \Theta([\mathcal{P}_1 \circ (X_{11} + Y_{12} + Z_{22}), \mathcal{P}_1]_\bullet) = [\Theta(\mathcal{P}_1) \circ (X_{11} + Y_{12} + Z_{22}), \mathcal{P}_1]_\bullet \\
+ \quad & [\mathcal{P}_1 \circ \Theta(X_{11} + Y_{12} + Z_{22}), \mathcal{P}_1]_\bullet + [\mathcal{P}_1 \circ (X_{11} + Y_{12} + Z_{22}), \Theta(\mathcal{P}_1)]_\bullet.
\end{aligned}$$

It follows from the last two expressions that $[\mathcal{P}_1 \circ \Omega, \mathcal{P}_1]_\bullet = 0$. In view of Lemma 2.2 this implies $\Omega_{11} = \Omega_{12} = 0$. Next, since $[\mathcal{P}_2 \circ X_{11}, \mathcal{P}_2]_\bullet = 0$. Following the same technique as above, we can get $\Omega_{22} = 0$ and thus $\Omega = 0$. Therefore, we obtain the desired result. □

Lemma 2.5. *For any $X_{12}, Y_{12} \in \mathcal{K}_{12}$, we have*

$$\Theta(X_{12} + Y_{12}) = \Theta(X_{12}) + \Theta(Y_{12}).$$

Proof. Let $X_{12} = A_{12} - A_{12}^* \in \mathcal{K}_{12}$ and $Y_{12} = B_{12} - B_{12}^* \in \mathcal{K}_{12}$ for $A_{12}, B_{12} \in \mathfrak{A}_{12}$. So,

$$\begin{aligned} & [(i\mathcal{P}_1 + iA_{12} + iA_{12}^*) \circ \frac{1}{2}\mathfrak{I}, (i\mathcal{P}_2 + iB_{12} + iB_{12}^*)]_\bullet \\ = \;& (A_{12} - A_{12}^*) + (B_{12} - B_{12}^*) + (A_{12}B_{12}^* + A_{12}^*B_{12} - B_{12}A_{12}^* - B_{12}^*A_{12}) \\ = \;& X_{12} + Y_{12} + X_{12}Y_{12}^* - Y_{12}X_{12}^*. \end{aligned}$$

Observe that $X_{12}Y_{12}^* - Y_{12}X_{12}^* = A_{12}B_{12}^* - B_{12}A_{12}^* + A_{12}^*B_{12} - B_{12}^*A_{12} = Z_{11} + W_{22}$, where $Z_{11} = A_{12}B_{12}^* - B_{12}A_{12}^* \in \mathcal{K}_{11}$ and $W_{22} = A_{12}^*B_{12} - B_{12}^*A_{12} \in \mathcal{K}_{22}$. Since $iA_{12} + iA_{12}^*, iB_{12} + iB_{12}^* \in \mathcal{K}_{12}$, then from Lemmas 2.3 and 2.4 it follows that

$$\begin{aligned} & \Theta(X_{12} + Y_{12}) + \Theta(Z_{11}) + \Theta(W_{22}) \\ = \;& \Theta(X_{12} + Y_{12} + Z_{11} + W_{22}) \\ = \;& \Theta(X_{12} + Y_{12} + X_{12}Y_{12}^* - Y_{12}X_{12}^*) \\ = \;& \Theta([(i\mathcal{P}_1 + iA_{12} + iA_{12}^*) \circ \frac{1}{2}\mathfrak{I}, (i\mathcal{P}_2 + iB_{12} + iB_{12}^*)]_\bullet) \\ = \;& [\Theta(i\mathcal{P}_1) + \Theta(iA_{12} + iA_{12}^*) \circ \frac{1}{2}\mathfrak{I}, (i\mathcal{P}_2 + iB_{12} + iB_{12}^*)]_\bullet \\ + \;& [(i\mathcal{P}_1 + iA_{12} + iA_{12}^*) \circ \Theta(\frac{1}{2}\mathfrak{I}), (i\mathcal{P}_2 + iB_{12} + iB_{12}^*)]_\bullet \\ + \;& [(i\mathcal{P}_1 + iA_{12} + iA_{12}^*) \circ \frac{1}{2}\mathfrak{I}, (\Theta(i\mathcal{P}_2) + \Theta(iB_{12} + iB_{12}^*))]_\bullet \\ = \;& \Theta([i\mathcal{P}_1 \circ \frac{1}{2}\mathfrak{I}, i\mathcal{P}_2]_\bullet) + \Theta([i\mathcal{P}_1 \circ \frac{1}{2}\mathfrak{I}, (iB_{12} + iB_{12}^*)]_\bullet) \\ + \;& \Theta([(iA_{12} + iA_{12}^*) \circ \frac{1}{2}\mathfrak{I}, i\mathcal{P}_2]_\bullet) + \Theta([(iA_{12} + iA_{12}^*) \circ \frac{1}{2}\mathfrak{I}, (iB_{12} + iB_{12}^*)]_\bullet) \\ = \;& \Theta(X_{12}) + \Theta(Y_{12}) + \Theta(X_{12}Y_{12}^* - Y_{12}X_{12}^*) \\ = \;& \Theta(X_{12}) + \Theta(Y_{12}) + \Theta(Z_{11}) + \Theta(W_{22}). \end{aligned}$$

This implies $\Theta(X_{12} + Y_{12}) = \Theta(X_{12}) + \Theta(Y_{12})$. Hence the proof. □

Lemma 2.6. *For every $X_{ii}, Y_{ii} \in \mathcal{K}_{ii}$ $(i = 1, 2)$, we have*

(i) $\Theta(X_{11} + Y_{11}) = \Theta(X_{11}) + \Theta(Y_{11})$;

(ii) $\Theta(X_{22} + Y_{22}) = \Theta(X_{22}) + \Theta(Y_{22})$.

Proof. Let $\Omega = \Theta(X_{11} + Y_{11}) - \Theta(X_{11}) - \Theta(Y_{11})$. We have to show that $\Omega = 0$. On the one hand, we have

$$\begin{aligned} & \Theta([\mathcal{P}_2 \circ (X_{11} + Y_{11}), \mathcal{P}_2]_\bullet) = \Theta([\mathcal{P}_2 \circ X_{11}, \mathcal{P}_2]_\bullet) + \Theta([\mathcal{P}_2 \circ Y_{11}, \mathcal{P}_2]_\bullet) \\ = \;& [\Theta(\mathcal{P}_2) \circ X_{11}, \mathcal{P}_2]_\bullet + [\mathcal{P}_2 \circ \Theta(X_{11}), \mathcal{P}_2]_\bullet \\ + \;& [\mathcal{P}_2 \circ X_{11}, \Theta(\mathcal{P}_2)]_\bullet + [\Theta(\mathcal{P}_2) \circ Y_{11}, \mathcal{P}_2]_\bullet \\ + \;& [\mathcal{P}_2 \circ \Theta(Y_{11}), \mathcal{P}_2]_\bullet + [\mathcal{P}_2 \circ Y_{11}, \Theta(\mathcal{P}_2)]_\bullet \\ = \;& [\Theta(\mathcal{P}_2) \circ (X_{11} + Y_{11}), \mathcal{P}_2]_\bullet + [\mathcal{P}_2 \circ (\Theta(X_{11}) + \Theta(Y_{11})), \mathcal{P}_2]_\bullet \\ + \;& [\mathcal{P}_2 \circ (X_{11} + Y_{11}), \Theta(\mathcal{P}_2)]_\bullet. \end{aligned}$$

On the other hand

$$\begin{aligned} & \Theta([\mathcal{P}_2 \circ (X_{11} + Y_{11}), \mathcal{P}_2]_\bullet) = [\Theta(\mathcal{P}_2) \circ (X_{11} + Y_{11}), \mathcal{P}_2]_\bullet \\ + \;& [\mathcal{P}_2 \circ \Theta(X_{11} + Y_{11}), \mathcal{P}_2]_\bullet + [\mathcal{P}_2 \circ (X_{11} + Y_{11}), \Theta(\mathcal{P}_2)]_\bullet. \end{aligned}$$

We obtain from the above two relations that $[\mathcal{P}_2 \circ \Omega, \mathcal{P}_2]_\bullet = 0$ and since $\Omega^* = -\Omega$, then we get $\Omega_{12} = \Omega_{22} = 0$. Further assume that $Z = A_{12} - A_{12}^* \in \mathcal{K}_{12}$ for $A_{12} \in \mathfrak{A}_{12}$. Then $[Z \circ \frac{1}{2}\mathfrak{I}, X_{11}]_\bullet, [Z \circ \frac{1}{2}\mathfrak{I}, Y_{11}]_\bullet \in \mathcal{K}_{12}$. Thus from Lemma 2.5, we can write

$$\begin{aligned} & [\Theta(Z) \circ \frac{1}{2}\mathfrak{I}, (X_{11} + Y_{11})]_\bullet + [Z \circ \Theta(\frac{1}{2}\mathfrak{I}), (X_{11} + Y_{11})]_\bullet \\ + & [Z \circ \frac{1}{2}\mathfrak{I}, (\Theta(X_{11} + Y_{11}))]_\bullet \\ = & \Theta([Z \circ \frac{1}{2}\mathfrak{I}, (X_{11} + Y_{11})]_\bullet) \\ = & \Theta([Z \circ \frac{1}{2}\mathfrak{I}, X_{11}]_\bullet) + \Theta([Z \circ \frac{1}{2}\mathfrak{I}, Y_{11}]_\bullet) \\ = & [\Theta(Z) \circ \frac{1}{2}\mathfrak{I}, (X_{11} + Y_{11})]_\bullet + [Z \circ \Theta(\frac{1}{2}\mathfrak{I}), (X_{11} + Y_{11})]_\bullet \\ + & [Z \circ \frac{1}{2}\mathfrak{I}, (\Theta(X_{11}) + \Theta(Y_{11}))]_\bullet. \end{aligned}$$

Reasoning as above, we get $[Z \circ \frac{1}{2}\mathfrak{I}, \Omega]_\bullet = 0$ which gives $\Omega_{11} = 0$. Thus $\Omega = 0$. Thereby the proof is completed. □

Remark 2.7. *The additivity of Θ on $\mathcal{K}$ can easily be observed from Lemmas 2.3–2.6.*

Lemma 2.8. $\Theta(\mathfrak{I}) = 0$.

Proof. Let $K \in \mathcal{K}$. From Lemma 2.2 and Remark 2.7, we have

$$4\Theta(K) = \Theta([K \circ \mathfrak{I}, \mathfrak{I}]_\bullet) = 4\Theta(K) + K\Theta(\mathfrak{I}) + \Theta(\mathfrak{I})^*K + 3(\Theta(\mathfrak{I})K + K\Theta(\mathfrak{I})^*).$$

This implies

$$3\Big(\Theta(\mathfrak{I})K + K\Theta(\mathfrak{I})^*\Big) + K\Theta(\mathfrak{I}) + \Theta(\mathfrak{I})^*K = 0. \tag{5}$$

Putting $K = i\mathfrak{I}$ in (5), we obtain

$$4i(\Theta(\mathfrak{I}) + \Theta(\mathfrak{I})^*) = 0.$$

Thus

$$\Theta(\mathfrak{I})^* = -\Theta(\mathfrak{I}). \tag{6}$$

It follows from (5) and (6) that

$$\Theta(\mathfrak{I})K = K\Theta(\mathfrak{I})$$

for any $K \in \mathcal{K}$. Since for any $\mathcal{A} \in \mathfrak{A}, \mathcal{A} = K_1 + iK_2$ with $K_1 = \frac{\mathcal{A}-\mathcal{A}^*}{2} \in \mathcal{K}$ and $K_2 = \frac{\mathcal{A}+\mathcal{A}^*}{2i} \in \mathcal{K}$. Thus

$$\Theta(\mathfrak{I})\mathcal{A} = \mathcal{A}\Theta(\mathfrak{I}) \tag{7}$$

for all $\mathcal{A} \in \mathfrak{A}$. For any $A_{12} \in \mathfrak{A}_{12}$, let $X = A_{12} - A_{12}^* \in \mathcal{K}$ observe that $[X \circ i\mathfrak{I}, \mathfrak{I}]_\bullet = 0$. It follows from Lemmas 2.1, 2.2 and Equations (6) and (7) that

$$\begin{aligned} 0 &= \Theta([X \circ i\mathfrak{I}, \mathfrak{I}]_\bullet) \\ &= [\Theta(X) \circ i\mathfrak{I}, \mathfrak{I}]_\bullet + [X \circ \Theta(i\mathfrak{I}), \mathfrak{I}]_\bullet + [X \circ i\mathfrak{I}, \Theta(\mathfrak{I})]_\bullet \\ &= [2i\Theta(X), \mathfrak{I}]_\bullet + [X\Theta(i\mathfrak{I}) + \Theta(i\mathfrak{I})X, \mathfrak{I}]_\bullet + [2iX, \Theta(\mathfrak{I})]_\bullet \\ &= -4i\Theta(\mathfrak{I})X. \end{aligned}$$

This implies

$$\Theta(\mathfrak{I})X = \Theta(\mathfrak{I})(A_{12} - A_{12}^*) = 0.$$

Multiply the above equation by $\mathcal{P}_2$ from the right and left respectively we get $\Theta(\mathfrak{I})A_{12} = 0$ and $\Theta(\mathfrak{I})A_{12}^* = 0$. Using equation (1) we obtain $\Theta(\mathfrak{I})\mathcal{P}_1 = 0$ and $\Theta(\mathfrak{I})\mathcal{P}_2 = 0$ and thus $\Theta(\mathfrak{I}) = \Theta(\mathfrak{I})\mathcal{P}_1 + \Theta(\mathfrak{I})\mathcal{P}_2 = 0$. □

Lemma 2.9. *For any $L \in \mathcal{L}$, $\Theta(L)^* = \Theta(L)$.*

Proof. Let $L \in \mathcal{L}$. Then $[\mathfrak{I} \circ L, \mathfrak{I}]_\bullet = 0$, so from Lemma 2.8, we can write

$$0 = \Theta([\mathfrak{I} \circ L, \mathfrak{I}]_\bullet) = [\mathfrak{I} \circ \Theta(L), \mathfrak{I}]_\bullet = 2(\Theta(L) - \Theta(L)^*). \tag{8}$$

Hence, we have $\Theta(L)^* = \Theta(L)$ for all $L \in \mathcal{L}$. □

Lemma 2.10. *For any $L \in \mathcal{L}$, $\Theta(iL) = i\Theta(L) + \Theta(i\mathfrak{I})L$.*

Proof. Observe that for any $L \in \mathcal{L}$, $[L \circ i\mathfrak{I}, i\mathfrak{I}]_\bullet = 0$. Therefore,

$$\begin{aligned} 0 &= \Theta([L \circ i\mathfrak{I}, i\mathfrak{I}]_\bullet) \\ &= [\Theta(L) \circ i\mathfrak{I}, i\mathfrak{I}]_\bullet + [L \circ \Theta(i\mathfrak{I}), i\mathfrak{I}]_\bullet + [L \circ i\mathfrak{I}, \Theta(i\mathfrak{I})]_\bullet \\ &= 2i(\Theta(i\mathfrak{I})L - L\Theta(i\mathfrak{I})) \end{aligned} \tag{9}$$

This implies $\Theta(i\mathfrak{I})L = L\Theta(i\mathfrak{I})$ for all $L \in \mathcal{L}$. Since for any $\mathcal{A} \in \mathfrak{A}, \mathcal{A} = L_1 + iL_2$ with $L_1 = \frac{\mathcal{A}+\mathcal{A}^*}{2} \in \mathcal{L}$ and $L_2 = \frac{\mathcal{A}-\mathcal{A}^*}{2i} \in \mathcal{L}$. Thus

$$\Theta(i\mathfrak{I})\mathcal{A} = \mathcal{A}\Theta(i\mathfrak{I}) \tag{10}$$

for all $\mathcal{A} \in \mathfrak{A}$. Now

$$\begin{aligned} 4\Theta(iL) &= \Theta([i\mathfrak{I} \circ \mathfrak{I}, L]_\bullet) \\ &= [\Theta(i\mathfrak{I}) \circ \mathfrak{I}, L]_\bullet + [i\mathfrak{I} \circ \mathfrak{I}, \Theta(L)]_\bullet \\ &= 4(i\Theta(L) + \Theta(i\mathfrak{I})L) \end{aligned}$$

Thus

$$\Theta(iL) = i\Theta(L) + \Theta(i\mathfrak{I})L.$$

□

Lemma 2.11. *Θ is additive on $\mathcal{L}$.*

Proof. Let $L_1, L_2 \in \mathcal{L}$. Then $iL_1, iL_2 \in \mathcal{K}$. Then, it follows from Remark 2.7 and Lemma 2.10 that

$$\begin{aligned} \Theta(iL_1 + iL_2) &= \Theta(iL_1) + \Theta(iL_2) \\ &= i\Theta(L_1) + i\Theta(L_2) + \Theta(i\mathfrak{I})(L_1 + L_2). \end{aligned} \tag{11}$$

Also

$$\Theta(i(L_1 + L_2)) = i\Theta(L_1 + L_2) + \Theta(i\mathfrak{I})(L_1 + L_2). \tag{12}$$

From (11) and (12), we obtain

$$\Theta(L_1 + L_2) = \Theta(L_1) + \Theta(L_2).$$

Hence the result. □

Lemma 2.12. *$\Theta(X^*) = \Theta(X)^*$ for all $X \in \mathfrak{A}$.*

Proof. Let $L_1, L_2 \in \mathcal{L}$. Then, in view of Remark 2.7, Lemmas 2.8, 2.10 and $[L_1 \circ \mathfrak{I}, \mathfrak{I}]_\bullet = 0$, we have

$$\begin{aligned} \Theta([L_1 + iL_2 \circ \mathfrak{I}, \mathfrak{I}]_\bullet) &= \Theta([L_1 \circ \mathfrak{I}, \mathfrak{I}]_\bullet) + \Theta([iL_2 \circ \mathfrak{I}, \mathfrak{I}]_\bullet) \\ &= 4\Big(i\Theta(L_2) + \Theta(i\mathfrak{I})L_2\Big). \end{aligned} \tag{13}$$

On the other hand

$$\Theta([L_1 + iL_2 \circ \mathfrak{I}, \mathfrak{I}]_\bullet) = 2(\Theta(L_1 + iL_2) - \Theta(L_1 + iL_2)^*). \tag{14}$$

From (13) and (14), we have

$$4\Big(i\Theta(L_2)+\Theta(i\mathfrak{I})L_2\Big)=2\Big(\Theta(L_1+iL_2)-\Theta(L_1+iL_2)^*\Big). \tag{15}$$

Since $[i\mathfrak{I}\circ iL_2,\mathfrak{I}]_\bullet=0$, then we have

$$\begin{aligned} 4\Big(i\Theta(L_1)+\Theta(i\mathfrak{I})L_1\Big) &= \Theta([i\mathfrak{I}\circ(L_1+iL_2),\mathfrak{I}]_\bullet) \qquad (16)\\ &= 2i\Big(\Theta(L_1+iL_2)+\Theta(L_1+iL_2)^*\Big)+4\Theta(i\mathfrak{I})L_1. \end{aligned}$$

From (15) and (16), we obtain

$$\Theta(L_1+iL_2)=\Theta(L_1)+i\Theta(L_2)+\Theta(i\mathfrak{I})L_2. \tag{17}$$

Let $X\in\mathfrak{A}$. Then $X=L+iM$ for $L,M\in\mathcal{L}$, so from Equation (17), Lemmas 2.9 and 2.11, we have

$$\begin{aligned} \Theta(X)^* &= \Theta(L+iM)^* \qquad (18)\\ &= (\Theta(L)+i\Theta(M)+\Theta(i\mathfrak{I})M)^*\\ &= \Theta(L)-i\Theta(M)-\Theta(i\mathfrak{I})M\\ &= \Theta(L-iM)\\ &= \Theta(X^*). \end{aligned}$$

This gives the assertion. □

Lemma 2.13. Θ *is additive on* $\mathfrak{A}$.

Proof. Let $X,Y\in\mathfrak{A}$ such that $X=L_1+iL_2$ and $Y=M_1+iM_2$ for all $L_1,L_2,M_1,M_2\in\mathcal{L}$. Then, in view of Equation (17) and Lemma 2.11, we have

$$\begin{aligned} \Theta(X+Y) &= \Theta((L_1+M_1)+i(L_2+M_2)) \qquad (19)\\ &= \Theta(L_1+M_1)+i\Theta(L_2+M_2)+\Theta(i\mathfrak{I})(L_2+M_2)\\ &= (\Theta(L_1)+i\Theta(L_2)+\Theta(i\mathfrak{I})L_2)\\ &+ (\Theta(M_1)+i\Theta(M_2)+\Theta(i\mathfrak{I})M_2)\\ &= \Theta(L_1+iL_2)+\Theta(M_1+iM_2)\\ &= \Theta(X)+\Theta(Y). \end{aligned}$$

Hence the result. □

Lemma 2.14. $\Theta(i\mathfrak{I})=0$.

Proof. In view of Lemmas 2.8, 2.9 and 2.12, let us assume that

$$\Theta(\mathcal{P}_1)=L \tag{20}$$

for some $L\in\mathcal{L}$, and

$$\Theta(i\mathcal{P}_1)=iL+\Theta(i\mathfrak{I})\mathcal{P}_1 \tag{21}$$

Also

$$\begin{aligned} 4\Theta(i\mathcal{P}_1) &= \Theta([i\mathcal{P}_1\circ\mathcal{P}_1,\mathfrak{I}]_\bullet)\\ &= [\Theta(i\mathcal{P}_1)\circ\mathcal{P}_1,\mathfrak{I}]_\bullet+[i\mathcal{P}_1\circ\Theta(\mathcal{P}_1),\mathfrak{I}]_\bullet\\ &= 4\Theta(i\mathfrak{I})\mathcal{P}_1+4i(\mathcal{P}_1L+L\mathcal{P}_1) \end{aligned}$$

This implies

$$\Theta(i\mathcal{P}_1)=\Theta(i\mathfrak{I})\mathcal{P}_1+i(\mathcal{P}_1L+L\mathcal{P}_1) \tag{22}$$

From Equations (21) and (22), we have

$$L = \mathcal{P}_1 L + L\mathcal{P}_1.$$

This gives

$$\mathcal{P}_1 L\mathcal{P}_1 = \mathcal{P}_2 L\mathcal{P}_2 = 0$$

and hence

$$\Theta(i\mathcal{P}_1) = \Theta(i\mathfrak{I})\mathcal{P}_1 + i\mathcal{P}_1 L\mathcal{P}_2 + i\mathcal{P}_2 L\mathcal{P}_1. \tag{23}$$

Observe, for any $X_{12} \in \mathfrak{A}_{12}$ that

$$\Theta([\mathfrak{I} \circ i\mathcal{P}_1, (X_{12} - X_{12}^*)]_\bullet) = -2\Theta(i(X_{12} + X_{12}^*)).$$

In view of Lemma 2.10, we have

$$-2\Theta(i(X_{12} + X_{12}^*)) = -2i(\Theta(X_{12}) + \Theta(X_{12})^*) - 2\Theta(i\mathfrak{I})(X_{12} + X_{12}^*).$$

Thus

$$\begin{aligned} \Theta([\mathfrak{I} \circ i\mathcal{P}_1, (X_{12} - X_{12}^*)]_\bullet) &= -2i(\Theta(X_{12}) + \Theta(X_{12})^*) \\ &- 2\Theta(i\mathfrak{I})(X_{12} + X_{12}^*). \end{aligned} \tag{24}$$

Alternatively, from (23) and Lemma 2.8, we have

$$\begin{aligned} & \Theta([\mathfrak{I} \circ i\mathcal{P}_1, (X_{12} - X_{12}^*)]_\bullet) \\ = \ & [\mathfrak{I} \circ \Theta(i\mathcal{P}_1), (X_{12} - X_{12}^*)]_\bullet + [\mathfrak{I} \circ i\mathcal{P}_1, \Theta(X_{12} - X_{12}^*)]_\bullet \\ = \ & [\mathfrak{I} \circ (\Theta(i\mathfrak{I})\mathcal{P}_1 + i\mathcal{P}_1 L\mathcal{P}_2 + i\mathcal{P}_2 L\mathcal{P}_1), (X_{12} - X_{12}^*)]_\bullet \\ + \ & [\mathfrak{I} \circ i\mathcal{P}_1, (\Theta(X_{12}) - \Theta(X_{12}^*))]_\bullet \\ = \ & 2(\Theta(i\mathfrak{I})\mathcal{P}_1 + i\mathcal{P}_1 L\mathcal{P}_2 + i\mathcal{P}_2 L\mathcal{P}_1)(X_{12}^* - X_{12}) \\ + \ & 2(X_{12} - X_{12}^*)(\Theta(i\mathfrak{I})\mathcal{P}_1 + i\mathcal{P}_1 L\mathcal{P}_2 + i\mathcal{P}_2 L\mathcal{P}_1) \\ + \ & 2i\mathcal{P}_1(\Theta(X_{12})^* - \Theta(X_{12})) + 2i(\Theta(X_{12}) - \Theta(X_{12})^*)\mathcal{P}_1. \end{aligned} \tag{25}$$

Now from (24) and (25), we obtain

$$\begin{aligned} - \ & i\Theta(X_{12}) - i\Theta(X_{12})^* - \Theta(i\mathfrak{I})(X_{12} + X_{12}^*) \\ = \ & (\Theta(i\mathfrak{I})\mathcal{P}_1 + i\mathcal{P}_1 L\mathcal{P}_2 + i\mathcal{P}_2 L\mathcal{P}_1)(X_{12}^* - X_{12}) \\ + \ & (X_{12} - X_{12}^*)(\Theta(i\mathfrak{I})\mathcal{P}_1 + i\mathcal{P}_1 L\mathcal{P}_2 + i\mathcal{P}_2 L\mathcal{P}_1) \\ + \ & i\mathcal{P}_1(\Theta(X_{12})^* - \Theta(X_{12})) + i(\Theta(X_{12}) - \Theta(X_{12})^*)\mathcal{P}_1. \end{aligned} \tag{26}$$

Multiply (26) by $\mathcal{P}_1$ from left and $\mathcal{P}_2$ from right, we get $\mathcal{P}_1\Theta(X_{12})^*\mathcal{P}_2 = 0$. Next, consider

$$\begin{aligned} & 2(\Theta(X_{12}) - \Theta(X_{12})^*) \\ = \ & \Theta([\mathfrak{I} \circ i\mathcal{P}_1, i(X_{12} + X_{12}^*)]_\bullet) \\ = \ & [\mathfrak{I} \circ \Theta(i\mathcal{P}_1), i(X_{12} + X_{12}^*)]_\bullet + [\mathfrak{I} \circ i\mathcal{P}_1, \Theta(i(X_{12} + X_{12}^*))]_\bullet \\ = \ & [\mathfrak{I} \circ (\Theta(i\mathfrak{I})\mathcal{P}_1 + i\mathcal{P}_1 L\mathcal{P}_2 + i\mathcal{P}_2 L\mathcal{P}_1), i(X_{12} + X_{12}^*)]_\bullet \\ + \ & [\mathfrak{I} \circ i\mathcal{P}_1, (i(\Theta(X_{12}) + \Theta(X_{12}^*)) + \Theta(i\mathfrak{I})(X_{12} + X_{12}^*))]_\bullet \\ = \ & -2i\Big(\Theta(i\mathfrak{I})\mathcal{P}_1 + i\mathcal{P}_1 L\mathcal{P}_2 + i\mathcal{P}_2 L\mathcal{P}_1\Big)(X_{12}^* + X_{12}) \\ + \ & 2i(X_{12}^* + X_{12})\Big(\Theta(i\mathfrak{I})\mathcal{P}_1 + i\mathcal{P}_1 L\mathcal{P}_2 + i\mathcal{P}_2 L\mathcal{P}_1\Big) \\ - \ & 2i\mathcal{P}_1\Big(i\Theta(X_{12})^* + i\Theta(X_{12}) + \Theta(i\mathfrak{I})(X_{12}^* + X_{12})\Big) \\ + \ & 2i\Big(i\Theta(X_{12}) + i\Theta(X_{12})^* + \Theta(i\mathfrak{I})(X_{12}^* + X_{12})\Big)\mathcal{P}_1. \end{aligned} \tag{27}$$

Multiply above relation by $\mathcal{P}_1$ from left and $\mathcal{P}_2$ from right, we obtain $\Theta(i\mathfrak{I})X_{12} = 0$ and so by Equation(1) we have $\Theta(i\mathfrak{I})\mathcal{P}_1 = 0$. Also by Equation (10) we get $\Theta(i\mathfrak{I})X_{12}^* = 0$ and thus by Equation(1) we obtain $\Theta(i\mathfrak{I})\mathcal{P}_2 = 0$. And hence, $\Theta(i\mathfrak{I}) = \Theta(i\mathfrak{I})\mathcal{P}_1 + \Theta(i\mathfrak{I})\mathcal{P}_2 = 0$. This completes the proof. □

Lemma 2.15. *$\Theta(iX) = i\Theta(X)$ for all $X \in \mathfrak{A}$.*

Proof. It follows from Lemmas 2.10 and 2.14 that $\Theta(iL) = i\Theta(L)$ for all $L \in \mathcal{L}$. Thus, for any $X \in \mathfrak{A}$ and $L_1, L_2 \in \mathcal{L}$ and using the fact that Θ is additive on $\mathfrak{A}$, we have

$$\Theta(iX) = \Theta(iL_1 - L_2)) = i\Theta(L_1) - \Theta(L_2) = i(\Theta(L_1) + i\Theta(L_2)) = i\Theta(X).$$

Hence the result. □

Lemma 2.16. *Θ is a derivation on $\mathfrak{A}$.*

Proof. Let $L_1, L_2 \in \mathcal{L}$. Then

$$\begin{aligned} 2\Theta(L_1L_2 - L_2L_1) &= \Theta([\mathfrak{I} \circ L_1, L_2]_\bullet) \\ &= [\mathfrak{I} \circ \Theta(L_1), L_2]_\bullet + [\mathfrak{I} \circ L_1, \Theta(L_2)]_\bullet \\ &= 2\Big(\Theta(L_1)L_2 - L_2\Theta(L_1) + L_1\Theta(L_2) \\ &- \Theta(L_2)L_1\Big). \end{aligned} \tag{28}$$

Also

$$\begin{aligned} 2i\Theta(L_1L_2 + L_2L_1) &= \Theta([\mathfrak{I} \circ iL_1, L_2]_\bullet) \\ &= [\mathfrak{I} \circ \Theta(iL_1), L_2]_\bullet + [\mathfrak{I} \circ iL_1, \Theta(L_2)]_\bullet \\ &= 2i\Big(\Theta(L_1)L_2 + L_2\Theta(L_1) + L_1\Theta(L_2) \\ &+ \Theta(L_2)L_1\Big). \end{aligned} \tag{29}$$

Addition of (28) and (29) gives $\Theta(L_1L_2) = \Theta(L_1)L_2 + L_1\Theta(L_2)$ for all $L_1, L_2 \in \mathcal{L}$. Further, for any $X, Y \in \mathfrak{A}$ assume that $X = L_1 + iL_2$ and $Y = M_1 + iM_2$ for $L_1, L_2, M_1, M_2 \in \mathcal{L}$. Then

$$\begin{aligned} \Theta(XY) &= \Theta\Big((L_1 + iL_2)(M_1 + iM_2)\Big) \\ &= \Theta(L_1M_1 + iL_1M_2 + iL_2M_1 - L_2M_2) \\ &= \Theta(L_1)M_1 + L_1\Theta(M_1) + i\Theta(L_1)M_2 \\ &+ iL_1\Theta(M_2) + i\Theta(L_2)M_1 + iL_2\Theta(M_1) \\ &- \Theta(L_2)M_2 - L_2\Theta(M_2) \end{aligned} \tag{30}$$

On the other hand

$$\begin{aligned} \Theta(X)Y + X\Theta(Y) &= \Theta(L_1 + iL_2)(M_1 + iM_2) \\ &+ (L_1 + iL_2)\Theta(M_1 + iM_2) \\ &= (\Theta(L_1) + i\Theta(L_2))(M_1 + iM_2) \\ &+ (L_1 + iL_2)(\Theta(M_1) + i\Theta(M_2)) \\ &= \Theta(L_1)M_1 + L_1\Theta(M_1) + i\Theta(L_1)M_2 \\ &+ iL_1\Theta(M_2) + i\Theta(L_2)M_1 + iL_2\Theta(M_1) \\ &- \Theta(L_2)M_2 - L_2\Theta(M_2) \end{aligned} \tag{31}$$

Comparing Equations (30) and (31), we conclude that Θ is a derivation on $\mathfrak{A}$. Therefore, the proof of our Main Theorem is completed. □

3. Corollaries

The following result [16, Theorem 1.1], is useful to describe the primeness of alternative rings.

Theorem 3.1. *Let R be a 3-torsion free alternative ring. So R is a prime ring if and only if $aR \cdot b = 0$ (or $a \cdot Rb = 0$) implies $a = 0$ or $b = 0$ for $a, b \in R$.*

Let $\mathfrak{A}$ be an associative $*$-algebra. Then $\mathfrak{A}$ is said to be prime if $IJ \neq (0)$ for any two nonzero ideals $I, J \subseteq \mathfrak{A}$. Theorem 3.1 can be applied to associative algebras over $\mathbb{C}$. In view of Theorem 3.1, we can say that prime $*$-algebras satisfy Equation (1). Then we have the following corollary.

Corollary 3.2. *Let $\mathfrak{A}$ be a unital prime $*$-algebra containing non-trivial projections $\mathcal{P}_1$ and $\mathcal{P}_2$. Then Θ is a non-linear mixed Jordan bi-skew Lie triple derivation on $\mathfrak{A}$ if and only if Θ is an additive $*$-derivation on $\mathfrak{A}$.*

A von Neumann algebra $\mathfrak{A}$ is a weakly closed self-adjoint algebra of operators on a complex Hilbert space $\mathcal{H}$ containing the identity operator $\mathfrak{I}$. $\mathfrak{A}$ is said to be a factor if its centre is trivial. Since a factor von Neumann algebra is a prime $*$-algebra, then we have the following corollary.

Corollary 3.3. *Let $\mathfrak{A}$ be a factor von Neumann algebra with $dim(\mathfrak{A}) \geq 2$. Then $\Theta : \mathfrak{A} \to \mathfrak{A}$ is a non-linear mixed Jordan bi-skew Lie triple derivation if and only if Θ is an additive $*$-derivation.*

Corollary 3.4. *Let $\mathfrak{A}$ be a von Neumann algebra with no central summands of type I_1. Then $\Theta : \mathfrak{A} \to \mathfrak{A}$ is a non-linear mixed Jordan bi-skew Lie triple derivation if and only if Θ is an additive $*$-derivation.*

Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. A subalgebra $\mathfrak{A}$ of $\mathcal{B}(\mathcal{H})$ is said to be a standard operator algebra if $\mathcal{F}(\mathcal{H}) \subseteq \mathfrak{A}$ where $\mathcal{F}(\mathcal{H})$ is the subalgebra of all finite rank operators on $\mathcal{H}$. As we know that a standard operator algebra is a prime $*$-algebra, thus we have the following corollary.

Corollary 3.5. *Let $\mathcal{H}$ be an infinite dimensional complex Hilbert space and $\mathfrak{A}$ be a standard operator algebra on $\mathcal{H}$ containing the identity operator $\mathfrak{I}$. Suppose that $\mathfrak{A}$ is closed under the adjoint operation. Then $\Theta : \mathfrak{A} \to \mathfrak{A}$ is a non-linear mixed Jordan bi-skew Lie triple derivation if and only if Θ is an additive $*$-derivation. Moreover, there exists an operator $Y \in B(\mathcal{H})$ satisfying $Y + Y^* = 0$ such that $\Theta(X) = XY - YX$ for all $X \in \mathfrak{A}$, i.e., Θ is inner.*

Proof. As Θ is an additive $*$-derivation on standard operator algebra $\mathfrak{A}$ from [15] it follows that Θ is an inner derivation, i.e., there exists $Y \in B(\mathcal{H})$ such that $\Theta(X) = XY - YX$ for all $X \in \mathfrak{A}$. Since $\Theta(X^*) = \Theta(X)^*$ for all $X \in \mathfrak{A}$, then we have

$$X^*Y - YX^* = \Theta(X^*) = Y^*X^* - X^*Y^*$$

for all $X \in \mathfrak{A}$. This implies $X^*(Y + Y^*) = (Y + Y^*)X^*$. Thus, $Y + Y^* = \alpha\mathfrak{I}$ for some $\alpha \in \mathbb{R}$. Let us set $Z = Y - \frac{1}{2}\alpha\mathfrak{I}$. One can check that $Z + Z^* = 0$ such that $\Theta(X) = XZ - ZX$. □

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