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On extremal solutions of weighted fractional hybrid differential equations

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Abstract. This research studies the existence of a solution for an initial value problem of nonlinear fractional hybrid differential equations involving Riemann-Liouville derivative in weighted space of continuous functions. An existence theorem for this equations is proved under mixed Lipschitz and Carathéodory conditions.

1. Introduction

Fractional calculus has evolved into an important and interesting field of research in view of into numerous applications in technical and applied sciences. The mathematical modeling of many real world phenomena based on fractional order operators is regarded as better and improved than the one depending on integer order operators. In particular, fractional calculus has played a significant role in the recent development of special functions and integral transforms, finance, stochastic processes, wave and diffusion phenomena, plasma physics, social sciences, for further details and applications, see [13, 16, 20, 21, 27]. Heymans and Podlubny [18] have demonstrated that it is possible to attribute physical meaning to initial conditions expressed in terms of Riemann-Liouville fractional derivatives or integrals on the field of viscoelasticity, and such initial condition are more appropriate than physically interpretable initial conditions. Zhao et al.[32] have discussed the following fractional hybrid differential equations involving Riemann-Liouville differential operators:

$$\begin{cases} D^q\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)) \quad \text{a.e.} \quad t \in J = [0,T], \\ \\ x(0) = 0, \end{cases}$$

where D^q is the Riemann-Liouville fractional derivative of order $0 < q < 1$, $f \in C(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$. The authors of established the existence theorem for fractional hybrid diferential equations and some fundamental differetial inequalities, they also established the existence of extremal solutions. For

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this reason, when $x(0) \neq 0$, the solutions to the functional fractional hybrid differential equations given in the mentioned papers may not be well-defined. In this paper, we continue the work for $x(0) \neq 0$, is to investigate the weighted fractional hybrid differential equations. We consider the initial value problems for hybrid differential equations with fractional order (IVPHDEF for short) involving Riemann-Liouville differential operators of order $0 < \alpha < 1$,

$$\begin{cases} {}^{RL}D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)) \quad \text{a.e.} \quad t \in J' = (0,T], \\ \\ I_{0^+}^{1-\alpha}\left(\frac{x(t)}{f(t,x(t))}\right)\Big|_{t=0} = x_0 \in \mathbb{R}, \end{cases} \tag{1}$$

where, ${}^{RL}D^{\alpha}$ is the Riemann-Liouville fractional derivative of order $0 < \alpha < 1$, $f \in C(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$, $g \in \mathcal{C}(J \times \mathbb{R}, \mathbb{R})$, by a solution of IVPFHDE (1) we mean a function $x \in C_{1-\alpha}(J, \mathbb{R})$ such that

(i) the function $t \to \frac{x}{f(t,x)}$ is continuous for each $x \in \mathbb{R}$, and

(ii) the function $x(t)$ satisfies the equations in (1).

2. Preliminary Results

In this section, we introduce notations, definitions, and preliminary facts which are used throught this paper.
Let $J = [0,T]$ and $J' = (0,T]$. $C(J, \mathbb{R})$ be the space of $\mathbb{R}$-valued continuous functions on J endowed with uniform norm topology

$$\|x\|_C = \sup\{|x(t)|, \ \ t \in J\}.$$

$L^1(J, \mathbb{R})$ denote the space of Lebesgue integrable $\mathbb{R}$-valued functions on J equipped with the norm $\|\cdot\|_{L^1}$ defined by

$$\|x\|_{L^1} = \int_0^T |x(s)| ds,$$

and let $\mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ denote the class of functions $g : J \times \mathbb{R} \to \mathbb{R}$ such that

(i) the map $t \to g(t,x)$ is measurable for each $x \in \mathbb{R}$, and

(ii) the map $x \to g(t,x)$ is continuous for each $t \in J$.

The class $\mathcal{C}(J \times \mathbb{R}, \mathbb{R})$ is called the Carathéodory class of functions on $J \times \mathbb{R}$.
By $E = C_{1-\alpha}(J, \mathbb{R})$ we denote the space of all continuous functions such that $C_{1-\alpha}(J, \mathbb{R}) = \{x \in C(J', \mathbb{R}) : \lim_{t\to 0^+} t^{1-\alpha}x(t) \quad \text{exists}\}$, where the norm in this space is given by

$$\|x\|_{\alpha} = \sup_{t \in J} t^{1-\alpha}|x(t)|,$$

it easy to see that $(C_{1-\alpha}(J, \mathbb{R}), \|\cdot\|_{\alpha})$ is a Banach space. The following lemma is a variant of classical Arzelà-Ascoli theorem. For Ω a subset of the space $C_{1-\alpha}(J, \mathbb{R})$, define Ω_{α} by $\Omega_{\alpha} = \{x_{\alpha} : x \in \Omega\}$,

$$\begin{cases} t^{1-\alpha}x(t), \quad \text{if} \quad t \in J', \\ \\ \lim\limits_{t\to 0^+} t^{1-\alpha}x(t), \quad \text{if} \quad t = 0. \end{cases}$$

It is clear that $x_{\alpha} \in C(J, \mathbb{R})$.

Lemma 2.1. *[4, 34] A set $\Omega \subset C_{1-\alpha}(J, \mathbb{R})$ is relatively compact if and only if Ω_{α} is relatively compact in $C(J, \mathbb{R})$.*

We give some concepts of fractional calculus. A function $x : J \to \mathbb{R}$ has a fractional integral if the following integral

$$I^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds,$$

is defined for $t \geq 0$, where $\Gamma(\cdot)$ is Gamma function. The Riemann-Liouville fractional derivative of x of order α is defined as

$$\left({}^{RL}D\right) x(t) = \frac{d}{dt}(I^{1-\alpha}x)(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} x(s) ds,$$

provided it is well defined for $t \geq 0$. The previous integral is taken in Lebesgue sense. Let $\phi_{\alpha}(t) : \mathbb{R} \to \mathbb{R}$, defined by

$$\begin{cases} \dfrac{t^{1-\alpha}}{\Gamma(\alpha)}, & \text{if} \quad t > 0, \\ \\ 0, & \text{if} \quad t \leq 0. \end{cases}$$

Then

$$I^{\alpha}x(t) = (\phi_{\alpha} * x)(t)$$

and

$${}^{RL}D^{\alpha}x(t) = \frac{d}{dt}(\phi_{1-\alpha} * x)(t).$$

3. Existence Results

In the present section, we considering the multiplication in E, such that $(xy)(t) = x(t)y(t)$ for $x, y \in E$.

Theorem 3.1. *[14] Let S be a nonempty, closed convex and bounded subset of the Banach algebra E, and let $A : E \to E$ and $B : E \to E$ be two operators such that*

- (a) *A is Lipschitzain with a Lipschitz constant k,*
- (b) *B is completely continuous,*
- (c) *$x = AxBx \Rightarrow x \in S$ for all $y \in S$, and*
- (d) *$Mk < 1$, where $M = \|B(S)\| = \sup\{\|B(x)\| : x \in S\}$.*

Then the operator equation $AxBx = x$ has a solution.

We make the following assumptions:

(A_0) function $x \to \dfrac{x}{f(t,x)}$ is increasing in $\mathbb{R}$ almost everywhere for $t \in J$.

(A_1) there exists a constant $L > 0$ such that

$$|f(t,x) - f(t,y)| \leq L|x-y|$$

for all $t \in J$ and $x, y \in \mathbb{R}$.

(A_2) there exists a function $h \in L^1(J, \mathbb{R})$ such that

$$|g(t,x)| \leq h(t) \ \text{ a.e. } \ t \in J$$

for all $x \in \mathbb{R}$.

(A_3) there exists a number $N > 0$ such that

$$N \geq \gamma\left(\|x_0\| + \frac{T\|h\|_{L^1}}{\Gamma(\alpha+1)}\right), \tag{2}$$

where $|f(t,x(t))| \leq \gamma, \quad \forall (t,x) \in J \times \mathbb{R}$.

Lemma 3.2. *[32] Let $0 < \alpha < 1$ and $x \in L^1(0,T)$.*

(H_1) *the equality ${}^{RL}D^\alpha I^\alpha x(t) = x(t)$ holds.*

(H_2) *the equality*

$$I^\alpha\, {}^{RL}D^\alpha x(t) = x(t) - \frac{I^{1-\alpha}x(0)}{\Gamma(\alpha)} t^{\alpha-1},$$

holds almost everywhere on J.

The following lemma is useful in what follows.

Lemma 3.3. *Assume that hypothesis (A_0) holds. Then for any $h \in L^1(J,\mathbb{R})$ and $0 < \alpha < 1$, the function $x \in C_{1-\alpha}(J,\mathbb{R})$ is a solution of IVPFHDE*

$$\begin{cases} {}^{RL}D^\alpha\left(\frac{x(t)}{f(t,x(t))}\right) = h(t), \quad a.e. \ \ t \in J', \\ \\ I_{0^+}^{1-\alpha}\left(\frac{x(t)}{f(t,x(t))}\right)\Big|_{t=0} = x_0, \end{cases} \tag{3}$$

if and only if x satisfies the hybrid integral equation HIE

$$x(t) = f(t,x(t))\left[t^{\alpha-1}x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^\alpha h(s)ds\right], \quad t \in J'. \tag{4}$$

Proof. Let x be a solution of the Cauchy problem (3). Since the Riemann-Liouville fractional integral I^α is a monotone operators, thus we apply fractional integral I^α by Lemma3.2, the initial value problem (1) is equivalent to the integral equation

$$I^\alpha\, {}^{RL}D^\alpha\left(\frac{x(t)}{f(t,x(t))}\right) = \frac{x(t)}{f(t,x(t))} - \frac{I^{1-\alpha}\left(\frac{x(t)}{f(t,x(t))}\right)\Big|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1} = I^\alpha h(t),$$

we get

$$\frac{x(t)}{f(t,x(t))} = \frac{I^{1-\alpha}\left(\frac{x(t)}{f(t,x(t))}\right)\Big|_{t=0}}{\Gamma(\alpha)} t^{\alpha-1} + I^\alpha h(t)$$

$$\frac{x(t)}{f(t,x(t))} = t^{\alpha-1}x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^\alpha h(s)ds,$$

i.e.,

$$x(t) = f(t,x(t))\left[t^{\alpha-1}x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^\alpha h(s)ds\right], \quad t \in J'.$$

Thus, (4) holds. Conversely, assume that x satisfies HIE (4). Then dividing by $f(t,x(t))$ and applying ${}^{RL}D^\alpha$ on both sides of (4), so (3) is satisfied. Again, substituting $t = 0$ in (4) yields $\lim_{t\to 0^+} t^{1-\alpha}\frac{x(t)}{f(t,x(t))} = x_0$. Hence (3) also holds. The proof is completed. □

Now we are in a position to prove the following existence thoerem for IVPFHDE.

Theorem 3.4. *Assume that hypotheses* $(A_0)-(A_3)$ *hold. Further, if*

$$L\left(\|x_0\|+\frac{T\|h\|_{L^1}}{\Gamma(\alpha+1)}\right)<1, \tag{5}$$

then the IVPFHDE (3) *has a solution on* J.

Proof. Set $E=C_{1-\alpha}(J,\mathbb{R})$ and define a subset S of E by $S=\{x\in E:\|x\|_\alpha\le N\}$, where N satisfies inequality (2). Clear S is a closed, convex and bounded subset of the Banach space E. By Lemma 3.3, IVPFHDE (3) is equivalent to the nonlinear HIE

$$x(t)=f(t,x(t))\left[t^{\alpha-1}x_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s,x(s))ds\right],\quad t\in J'. \tag{6}$$

Define two operators $A:E\to E$ and $B:S\to E$ by

$$Ax(t)=f(t,x(t)),\quad t\in J \tag{7}$$

and

$$Bx(t)=t^{\alpha-1}x_0+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}g(s,x(s))ds,\quad t\in J'. \tag{8}$$

Then the hybrid integral equation (6) is transformed into the operator equation as

$$x(t)=Ax(t)Bx(t),\quad t\in J'. \tag{9}$$

We shall show that the operators A and B satisfy all the conditions of Theorem 3.1.
`Claim1`. Let $x,y\in E$, then by hypothesis (A_1)

$$|Ax(t)-Ay(t)|=|f(t,x(t))-f(t,y(t))|\le L|x-y|,$$

so, that

$$\begin{aligned}t^{1-\alpha}|Ax(t)-Ay(t)|&=t^{1-\alpha}|f(t,x(t))-f(t,y(t)|\\&\le Lt^{1-\alpha}|x(t)-y(t)|\\&\le L\|x-y\|_\alpha\end{aligned}$$

for all $t\in J$. Taking the supremum over the interval J we obtain

$$\|Ax-Ay\|_\alpha\le L\|x-y\|_\alpha,$$

for all $x,y\in E$. So A is a Lipschitz on E with Lipschitz constant L.
`Claim2`. The operator B is completely continuous on S, i.e.,(b) of Theorem 3.1 holds.
First we show that B is continuous on S. Let $\{x_n\}$ be a sequence in S converging to a point $x\in S$. Then by

Lebesgue dominated convergence theorem

$$\begin{aligned}
\lim_{n\to+\infty} t^{1-\alpha}Bx_n(t) &= \lim_{n\to+\infty}\left(x_0 + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,x_n(s))ds\right)\\
&= x_0 + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lim_{n\to+\infty} g(s,x(s))ds\\
&= x_0 + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,x(s))ds\\
&= t^{1-\alpha}Bx(t),
\end{aligned}$$

for all $t \in J$. This shows that B is a continuous operator on S.
`Claim3`. B is a compact operator on S.
We show that $B(S)$ is a uniformly bounded set in E. Let $x \in S$. Then by hypothesis (A_2),

$$\begin{aligned}
t^{1-\alpha}|Bx(t)| &= \left|x_0 + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,x(s))ds\right|\\
&\le \|x_0\| + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|g(s,x(s))|ds\\
&\le \|x_0\| + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|h(s)|ds\\
&\le \|x_0\| + \frac{T}{\Gamma(\alpha+1)}\|h\|_{L^1},
\end{aligned}$$

for all $x \in S$. This shows that B is uniformly bounded on S.

On the other hand, let $t_1, t_2 \in J'$ with $0 < t_1 < t_2$, then for any $x \in S$, one has

$$\left|t_2^{1-\alpha}Bx(t_2) - t_1^{1-\alpha}Bx(t_1)\right|$$

$$\leq \frac{(t_2^{1-\alpha} - t_1^{1-\alpha})}{\Gamma(\alpha)}\left|\int_0^{t_1}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]g(s,x(s))ds\right|$$

$$+\frac{t_2^{1-\alpha}}{\Gamma(\alpha)}\left|\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}g(s,x(s))ds\right|$$

$$\leq \frac{(t_2^{1-\alpha} - t_1^{1-\alpha})}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}]|h(s)|ds$$

$$+\frac{t_2^{1-\alpha}}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}|h(s)|ds$$

$$\left|t_2^{1-\alpha}Bx(t_2) - t_1^{1-\alpha}Bx(t_1)\right| \leq \frac{(t_2^{1-\alpha} - t_1^{1-\alpha})}{\Gamma(\alpha+1)}\|h\|_{L^1}[(t_2-t_1)^\alpha + (t_1^\alpha - t_2^\alpha)]$$

$$+\frac{t_2^{1-\alpha}\|h\|_{L^1}}{\Gamma(\alpha+1)}(t_2-t_1)^\alpha.$$

As $t_2 \to t_1$ the right-hand side of above expression tends to zero independently of $x \in S$. This shows that $B(S)$ is an equicontinuous set in E. Now the set $B(S)$ is uniformly bounded and equicontinuous set in E so it is compact by the Arzelá-Ascoli theorem. As a result B is a complete continuous operator on S.
Claim4. The hypothesis (c) of Theorem 3.1 is satisfied.
Let $x \in E$ and $y \in S$ be arbitray such that $x = AxBx$. Then

$$t^{1-\alpha}|x(t)| = t^{1-\alpha}|Ax(t)By(t)|$$

$$= t^{1-\alpha}|Ax(t)||By(t)|$$

$$= |f(t,x(t))|\left|x_0 + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s,x(s))ds\right|$$

$$\leq \gamma\left(\|x_0\| + \frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|h(s)|ds\right)$$

$$\leq \gamma\left(\|x_0\| + \frac{T\|h\|_{L^1}}{\Gamma(\alpha+1)}\right).$$

Taking supremum for $t \in J$, we obtain

$$\|x\|_\alpha \leq \gamma\left(\|x_0\| + \frac{T\|h\|_{L^1}}{\Gamma(\alpha+1)}\right) \leq N,$$

that is, $x \in S$.
Claim.5 Now we show that $Mk < 1$ that is, (d) of Theorem 3.1 holds.

This is obvious by (5), since we have $M = \|B(S)\| = \sup\{\|Bx\| : x \in S\} \leq \left(\|x_0\| + \dfrac{T\|h\|_{L^1}}{\Gamma(\alpha+1)}\right)$ and $k = L$. Thus, all the conditions of Theorem 3.1 are satisfied and hence the operator equation $AxBx = x$ has a solution in S. As a result, IVPFHDE (1) has a solution defined on J. This completes the proof. □

4. Weighted fractional hybrid differential inequalities

We discuss a fundamental result relative to strict inequalities for IVPFHDE (1).

Lemma 4.1. *[31] Let $m \in E$. Suppose that for any $t_1 \in (0, +\infty)$, we have $m(t_1) = 0$ and $m(t) \leq 0$ for $0 \leq t \leq t_1$. Then it follows that*

$$^{RL}D^{\alpha} m(t_1) \geq 0.$$

Theorem 4.2. *Assume that hypothesis (A_0) holds. Suppose that there exist functions $y, z \in E$ such that*

$$^{RL}D^{\alpha}\left(\frac{y(t)}{f(t, y(t))}\right) \leq g(t, y(t)) \quad a.e.\ t \in J' \tag{10}$$

and

$$^{RL}D^{\alpha}\left(\frac{z(t)}{f(t, z(t))}\right) \geq g(t, z(t)) \quad a.e.\ t \in J' \tag{11}$$

$0 < t \leq T$, with one of the inequality being strict. Then

$$y^0 < z^0,$$

where $y^0 = \lim_{t\to 0^+} t^{1-\alpha}y(t)$ and $z^0 = \lim_{t\to 0^+} t^{1-\alpha}z(t)$, implies

$$y(t) < z(t)$$

for all $t \in J$.

Proof. Suppose that inequality (11) holds. Assume that the claim is false. Then, since $y^0 < z^0$ and $t^{1-\alpha}y(t)$ and $t^{1-\alpha}z(t)$ are continuous functions, there exists t_1 such that $0 < t_1 \leq T$ with $y(t_1) = z(t_1)$ and $y(t) < z(t)$, $0 \leq t < t_1$. Define

$$Y(t) = \frac{y(t)}{f(t, y(t))} \quad \text{and} \quad Z(t) = \frac{z(t)}{f(t, z(t))}.$$

Then we have $Y(t_1) = Z(t_1)$, and by virtue of hypothesis (A_0), we get $Y(t) < Z(t)$ for all $0 \leq t < t_1$.
Setting $m(t) = Y(t) - Z(t)$, $0 \leq t \leq t_1$, we find that $m(t) < 0$, $0 \leq t < t_1$ and $m(t_1) = 0$ with $m \in E$. Then, by Lemma 4.1, we have $^{RL}D^{\alpha}m(t_1) \geq 0$. By (10) and (11), we obtain

$$g(t_1, y(t_1)) \geq\ ^{RL}D^{\alpha}Y(t_1) \geq\ ^{RL}D^{\alpha}Z(t_1) > g(t_1, z(t_1)).$$

This is contradiction with $y(t_1) = z(t_1)$. thus the conclusion of the theorem holds and the proof is complete. □

Theorem 4.3. *Assume that hypothesis (A_0) holds. Suppose that there exist functions $y, z \in C_{1-\alpha}(J, \mathbb{R})$ such that*

$$^{RL}D^{\alpha}\left(\frac{y(t)}{f(t, y(t))}\right) \leq g(t, y(t)) \quad a.e.\ t \in J' \tag{12}$$

and

$$^{RL}D^{\alpha}\left(\frac{z(t)}{f(t,y(t))}\right) \geq g(t,z(t)) \quad a.e.\ t \in J', \tag{13}$$

one of the inequalities being strict, and $\lim_{t\to 0^+} t^{1-\alpha}y(t) < \lim_{t\to 0^+} t^{1-\alpha}z(t)$ *, then*

$$\lim_{t\to 0^+} t^{1-\alpha}\frac{y(t)}{f(t,y(t))} < \lim_{t\to 0^+} t^{1-\alpha}\frac{z(t)}{f(t,z(t))} \tag{14}$$

implies

$$y(t) < z(t) \tag{15}$$

for all $t \in J$.

Proof. We have $\lim_{t\to 0^+} t^{1-\alpha}\frac{y(t)}{f(t,y(t))} < \lim_{t\to 0^+} t^{1-\alpha}\frac{z(t)}{f(t,z(t))}$.
This implies $\lim_{t\to 0^+} t^{1-\alpha}\left(\frac{y(t)}{f(t,y(t))} - \frac{z(t)}{f(t,z(t))}\right) < 0$, and by hypothesis (A_0) we have $\lim_{t\to 0^+} t^{1-\alpha}y(t) < \lim_{t\to 0^+} t^{1-\alpha}z(t)$. Hence the application of Theorem 4.2 yields that $y(t) < z(t)$. □

Theorem 4.4. *Assume that the conditions Theorem 4.3 hold with inequalities* (10) *and* (11). *Suppose that there exists a real number* $M > 0$ *such that*

$$g(t,x_1) - g(t,x_2) \leq \frac{M}{1+t^{\alpha}}\left(\frac{x_1}{f(t,x_1)} - \frac{x_2}{f(t,x_2)}\right) \quad a.e.\ t \in J \tag{16}$$

for all $x_1, x_2 \in \mathbb{R}$ *with* $x_1 \geq x_2$. *Then*

$$y^0 < z^0$$

implies, provided $M \leq \Gamma(\alpha+1)$,

$$y(t) < z(t)$$

for all $t \in J$.

Proof. We set $\frac{z_\varepsilon(t)}{f(t,z_\varepsilon(t))} + \varepsilon(1+t^{\alpha})$ for small $\varepsilon > 0$ and let $Z_\varepsilon(t) = \frac{z_\varepsilon(t)}{f(t,z_\varepsilon(t))}$ and $Z(t) = \frac{z(t)}{f(t,z(t))}$ for $t \in J$. So that we have

$$Z_\varepsilon(t) > Z(t) \Rightarrow Z_\varepsilon(t) > z(t).$$

Since $g(t,x_1) - g(t,x_2) \leq \frac{M}{1+t^{\alpha}}\left(\frac{x_1}{f(t,x_1)} - \frac{x_2}{f(t,x_2)}\right)$ and $^{RL}D^{\alpha}\left(\frac{z(t)}{f(t,z(t))}\right) \geq g(t,z(t))$ for all $t \in J$, one has

$$\begin{aligned}
^{RL}D^{\alpha}Z_\varepsilon(t) &= {}^{RL}D^{\alpha}Z(t) + \varepsilon\, {}^{RL}D^{\alpha}t^{\alpha} \\
&\geq g(t,z(t)) + \varepsilon\Gamma(\alpha+1) \\
&\geq g(t,z_\varepsilon(t)) - \frac{M}{1+t^{\alpha}}(Z_\varepsilon - Z) + \varepsilon\Gamma(\alpha+1) \\
&\geq g(t,z_\varepsilon(t)) + \varepsilon(\Gamma(\alpha+1) - M) \\
&> g(t,z_\varepsilon(t))
\end{aligned}$$

provided $M \leq \Gamma(\alpha+1)$. Also, we have $z_\varepsilon^0 > z^0 \geq y^0$. Hence, the application of Theorem 4.2 yields that $y(t) < z_\varepsilon(t)$ for all $t \in J$. By the arbitrariness of $\varepsilon > 0$, taking the limits as $\varepsilon \to 0$, we have $y(t) \leq z(t)$ for all

$t \in J$.
This completes the proof. $\square$

5. Existence of maximal and minimal solutions

In this section, we shall prove the existence of maximal and minimal solutions for IVPFHDE (1) on J. We need the following definition in what follows.

Definition 5.1. *A solution r of IVPFHDE (1) is said to be maximal if for any other solution x to IVPFHDE (1) one has $x(t) \leq r(t)$ for all $t \in J$. Similarly, a solution ρ of IVPFHDE (1) is said to be minimal if $\rho(t) \leq x(t)$ for all $t \in J$, where x is any solution of IVPFHDE (1) on J.*

We discuss the case of maximal solution only, as the case of minimal solution is similar and can be obtained with the same arguments with appropriate modifications. Given an arbitrarily small real number $\varepsilon > 0$, consider the following initial value problem of IVPFHDE of order $0 < \alpha < 1$:

$$\begin{cases} {}^{RL}D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)) + \varepsilon \quad \text{a.e. } t \in J', \\ \\ I^{1-\alpha}\left(\frac{x(t)}{f(t,x(t))}\right)\Big|_{t=0} = x_0 + \varepsilon, \end{cases} \tag{17}$$

where $f \in C(J \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and $g \in C(J \times \mathbb{R}, \mathbb{R})$.
An existence theorem for IVPFHDE (17) can be stated as follows.

Theorem 5.2. *Assume that hypotheses (A_0)-(A_3) hold. Suppose that inequality (5) holds. Then, for every small $\varepsilon > 0$, IVPFHDE (17) has a solution defined on J.*

Proof. By hypothesis, since

$$L\left(\|x_0\| + \frac{T\|h\|_{L^1}}{\Gamma(\alpha+1)}\right) < 1$$

there exists $\varepsilon_0 > 0$ such that

$$L\left(\|x_0\| + \frac{T(\|h\|_{L^1} + \varepsilon)}{\Gamma(\alpha+1)}\right) < 1$$

for all $0 < \varepsilon \leq \varepsilon_0$. Now the rest of the proof is similar to Theorem 4.4. $\square$

Our main existence theorem for maximal solution for IVPFHDE (1) is following.

Theorem 5.3. *Assume that hypotheses (A_0)-(A_3) hold with the conditions of Theorem 4.2. Furthermore, if condition (5) holds, then IVPFHDE (1) has a maximal solution defined on J.*

Proof. Let $\{\varepsilon\}_0^{\infty}$ be a decreasing sequence of positive real numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$, where ε_0 is a positive real number satisfying the inequality

$$L\left(\|x_0\| + \frac{T(\|h\|_{L^1} + \varepsilon)}{\Gamma(\alpha+1)}\right) < 1.$$

The number ε_0 exists in view of inequality (5). By Theorem 5.2, there exists a solution $r(t, \varepsilon_n)$ defined on J of the IVPFHDE

$$\begin{cases} {}^{RL}D^{\alpha}\left(\frac{x(t)}{f(t,x(t))}\right) = g(t,x(t)) + \varepsilon_n \quad \text{a.e. } t \in J', \\ \\ I^{1-\alpha}\left(\frac{x(t)}{f(t,x(t))}\right)\Big|_{t=0} = x_0 + \varepsilon_n. \end{cases} \tag{18}$$

Then any solution u of IVPFHDE (1) satisfies

$$^{RL}D^{\alpha}\left(\frac{x(t)}{f(t,u(t))}\right) \leq g(t,u(t)),$$

and any solution of auxiliary problem (18) satisfies

$$^{RL}D^{\alpha}\left(\frac{r(t,\varepsilon_n)}{f(t,r(t,\varepsilon_n))}\right) = g(t,r(t,\varepsilon_n)) + \varepsilon_n > g(t,r(t,\varepsilon_n)),$$

where $\lim_{t\to 0^+}\left(\frac{x(t)}{f(t,x(t))}\right) = x_0 \leq x_0 + \varepsilon_n = \lim_{t\to 0^+}\left(\frac{r(t,\varepsilon_n)}{f(t,r(t,\varepsilon_n))}\right)$. By Theorem 4.2, we infer that

$$u(t) \leq r(t,\varepsilon_n) \tag{19}$$

for all $t \in J$ and $n \in \mathbb{N}$. Since

$$\begin{aligned} x_0 + \varepsilon_2 &= \lim_{t\to 0^+}\left(\frac{r(t,\varepsilon_2)}{f(t,r(t,\varepsilon_2))}\right) \\ &\leq \lim_{t\to 0^+}\left(\frac{r(t,\varepsilon_1)}{f(t,r(t,\varepsilon_1))}\right) = x_0 + \varepsilon_1, \end{aligned}$$

then by Theorem 4.2, we infer that $r(t,\varepsilon_2) \leq r(t,\varepsilon_1)$. Therefore, $r(t,\varepsilon_n)$ is decreasing sequence of positive real numbers, and limit

$$r(t) = \lim_{n\to\infty} r(t,\varepsilon_n) \tag{20}$$

exists. We show that the convergence in (20) is uniform on J. To finish, it is enough to prove that sequence $r(t,\varepsilon_n)$ is equicontinuous in $C_{1-\alpha}(J,\mathbb{R})$. Let $t_1, t_2 \in J$ with $t_1 < t_2$ be arbitrary. Then

$$\begin{aligned} &|t_2^{1-\alpha} r(t_2,\varepsilon_n) - t_1^{1-\alpha} r(t_1,\varepsilon_n)| \\ &= \Bigg|[f(t_2,r(t_2,\varepsilon_n))]\left((\|x_0\| + \varepsilon_n) + \frac{t_2^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_2}(t_2-s)^{\alpha-1}(g(s,r(s,\varepsilon_n)) + \varepsilon_n)ds\right) \\ &-[f(t_1,r(t_1,\varepsilon_n))]\left((\|x_0\| + \varepsilon_n) + \frac{t_1^{1-\alpha}}{\Gamma(\alpha)}\int_0^{t_1}(t_1-s)^{\alpha-1}(g(s,r(s,\varepsilon_n)) + \varepsilon_n)ds\right)\Bigg|. \end{aligned}$$

Thus

$$
\begin{aligned}
&|t_2^{1-\alpha} r(t_2,\varepsilon_n) - t_1^{1-\alpha} r(t_1,\varepsilon_n)| \\
&= \Big| (f(t_2, r(t_2,\varepsilon_n)) - f(t_1, r(t_1,\varepsilon_n)))(\|x_0\| + \varepsilon_n) \\
&+ [f(t_2, r(t_2,\varepsilon_n))] \left(\frac{t_2^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} (g(s, r(s,\varepsilon_n)) + \varepsilon_n) ds \right) \\
&- [f(t_1, r(t_1,\varepsilon_n))] \left(\frac{t_1^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} (g(s, r(s,\varepsilon_n)) + \varepsilon_n) ds \right) \Big| \\
&\leq |(f(t_2, r(t_2,\varepsilon_n)) - f(t_1, r(t_1,\varepsilon_n)))(\|x_0\| + \varepsilon_n)| \\
&+ \Big| [f(t_2, r(t_2,\varepsilon_n))] \left(\frac{t_2^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} (g(s, r(s,\varepsilon_n)) + \varepsilon_n) ds \right) \\
&- [f(t_1, r(t_1,\varepsilon_n))] \left(\frac{t_1^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} (g(s, r(s,\varepsilon_n)) + \varepsilon_n) ds \right) \Big| \\
&\leq |(f(t_2, r(t_2,\varepsilon_n)) - f(t_1, r(t_1,\varepsilon_n)))| \left[\|x_0\| + \varepsilon_n + \frac{T(\|h\|_\alpha + \varepsilon_n)}{\Gamma(\alpha+1)} \right] \\
&+ F \frac{(t_2^{1-\alpha} - t_1^{1-\alpha})}{\Gamma(\alpha+1)} (\|h\|_{L^1} + \varepsilon_n)[(t_2 - t_1)^\alpha + (t_1^\alpha - t_2^\alpha)] \\
&+ F \frac{t_2^{1-\alpha}}{\Gamma(\alpha+1)} (\|h\|_{L^1} + \varepsilon_n)(t_2 - t_1)^\alpha,
\end{aligned}
$$

where $F = \sup_{(t,x)\in J\times[-N,N]} |f(t,x)|$.
Since f is continuous on a compact set $J \times [-N, N]$, it is uniformly continuous there. Hence

$$|f(t_2, r(t_2,\varepsilon_n)) - f(t_1, r(t_1,\varepsilon_n))| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $n \in \mathbb{N}$. Therefore, from the above inequality, it follows that

$$|t_2^{1-\alpha} r(t_2,\varepsilon_n) - t_1^{1-\alpha} r(t_1,\varepsilon_n)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $n \in \mathbb{N}$. Therefore,

$$r(t,\varepsilon_n) \to r(t) \quad \text{as} \quad n \to \infty \ \text{ for all } t \in J$$

Next, we show that function $r(t)$ is a solution of IVPFHDE (1) defined on J. Now, since $r(t,\varepsilon_n)$ is a solution of IVPFHDE (18), we have

$$r(t,\varepsilon_n) = [f(t,r(t,\varepsilon_n))]\Big(t^{\alpha-1}(x_0+\varepsilon_n)$$
$$+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(g(s,r(s,\varepsilon_n))+\varepsilon_n)ds\Big)$$

for all $t \in J'$. Taking the limit as $n \to \infty$ in the above equation yields

$$r(t) = [f(t,r(t))]\left(t^{\alpha-1}x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(g(s,r(s))ds\right)$$

for all $t \in J'$. Thus, the function r is a solution of IVPFHDE (1). Finally, from inequality (19) it follows that $u(t) \le r(t)$ for all $t \in J$. Hence, IVPFHDE (1) has a maximal solution on J. This completes the proof. □

6. Comparison theorems

The main problem of the differential inequalities is to estimate a bound for the solution set for the differential inequality related to IVPFHDE (1). In this section, we prove that the maximal and minimal solutions serve as bounds for the solutions of the related differential inequality to IVPFHDE (1) on J.

Theorem 6.1. *Assume that hypotheses* (A_0)-(A_3) *and condition* (5) *hold. Suppose that there exists a real number* $M > 0$ *such that*

$$g(t,x_1) - g(t,x_2) \le \frac{M}{1+t^\alpha}\left(\frac{x_1}{f(t,x_1)} - \frac{x_2}{f(t,x_2)}\right) \quad a.e.\ t \in J$$

for all $x_1, x_2 \in \mathbb{R}$ *with* $x_1 \ge x_2$, *where* $M \le \Gamma(\alpha+1)$. *Furthermore, if there exists a function* $u \in C_{1-\alpha}(J, \mathbb{R}$ *such that*

$$\begin{cases} {}^{RL}D^\alpha\left(\frac{u(t)}{f(t,u(t))}\right) \le g(t,u(t)) \quad a.e.\ t \in J' \\ I^{1-\alpha}\left(\frac{u(t)}{f(t,u(t))}\right)\Big|_{t=0} \le x_0, \end{cases} \tag{21}$$

then

$$u(t) \le r(t) \tag{22}$$

for all $t \in J$, *where* r *is a maximal solution of IVPFHDE* (1) *on* J.

Proof. Let $\varepsilon > 0$ be arbitrarily small. By Theorem 5.3, $r(t,\varepsilon)$ is a maximal solution of IVPFHDE (17) so that the limit

$$r(t) = \lim_{\varepsilon \to 0} r(t,\varepsilon) \tag{23}$$

is uniform on J and the function r is a maximal solution of IVPFDE (1) on J. Hence, we obtain

$$\begin{cases} {}^{RL}D^\alpha\left(\frac{r(t,\varepsilon)}{f(t,r(t,\varepsilon))}\right) = g(t,r(t,\varepsilon)) + \varepsilon \quad \text{a.e. } t \in J' \\ I^{1-\alpha}\left(\frac{r(t,\varepsilon)}{f(t,r(t,\varepsilon))}\right)\Big|_{t=0} = x_0 + \varepsilon. \end{cases} \tag{24}$$

Form the above inequality it follows that

$$\begin{cases} {}^{RL}D^{\alpha}\left(\frac{r(t,\varepsilon)}{f(t,r(t,\varepsilon))}\right) > g(t, r(t,\varepsilon)) \quad \text{a.e. } t \in J' \\ I^{1-\alpha}\left(\frac{r(t,\varepsilon)}{f(t,r(t,\varepsilon))}\right)\Big|_{t=0} = x_0 + \varepsilon. \end{cases} \tag{25}$$

Now we apply Theorem 4.4 to inequalities (21) and (25), conclude that $u(t) < r(t,\varepsilon)$ for all $t \in J$. This, in view of limit (23), further implies that inequality (22) holds on J. This completes the proof. □

Theorem 6.2. *Assume that hypotheses* (A_0)-(A_3) *and condition* (5) *hold. Suppose that there exists a real number* $M > 0$ *such that*

$$g(t,x_1) - g(t,x_2) \le \frac{M}{1+t^{\alpha}}\left(\frac{x_1}{f(t,x_1)} - \frac{x_2}{f(t,x_2)}\right) \quad \text{a.e. } t \in J$$

for all $x_1, x_2 \in \mathbb{R}$ *with* $x_1 \ge x_2$, *where* $M \le \Gamma(\alpha+1)$. *Furthermore, if there exists a function* $u \in C_{1-\alpha}(J, \mathbb{R}$ *such that*

$$\begin{cases} {}^{RL}D^{\alpha}\left(\frac{v(t)}{f(t,v(t))}\right) \ge g(t, v(t)) \quad \text{a.e. } t \in J' \\ I^{1-\alpha}\left(\frac{v(t)}{f(t,v(t))}\right)\Big|_{t=0} > x_0, \end{cases} \tag{26}$$

then

$$\rho(t) \le v(t)$$

for all $t \in J$, *where* ρ *is a minimal solution of IVPFHDE* (1) *on* J.

7. Existence of extremal solutions in vector segment

Sometimes it is desirable to have knowlege of the existence of extremal positive solutions for IVPFHDE (1) on J. In this section, we shall prove the existence maximal and minimal positive solutions for IVPFHDE (1) between the given upper and lower solutions on J. We use a hybrid fixed point theorem of Dhage [15] in ordered Banach spaces for establishing our results. We need the following preliminaries in what follows. A nonempty closed set K in a Banach algebra E is called a cone with vertex 0 if

(*i*) $K + K \subseteq K$,
(*ii*) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}$, $\lambda \ge 0$,
(*iii*) $(-K) \cap (K) = 0$, where 0 is the zero element of E.
(*iv*) a cone K is called positive if $K \circ K \subseteq K$, where $\circ$ is a multiplication composition in E.

We introduce an order relation $\le$ in E as follows. Let $x, y \in E$. Then $x \le y$ if and only if $y - x \in K$. A cone K is said to be normal if the norm $\|\cdot\|_{\alpha}$ is semi-monotone increasing on K, that is there is a constant $N > 0$ such that $\|x\| \le N\|y\|_{\alpha}$ for all $x, y \in K$ with $x \le y$. It is known that if the cone K is normal in E, then every order-bounded set in E is norm-bounded. The details of cones and their properties appear in Heikkila al.[17].

Lemma 7.1. *[15] Let K be a positive cone in a real Banach algebra E and let* $u_1, u_2, v_1, v_2 \in K$ *be such that* $u_1 \le v_1$ *and* $u_2 \le v_2$. *Then* $u_1u_2 \le v_1v_2$. *For any* $a, b \in E$, *the order interval* $[a,b]$ *is a set in E. given by*

$$[a,b] = \{x \in E : \ a \le x \le b\}.$$

Definition 7.2. *[32] A mapping* $Q : [a,b] \to E$ *is said to be nondecreasing or monotone increasing if* $x \le y$ *implies* $Qx \le Qy$ *for all* $x, y \in [a,b]$.

We use the following fixed point theorems of Dhage [15] for proving the existence of extremal solutions for problem (1) under certain monotonicity conditions.

Lemma 7.3. *[15] Let K be a cone in a Banach algebra E and let $a, b \in E$ such that $a \leq b$. Suppose that $A, B : [a, b] \to K$ are two nondecreasing operators such that*

- (a) *A is Lipschitzian with a Lipschitz constant k,*
- (b) *B is complete,*
- (c) *$AxBx \in [a, b]$ for each $x \in [a, b]$.*

Further, if the cone K is positive and normal, then the operator equation $AxBx = x$ has the least and the greatest positive solution in$[a, b]$, whenever $kM < 1$, where $M = \|B([a, b])\| = \sup\{\|Bx\| : x \in [a, b]\}$.

We equip the space $C_{1-\alpha}(J, \mathbb{R})$ with the order relation $\leq$ with the help of cone K defined by

$$K = \{x \in C_{1-\alpha}(J, \mathbb{R}) : \ x(t) \geq 0, \ \forall t \in J\}. \tag{27}$$

It is well known that the cone K is positive and normal in $C_{1-\alpha}(J, \mathbb{R})$. We need the following definitions in what follows.

Definition 7.4. *A function $a \in C_{1-\alpha}(J, \mathbb{R})$ is called a lower solution of IVPFHDE* (1) *defined on J if it satisfies* (12). *Similarly, a function $a \in C_{1-\alpha}(J, \mathbb{R})$ is called an upper solution of IVPFHDE* (1) *defined on J if it satisfies* (13). *A solution to IVPFHDE* (1) *is a lower as well as an upper solution for IVPFHDE* (1) *defined on J and vice versa.*

We consider the following set of assumptions:

(F_0) $f : J \times \mathbb{R} \to \mathbb{R}_+ - 0$, $g : J \times \mathbb{R} \to \mathbb{R}_+$.

(F_1) IVPFHDE (1) has a lower solution a and an upper solution b defined on J with $a \leq b$.

(F_2) the function $x \to \frac{x}{f(t,x(t))}$ is increasing in the iterval $[\min_{t\in J} a(t), \max_{t \in J} b(t)]$ almost everywhere for $t \in J$.

(F_3) the functions $f(t, x)$ and $g(t, x)$ are nondecreasing in x almost everywhere for $t \in J$.

(F_4) there exists a function $k \in L^1(J, \mathbb{R})$ such that $g(t, b(t)) \leq k(t)$.

We remark that hypothesis (F_4) holds in particular if f is continuous and $g \in L^1$-Carathéodory on $J \times \mathbb{R}$.

Theorem 7.5. *Suppose that assumptions* (A_1) *and* (F_0)-(F_4) *hold. Furthermore, if*

$$L\left(x_0 + \frac{T\|k\|_{L^1}}{\Gamma(\alpha+1)}\right) < 1, \quad \textit{and} \ \ x_0 > 0, \tag{28}$$

then IVPFHDE (1) *has a minimal and a maximal positive solution defined on J.*

Proof. Now, IVPFHDE (1) is equivalent to integral equation (6) defined on J. Let $E = C_{1-\alpha}(J, \mathbb{R})$. Defined two operators A and B on E by (7) and (8), respectively. Then the intgral equations (6) is transformed into an operator equation $Ax(t)Bx(t) = x(t)$ in the Banach algebra E. Notice that hypothesis (F_0) implies $A, B : [a, b] \to K$. Since the cone K in E is normal, $[a, b]$ is a norm-bounded set in E. Now it is shown, as in the proof of Theorem 4.4, that A is a Lipschitzian with the Lipschitz constant L and B is a completely continuous operator on $[a, b]$. Again, hypothesis (F_3) implies that A and B are nondecreasing on $[a, b]$. To see this, let $x, y \in [a, b]$ be such that $x \leq y$. Then, by hypothesis (F_3),

$$Ax(t) = f(t, x(t)) \leq f(t, y(t)) = Ay(t)$$

for all $t \in J$. Similarly, we have

$$Bx(t) = t^{\alpha-1}x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,x(s))ds$$

$$\leq t^{\alpha-1}x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,y(s))ds$$

$$= By(t)$$

for all $t \in J'$. So A and B are nondecreasing operators on $[a,b]$. Lemma 7.3 and hypothesis (F_3) together imply that

$$a(t) \leq f(t,a(t))\left(t^{\alpha-1}x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,x(s))ds\right)$$

$$\leq (f(t,x(t))\left(t^{\alpha-1}x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,x(s))ds\right)$$

$$\leq (f(t,b(t))\left(t^{\alpha-1}x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s,x(s))ds\right)$$

$$\leq b(t),$$

for all $t \in J'$ and $x \in [a,b]$. As result $a(t) \leq Ax(t)Bx(t) \leq b(t)$ and $x \in [a,b]$. Hence, $AxBx \in [a,b]$ for all $x \in [a,b]$. Again,

$$M = \|B([a,b])\| = \sup\{\|Bx\| : \ x \in [a,b]\} \leq L\left(x_0 + \frac{T\|k\|_{L^1}}{\Gamma(\alpha+1)}\right) < 1.$$

Now, we apply Lemma 7.3 to operator equation $AxBx = x$ to yield that IVPFHDE (1) has a minimal and a maximal positive solutions in $[a,b]$ defined on J. This completes the proof. □

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