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$p(x,\cdot)$-Kirchhoff type problem involving the fractional $p(x)$-Laplacian operator with discontinuous nonlinearities

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Abstract. The purpose of this paper is mainly to investigate the existence of weak solution of the stationary Kirchhoff type equations driven by the fractional $p(x)$-Laplacian operator with discontinuous nonlinearities for a class of elliptic Dirichlet boundary value problems. By using the topological degree based on the abstract Hammerstein equation, we conduct our existence analysis. The fractional Sobolev space with variable exponent provides an effective functional framework for these situations.

1. Introduction and main result

In this paper we deal with the following fractional Kirchhoff type problem

$$\begin{cases} M([u]^{p(x)}_{s,p(x)})(-\Delta)^s_{p(x)}u(x) + |u(x)|^{q(x)-2}u(x) + \lambda H(x,u) \in -\left[\underline{\phi}(x,u), \overline{\phi}(x,u)\right] & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded open set with Lipschitz boundary and $p : \overline{\Omega} \times \overline{\Omega} \to (1, +\infty)$ be a continuous variable exponent and $0 < s < 1$ and $(-\Delta)^s_{p(x)}$ is the fractional $p(x)$-Laplacian operator defined by

$$(-\Delta)^s_{p(x)}u(x) = p.v. \int_{\mathbb{R}^N \backslash B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))}{|x - y|^{N+sp(x,y)}} dy, \quad x \in \mathbb{R}^N \tag{2}$$

where $p.v.$ is a commonly used abbreviation in the principal value sense and let $p \in C(\mathbb{R}^N \times \mathbb{R}^N)$ satisfying

$$1 < p^- = \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x, y) \leq p(x, y) \leq p^+ = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x, y) < +\infty, \tag{3}$$

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and p is symmetric i.e.

$$p(x,\ y) = p(y,\ x),\ \forall (x,\ y) \in \overline{\Omega} \times \overline{\Omega}; \tag{4}$$

and

$$B_\varepsilon(x) := \Big\{ y \in \mathbb{R}^N \ :\ |x-y| < \varepsilon \Big\}.$$

Let denote by:

$$\widetilde{p}(x) = p(x,\ x),\ \forall x \in \overline{\Omega}.$$

The Kirchhoff function $M : \mathbb{R}^+ \to \mathbb{R}^+$ is assumed to be continuous, nondecreasing and to verify the structural assumption:

(M_1) The Kirchhoff function $M : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous and non-decreasing function, for which there exist two positive constant m_0 and m_1 such that,

$$m_0 t^{\mu(x)-1} \leq M(t) \leq m_1 t^{\mu(x)-1}, \tag{5}$$

where $\mu(x) \in C(\overline{\Omega})$ and $1 \leq \mu^- \leq \mu(x) \leq \mu^+ \leq p^- \leq p(x) \leq p^+$, for all $t \in [0, +\infty[$.

Of course, condition (5) trivially holds in the non-degenerate case, that is, when $M(0) > 0$.
Furthermore, the Carathéodory's functions H is satisfies only the growth condition, for all $s \in \mathbb{R}$ and a.e. $x \in \Omega$.

(H_0) $\qquad |H(x,s)| \leq \varrho(e(x) + |s|^{q(x)-1}),$

where ϱ is a positive constant, $e(x)$ is a positive function in $L^{p'(x)}(\Omega)$.

Now we turn to the main advance of our problem. Kirchhoff [27] investigated an equation of the form

$$\rho \frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0. \tag{6}$$

The Equation (6) is an extension of the classical D'Alembert wave equation and that by considering the effect of the changing in the length of the string during the vibration. Since $\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx$ is a nonlocal coefficient which depends on the average $\frac{1}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx$, (6) is no longer a pointwise equation. The Equation (1) is called a nonlocal problem because of the term M.

Corrêa and Figueieredo [4] proved the existence of positive solutions to the class of nonlocal boundary problems

$$-\left[M\left(\int_\Omega |\nabla u|^p dx\right)\right]^{p-1} \Delta_p u = f(x,u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega$$

and

$$-\left[M\left(\int_\Omega |\nabla u|^p dx\right)\right]^{p-1} \Delta_p u = f(x,u) + \lambda |u|^{s-2} u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

via variational methods (see also [1, 5]). An existence result of three nontrivial weak solutions for the following fractional $p(x,.)$-Kirchhoff type problem

$$\begin{aligned} M\Big(\int_{\Omega\times\Omega} \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}} \quad dxdy\Big)(-\Delta)^s_{p(x,\cdot)} u & \\ = \lambda B\left(\int_\Omega F(x,u)dx\right) f(x,u) + \mu g(x,u) & \qquad \text{in } \Omega \\ u \quad = 0 \qquad\qquad & \qquad \text{in } \mathbb{R}^N \backslash \Omega, \end{aligned} \tag{7}$$

was proved by Azroul et al. in [2], based on the general three critical points theorem obtained by B. Ricceri. In [16] we have proved the existence of weak solution via topological degree based on the abstract Hammerstein equation for the fractional $p(x)$-Laplacian problems with discontinuous nonlinearities. In the present paper we extend the results of [16] to problem (1), overcoming several diffculties which arise from the facts that the problem is nonlocal and that $M(0)$ could be zero, that is, problem (1) could be degenerate. Hence, the results of this paper are new even in the study of Kirchhoff type problems. In the large literature of degenerate Kirchhoff problems, the transverse oscillations of a stretched string, with nonlocal flexural rigidity, depends continuously on the Sobolev deflection norm of u via $M(\|u\|^2)$. From a physical point of view, the fact that $M(0) = 0$ means that the base tension of the string is zero, a very realistic model. More specifically, M measures the change of the tension on the string caused by the change of its length during the vibration. The presence of the nonlinear coeffecient M is crucial to be considered when the changes in tension during the motion cannot be neglected.

Further, the main challenges in proving the presence of nontrivial weak solutions are represented in the following aspect: we cannot naturally employ topological degree methods because the nonlinear term ϕ is discontinuous. We shall adapt this Dirichlet boundary value issue involving the fractional $p(.)$-Laplacian operator with discontinuous nonlinearities into a new one guided by a Hammerstein equation to overcome the discontinuous difficulty. Then, based on the Berkovits-Tienari degree [8], we will use Kim's topological degree theory for a class of weakly upper semi-continuous locally bounded set-valued operators of (S_+) type in the framework of real reflexive separable Banach spaces [25, 26]. To this end, we always assume that $\phi : \Omega \times \mathbb{R} \to \mathbb{R}$ is a possibly discontinuous function, we "fill the discontinuity gaps" of ϕ, replacing ϕ by an interval $\left[\underline{\phi}(x,u), \overline{\phi}(x,u)\right]$, where

$$\underline{\phi}(x,s) = \liminf_{\eta \to s} \phi(x,\eta) = \lim_{\delta \to 0^+} \inf_{|\eta - s| < \delta} \phi(x,\eta),$$

$$\overline{\phi}(x,s) = \limsup_{\eta \to s} \phi(x,\eta) = \lim_{\delta \to 0^+} \sup_{|\eta - s| < \delta} \phi(x,\eta),$$

such that

(H_1) $\overline{\phi}$ and $\underline{\phi}$ are super-positionally measurable (i.e, $\overline{\phi}(\cdot, u(\cdot))$ and $\underline{\phi}(\cdot, u(\cdot))$ are measurable on Ω for every measurable function $u : \Omega \to \mathbb{R}$).

(H_2) ϕ satisfies the growth condition:

$$|\phi(x,s)| \leq d(x) + c(x)|s|^{\zeta(x)-1},$$

for almost all $x \in \Omega$ and all $s \in \mathbb{R}$, where $b \in L^{\zeta'(x)}(\Omega)$, $c \in L^\infty(\Omega)$, where $1 < \zeta(x) < p(x)$ for all $x \in \overline{\Omega}$.

First of all, we define the operator $\mathcal{N}$ acting from $W_0^{s,p(x,y)}(\Omega)$ into $2^{\left(W_0^{s,p(x,y)}(\Omega)\right)^*}$ by

$$\mathcal{N}u = \Big\{\vartheta \in \left(W_0^{s,p(x,y)}(\Omega)\right)^* \mid \exists\, h \in L^{p'(x)}(\Omega) \text{ such that } \underline{\phi}(x,u(x)) \leq h(x) \leq \overline{\phi}(x,u(x)) \ \text{ a.e. } x \in \Omega$$
$$\text{and } \ \langle \vartheta, v\rangle = \int_\Omega hv\,dx \quad \forall\, v \in W_0^{s,p(x,y)}(\Omega)\Big\}.$$

In this spirit, we consider $Q : W_0^{s,p(x,y)}(\Omega) \longrightarrow \left(W_0^{s,p(x,y)}(\Omega)\right)^*$ such that

$$\langle Qu, v\rangle = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp(x,y)}}\,dxdy, \tag{8}$$

for all $v \in W_0^{s,p(x,y)}(\Omega)$ and the operator $A : W_0 \to W_0^*$ setting by

$\langle Au, v\rangle = \displaystyle\int_\Omega |u(x)|^{q(x)-2}(u(x)v(x) + \lambda G(x,u))v(x)dx, \forall u, v \in W_0$, where spaces $W_0 := W_0^{s,p(x,y)}(\Omega)$ will be introduced in Section 2.

Next, we give the definition of weak solutions for problem (1).

Definition 1.1. *A function $u \in W_0^{s,p(x,y)}(\Omega)$ is called a weak solution to problem* (1)*, if there exists an element $\vartheta \in \mathcal{N}u$ verifying*

$$M([u]^{p(x)}_{s,p(x)})\langle Qu, v\rangle + \langle Au, v\rangle + \langle \vartheta, v\rangle = 0, \quad \textit{for all} \quad v \in W_0^{s,p(x,y)}(\Omega).$$

2. Preliminaries and useful properties

To deal with this situation, we introduce the fractional Sobolev space to investigate problem (1). Let us recall some definitions and elementary properties of these spaces. We refer the reader to [6, 14, 15, 20, 21, 24, 30, 32] for further reference.

2.1. Variable exponent Lebesgue spaces.

Consider the set,

$$C_+(\overline{\Omega}) = \{f \in C(\overline{\Omega}) | \inf_{x\in\overline{\Omega}} f(x) > 1\}.$$

For any $f \in C_+(\overline{\Omega})$, we define

$$f^+ := \max\{f(x),\ x \in \overline{\Omega}\}, \quad f^- := \min\{f(x),\ x \in \overline{\Omega}\}.$$

For any $p \in C_+(\overline{\Omega})$ we define the variable exponent Lebesgue spaces

$$L^{p(x)}(\Omega) = \{u;\ u : \Omega \to \mathbb{R} \textit{ is measurable and } \int_\Omega |u(x)|^{p(x)}dx < +\infty\}.$$

Endowed with Luxemburg norm

$$||u||_{p(x)} = \inf\{\lambda > 0 \ :\ \rho_{p(\cdot)}(\frac{u}{\lambda}) \leq 1\}$$

where

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}dx,\ \forall u \in L^{p(x)}.$$

Note that $\left(L^{p(x)}(\Omega), ||.||_{p(x)}\right)$ is a Banach space, separable and reflexive. Its conjugate space is $L^{p'(x)}(\Omega)$ where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1$ for all $x \in \Omega$. In addition, we have the following result.

Proposition 2.1. *([17, 23]) For any $u \in L^{p(x)}(\Omega)$, we have*

(i) $||u||_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \rho_{p(\cdot)}(u) < 1 (= 1; > 1)$,

(ii) $||u||_{p(x)} \geq 1 \Rightarrow ||u||^{p^-}_{p(x)} \leq \rho_{p(\cdot)}(u) \leq ||u||^{p^+}_{p(x)}$,

(iii) $||u||_{p(x)} \leq 1 \Rightarrow ||u||^{p^+}_{p(x)} \leq \rho_{p(\cdot)}(u) \leq ||u||^{p^-}_{p(x)}$.

From this proposition, we can deduce the inequalities

$$||u||_{p(x)} \leq \rho_{p(\cdot)}(u) + 1, \tag{9}$$

$$\rho_{p(\cdot)}(u) \leq ||u||^{p^-}_{p(x)} + ||u||^{p^+}_{p(x)}. \tag{10}$$

If $p, q \in C_+(\overline{\Omega})$ such that $p(x) \leq q(x)$ for any $x \in \overline{\Omega}$, then there exists the continuous embedding

$$L^{q(x)}(\Omega) \to L^{p(x)}(\Omega).$$

2.2. Fractional Sobolev spaces with variable exponent

The definition and some results for fractional Sobolev spaces with variable exponent that were introduced in [6, 24] are presented below.

Let s be a fixed real number such that $0 < s < 1$, and let $q : \overline{\Omega} \to (0, \infty)$ and $p : \overline{\Omega} \times \overline{\Omega} \to (0, \infty)$ be two continuous functions. Furthermore, we suppose that the assumptions (3) and (4) be satisfied, we define the fractional Sobolev space with variable exponent via the Gagliardo approach as follows:

$$W = W^{s,q(x),p(x,y)}(\Omega) = \Big\{u \in L^{q(x)}(\Omega) : \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{N+sp(x,y)}} dxdy < +\infty,\ for\ some\ \lambda > 0\Big\}.$$

We equiped the space W with the norm

$$\|u\|_W = \|u\|_{q(x)} + [u]_{s,p(x,y)},$$

where $[\cdot]_{s,p(x,y)}$ is a Gagliardo semi norm with variable exponent, which is defined by

$$[u]_{s,p(x,y)} = \inf\Big\{\lambda > 0 : \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{\lambda^{p(x,y)}|x - y|^{N+sp(x,y)}} dxdy \leq 1\Big\}.$$

Theorem 2.2. *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and $s \in (0, 1)$. Let $p : \bar{\Omega}\times\bar{\Omega} \longrightarrow (1, +\infty)$ be a continuous function satisfies (2) and (3) with $\mathrm{sp}^+ < N$. Let $r : \bar{\Omega} \longrightarrow (1, +\infty)$ be a continuous variable exponent such that*

$$1 < r^- = \min_{x\in\bar{\Omega}} r(x) \leqslant r(x) < p_s^*(x) = \frac{N\bar{p}(x)}{N - s\bar{p}(x)}\ for\ all\ x \in \bar{\Omega}.$$

Then, there exists a constant $C = C(N, s, p, r, \Omega) > 0$ such that, for any $u \in W$,

$$\|u\|_{L^{r(x)}(\Omega)} \leqslant C\|u\|_W.$$

Thus, the space W is continuously embedded in $L^{r(x)}(\Omega)$. Moreover, this embedding is compact.

The space $(W, \|\cdot\|_W)$ is a Banach space (see [12]), separable and reflexive (see [6, Lemma 3.1]).

Remark 2.3. *Let W_0 denote the closure of $C_0^\infty(\Omega)$ in W with respect to the norm $\|\cdot\|_W$.*

(i) *Theorem 2.2 remains true if we replace W by W_0.*

(ii) *Since $1 < p^- \leqslant \bar{p}(x) < p_s^*(x)$ for all $x \in \bar{\Omega}$, then Theorem 2.2 implies that $[.]s, p(x, y)$ is a norm on W_0, which is equivalent to the norm $\|\cdot\|_W$. So $\big(W_0, [\cdot]_{s,p(x,y)}\big)$ is a Banach space.*

We defne the modular $\rho_{p(\cdot,\cdot)} : W_0 \to \mathbb{R}$ by

$$\rho_{p(\cdot,\cdot)}(u) = \int_{\Omega\times\Omega} \frac{|u(x) - u(y)|^{p(x,y)}}{|x - y|^{N+sp(x,y)}} dxdy, \tag{11}$$

and

$$\|u\|_{p(x,y)} = \inf\left\{\lambda > 0 : \rho_{p(x,y)}\left(\frac{u}{\lambda}\right) \leqslant 1\right\} = [\cdot]_{s,p(x,y)}. \tag{12}$$

The modular ρ_p checks the following results, which is similar to Proposition 2.1(see [33, Lemma 2.1])

Proposition 2.4. *([28]) For any $u \in W_0$ we have*

(i) $\|u\|_{W_0} \geq 1 \Rightarrow \|u\|_{W_0}^{p^-} \leq \rho_{p(\cdot,\cdot)}(u) \leq \|u\|_{W_0}^{p^+}$,

(ii) $\|u\|_{W_0} \leq 1 \Rightarrow \|u\|_{W_0}^{p^+} \leq \rho_{p(\cdot,\cdot)}(u) \leq \|u\|_{W_0}^{p^-}$.

2.3. A brief overview of topological degree theory

Now we will go over topological degree theory, which is one of the most important tools we'll use to analyze our results. We start by defining some classes of mappings.
Let X be a real separable reflexive Banach space with dual X^* and with continuous dual pairing $\langle \cdot, \cdot \rangle$ between X^* and X in this order. The symbol $\rightharpoonup$ stands for weak convergence. Let Y be another real Banach space.

Definition 2.5. *The set-valued operator* $\mathrm{F} : \Omega \subset X \to 2^{\mathrm{Y}}$ *is :*

1. *bounded, if* F *maps bounded sets into bounded sets;*
2. *locally bounded at the point* $u \in \Omega$, *if there is a neighborhood* V *of* u *such that the set* $F(\mathrm{V}) = \bigcup_{u \in V} Fu$ *is bounded.*
3. *upper semicontinuous (u.s.c.) at the point* u, *if, for any open neighborhood* V *of the set* Fu, *there is a neighbhorhood* U *of the point* u *such that* $\mathrm{F}(U) \subseteq V$.
4. *upper semicontinuous (u.s.c) if it is u.s.c at every* $u \in X$.
5. *weakly upper semicontinuous (w.u.s.c.), if* $F^{-1}(U)$ *is closed in* X *for all weakly closed set* U *in* Y.

Definition 2.6. *Let* Ω *be a nonempty subset of* X, $(u_n)_{n\geq 1} \subseteq \Omega$ *and* $F : \Omega \subset X \to 2^{X^*} \backslash \emptyset$. *Then, the set-valued operator* F *is*

1. *of type* (S_+), *if* $u_n \rightharpoonup u$ *in* X *and for each sequence* (h_n) *in* X^* *with* $h_n \in Fu_n$ *such that*
$$\limsup_{n\to\infty} \langle h_n, u_n - u \rangle \leq 0, \tag{13}$$
we get $u_n \to u$ *in* X.
2. *quasi-monotone, if* $u_n \rightharpoonup u$ *in* X *and for each sequence* (w_n) *in* X^* *such that* $w_n \in Fu_n$ *yield*
$$\liminf_{n\to\infty} \langle w_n, u_n - u \rangle \geq 0.$$

Definition 2.7. *Let* Ω *be a nonempty subset of* X *such that* $\Omega \subset \Omega_1$, $(u_n)_{n\geq 1} \subseteq \Omega$ *and* $E : \Omega_1 \subset X \to X^*$ *be a bounded operator. Then, the set-valued operator* $F : \Omega \subset X \to 2^X \backslash \emptyset$ *is of type* $(S_+)_E$, *if*

$$\begin{cases} u_n \rightharpoonup u \text{ in } X, \\ Eu_n \rightharpoonup y \text{ in } X^*, \end{cases}$$

and for any sequence (h_n) *in* X *with* $h_n \in Fu_n$ *such that*

$$\limsup_{n\to\infty} \langle h_n, Eu_n - y \rangle \leq 0,$$

we have $u_n \to u$ *in* X.

Next, we consider the following sets :

$$\mathcal{F}_1(\Omega) := \{F : \Omega \to X^* |\, F \text{ is bounded, demicontinuous and of type } (S_+)\},$$
$$\mathcal{F}_E(\Omega) := \{F : \Omega \to 2^X |\, F \text{ is locally bounded, w.u.s.c. and of type } (S_+)_E\},$$

for any $\Omega \subset D_F$ and each bounded operator $E : \Omega \to X^*$, where D_F denotes the domain of F.

Remark 2.8. *We say that the operator* E *is an essential inner map of* F, *if* $E \in \mathcal{F}_1(\overline{G})$.

Lemma 2.9. *([26, Lemma 1.4]) Let* X *be a real reflexive Banach space and* $G \subset X$ *is a bounded open set. Assume that* $E \in \mathcal{F}_1(\overline{G})$ *is continuous and* $S : D_S \subset X^* \to 2^X$ *weakly upper semicontinuous and locally bounded with* $E(\overline{G}) \subset D_s$. *Then the following alternative holds:*

1. *If S is quasi-monotone, yield $I + S \circ E \in \mathcal{F}_E(\overline{G})$, where I denotes the identity operator.*
2. *If S is of type (S_+), yield $S \circ E \in \mathcal{F}_E(\overline{G})$.*

Definition 2.10. *([26]) Let $E : \overline{G} \subset X \to X^*$ is to be a bounded operator, a homotopy $H : [0,1] \times \overline{G} \to 2^X$ is called of type $(S_+)_E$, if for every sequence (t_k, u_k) in $[0,1] \times \overline{G}$ and each sequence (a_k) in X with $a_k \in H(t_k, u_k)$ such that*

$$u_k \rightharpoonup u \in X, \quad t_k \to t \in [0,1], \quad Eu_k \rightharpoonup y \quad \text{in} \quad X^* \text{ and } \quad \limsup_{k\to\infty} \langle a_k, Eu_k - y\rangle \leq 0,$$

we get $u_k \to u$ in X.

Lemma 2.11. *([26]) Let X be a real reflexive Banach space and $G \subset X$ is a bounded open set, $E : \overline{G} \to X^*$ is continuous and bounded. If F, S are bounded and of class $(S_+)_E$, then an affine homotopy $H : [0,1] \times \overline{G} \to 2^X$ giving by*

$$H(t,u) := (1-t)Fu + tSu, \quad \text{for } (t,u) \in [0,1] \times \overline{G},$$

is of type $(S_+)_E$.

Now we introduce the topological degree for a class of locally bounded, w.u.s.c. operators that satisfy condition $(S_+)_E$ for more details see [18, 19, 22, 26] and properties of operators.

Theorem 2.12. *Let*

$$L = \left\{(F,G,g) |\, G \in \mathcal{O},\ E \in \mathcal{F}_1(\overline{G}),\ F \in \mathcal{F}_E(\overline{G}),\ g \notin F(\partial G)\right\},$$

where $\mathcal{O}$ denotes the collection of all bounded open set in X. There exists a unique (Hammerstein type) degree function

$$d : L \longrightarrow \mathbb{Z}$$

such that the following alternative holds:

1. *(* Normalization*) For each $g \in G$, we have $d(I, G, g) = 1$.*
2. *(* Domain additivity*) Let $F \in \mathcal{F}_E(\overline{G})$. We have*

$$d(F,G,g) = d(F,G_1,g) + d(F,G_2,g),$$

   with G_1, $G_2 \subseteq G$ disjoint open such that $g \notin F(\overline{G} \backslash (G_1 \cup G_2))$.
3. *(* Homotopy invariance*) If $H : [0,1] \times \overline{G} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and g: $[0,1] \to X$ is a continuous path in X such that $g(t) \notin H(t, \partial G)$ for all $t \in [0,1]$, then the value of $d(H(t,\cdot), G, g(t))$ is constant for any $t \in [0,1]$.*
4. *(* Solution property*) if $d(F,G,g) \neq 0$, then the equation $g \in Fu$ has a solution in G.*

Now, define the functional $K : W_0^{s,p(x,y)}(\Omega) \to \mathbb{R}$ by

$$K(u) = \frac{1}{p(x)} M\left([u]_{s,p(x)}^{p(x)}\right),$$

for all $u \in W_0^{s,p(x,y)}(\Omega)$ and let F be its derivative operator, i.e., $F = K' : W_0^{s,p(x,y)}(\Omega) \longrightarrow \left(W_0^{s,p(x,y)}(\Omega)\right)^*$. Moreover, F can be represented as

$$\langle Fu, v\rangle = M\left([u]_{s,p(x)}^{p(x)}\right) \langle Qu, v\rangle_{s,p(x)}. \tag{14}$$

Lemma 2.13. *The functional K is convex, of class $C^1(W_0^{s,p(x,y)}(\Omega))$ and*

$$\langle Fu, v\rangle = M\left([u]_{s,p(x)}^{p(x)}\right) \langle Qu, v\rangle_{s,p(x)} \tag{15}$$

for all $u, v \in W_0^{s,p(x,y)}(\Omega)$. Moreover, K is sequentially weakly lower semicontinuous in $W_0^{s,p(x,y)}(\Omega)$.

Proof. Standard arguments (see, for instance [3, Lemma 3.1] and the continuity of M imply that K is well defined and $K \in C^1(W_0^{s,p(x,y)}(\Omega), \mathbb{R})$. Moreover, for all $u, v \in W_0^{s,p(x,y)}$, its Gâteaux derivative is given by

$$\langle K'u, v\rangle = M\left([u]_{s,p(x)}^{p(x)}\right) \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+sp(x,y)}} dxdy = M\left([u]_{s,p(x)}^{p(x)}\right) \langle Qu, v\rangle_{s,p(x)}.$$

Now, let $\{u_n\} \subset W_0^{s,p(x,y)}$, $u \in W_0^{s,p(x,y)}$ satisfy $u_n \to u$ strongly in $W_0^{s,p(x,y)}$ as $n \to \infty$. Without loss of generality, we assume that $u_n \to u$ a.e. in $\mathbb{R}^N$. Then, the sequence

$$\left\{ \frac{\left|u_n(x)-u_n(y)\right|^{p(x,y)-2}(u_n(x)-u_n(y))}{|x-y|^{(N+sp(x,y))}} \right\}_{n=1}^{\infty}$$

is bounded in $L^{p'}\left(\mathbb{R}^{2N}\right)$ and

$$\mathcal{U}_n(x,y) := \frac{\left|u_n(x)-u_n(y)\right|^{p(x,y)-2}(u_n(x)-u_n(y))}{|x-y|^{(N+sp(x,y))}} \to \mathcal{U}(x,y) := \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{(N+sp(x,y))}},$$

a.e. in $\mathbb{R}^{2N}$. Thus, the Brézis-Lieb lemma (see [7]) implies that

$$\lim_{n\to\infty} \iint_{\mathbb{R}^{2N}} \left|\mathcal{U}_n(x,y) - \mathcal{U}(x,y)\right|^{p'(x)} dxdy = \lim_{n\to\infty} \left([u_n]_{s,p(x)}^{p(x)} - [u]_{s,p(x)}^{p(x)}\right) = 0,$$

since $u_n \to u$ strongly in $W_0^{s,p(x,y)}$. Moreover, $M\left([u_n]_{s,p(x)}^{p(x)}\right) \to M\left([u]_{s,p(x)}^{p(x)}\right)$ by the continuity of M. Hence,

$$\lim_{n\to\infty} M\left([u_n]_{s,p(x)}^{p(x)}\right) \iint_{\mathbb{R}^{2N}} \left|\mathcal{U}_n(x,y) - \mathcal{U}(x,y)\right|^{p'(x)} dxdy = 0. \tag{16}$$

Combining (16), $M\left([u_n]_{s,p(x)}\right) \to M\left([u]_{s,p(x)}^{p(x)}\right)$ with the Hölder inequality, we have as $n \to \infty$

$$\|F(u_n) - F(u)\|_{W'} = \sup_{\varphi\in W, \|\varphi\|_W=1} \left|\langle F(u_n) - F(u), \varphi\rangle\right| \to 0.$$

Hence, $K \in C^1(W_0^{s,p(x,y)}(\Omega))$.
Next, we show that K is convex in $W_0^{s,p(x,y)}(\Omega)$. Since M is nondecreasing and continuous on $\mathbb{R}_0^+$, we have that $\mathcal{M}(t) = \int_0^t M(\tau)d\tau$ is increasing and convex in $\mathbb{R}_0^+$. Moreover, the map $W_0^{s,p(x,y)} \ni v \mapsto [v]_{s,p(x)}^{p(x)}$ is convex, being a seminorm in $W_0^{s,p(x,y)}$. Therefore, for all $u, v \in W_0^{s,p(x,y)}$, we obtain

$$\mathcal{M}\left(\left[\frac{u+v}{2}\right]_{s,p(x)}^{p(x)}\right) \le \mathcal{M}\left(\frac{1}{2}[u]_{s,p(x)}^{p(x)} + \frac{1}{2}[v]_{s,p(x)}^{p(x)}\right) \le \frac{1}{2}\mathcal{M}\left([u]_{s,p(x)}^{p(x)}\right) + \frac{1}{2}\mathcal{M}\left([v]_{s,p(x)}^{p(x)}\right).$$

This means that K is a convex functional in $W_0^{s,p(x,y)}$. Furthermore, [10, Proposition 1.1] implies that K is subdifferentiable and its subdifferential, denoted by ∂K, satisfies $\partial K(v) = \{F(v)\}$ for all $v \in W_0^{s,p(x,y)}$. Now, let $\{v_n\}_n \subset W_0^{s,p(x,y)}$, $v \in W_0^{s,p(x,y)}$, with $v_n \to v$ weakly in $W_0^{s,p(x,y)}$ as $n \to \infty$. Then, it follows from the definition of subdifferential that

$$K(v_n) - K(v) \ge \langle F(v), v_n - v\rangle.$$

Hence, we obtain $K(v) \le \liminf_{n\to\infty} K(v_n)$, that is, K is sequentially weakly lower semi-continuous in $W_0^{s,p(x,y)}$.
□

3. Existence of weak solutions

The existence of weak solutions for the issue (1) in fractional Sobolev spaces is proved using compactness methods (see [13, 29]). We convert elliptic Dirichlet boundary value issues with discontinuous nonlinearities involving the fractional $p(.)$-Laplacian operator into a new problem governed by a Hammerstein equation in this way. We prove the existence of weak solutions to the state issue, which holds under proper assumptions, by using the topological degree theory introduced in Section 2. Now we are in a position to present our main result.

Theorem 3.1. *Assume that ϕ satisfies $(H_1), (H_2)$ and H satisfies (H_0). Then, the problem* (1) *has a weak solution u in $W_0^{s,p(x,y)}(\Omega)$.*

Proof of Theorem 3.1

First, we give several lemmas.

Lemma 3.2. *Let $0 < s < 1$ and $1 < p(x,y) < +\infty$, (or $sp_+ < N$) the operator F defined in* (14) *is*

(a) continuous, bounded and strictly monotone operator.

(b) of type (S_+).

Proof. **(a)** From Lemma 2.13 F is continuous bounded operator. Next we show that F is a strictly monotone operator. To this end, let us now recall the well-known Simon inequalities (see [31]): for all $\xi, \eta \in \mathbb{R}^N$,

$$\begin{cases} |\xi-\eta|^p \le c_p\Big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\Big)(\xi-\eta); & p \ge 2 \\ |\xi-\eta|^p \le C_p\Big[\Big(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\Big)(\xi-\eta)\Big]^{\frac{p}{2}}\left(|\xi|^p+|\eta|^p\right)^{\frac{2-p}{2}}, & 1<p<2, \end{cases}$$

where $c_p = \left(\frac{1}{2}\right)^{-p}$ and $C_p = \frac{1}{p-1}$.

Using these two inequalities and the convexity of K and the property of subdifferentiability imply that F is a monotone operator. Hence,

$$\langle Fu - Fv, u - v\rangle > 0 \tag{17}$$

for all $u, v \in W_0^{s,p(x,y)}(\Omega)$ with $u \neq v$ a.e. in $\mathbb{R}^N$, that is, F is a strictly monotone operator in $W_0^{s,p(x,y)}(\Omega)$ as claimed.

(b) Let $(u_n) \in W_0^{s,p(x,y)}(\Omega)$ be a sequence such that $u_n \rightharpoonup u$ and $\limsup\limits_{n\to\infty}\langle Fu_n - Fu, u_n - u\rangle \le 0$. In view of (a), we get

$$\lim_{n\to\infty}\langle Fu_n - Fu, u_n - u\rangle = 0.$$

Thanks to Proposition 2.1, we obtain

$$u_n(x) \to u(x), \text{ a.e. } x \in \Omega. \tag{18}$$

In the sequel, we denote by $L(x,y) = |x-y|^{-N-sp(x,y)}$.

By Fatou's lemma and (18), we get

$$\liminf_{n\to+\infty}\int_{\Omega\times\Omega}|u_n(x)-u_n(y)|^{p(x,y)}L(x,y)dxdy \ge \int_{\Omega\times\Omega}|u(x)-u(y)|^{p(x,y)}L(x,y)dxdy. \tag{19}$$

On the other hand, from $u_n \rightharpoonup u$ we have

$$\lim_{n\to+\infty}\langle Qu_n,\ u_n - u\rangle = \lim_{n\to+\infty}\langle Qu_n - Qu, u_n - u\rangle = 0. \tag{20}$$

Now, by using Young's inequality, there exists a positive constant c such that

$$\begin{aligned}\langle Qu_n, u_n - u\rangle &= \int_{\Omega\times\Omega} |u_n(x) - u_n(y)|^{p(x,y)} L(x,y)dxdy \\ &- \int_{\Omega\times\Omega} |u_n(x) - u_n(y)|^{p(x,y)-2}(u_n(x) - u_n(y))(u(x) - u(y))L(x,y)dxdy \\ &\geq c\int_{\Omega\times\Omega} |u_n(x) - u_n(y)|^{p(x,y)} L(x,y)dxdy \\ &- c\int_{\Omega\times\Omega} |u(x) - u(y)|^{p(x,y)} L(x,y)dxdy.\end{aligned} \tag{21}$$

Combine (19), (20) and (21), we obtain

$$\lim_{n\to+\infty} \int_{\Omega\times\Omega} |u_n(x) - u_n(y)|^{p(x,y)} L(x,y)dxdy = \int_{\Omega\times\Omega} |u(x) - u(y)|^{p(x,y)} L(x,y)dxdy. \tag{22}$$

And in particular, since the sequence $\left\{M\left([u_n]_{s,p(x)}^{p(x)}\right)\right\}_n$ is bounded,

$$\lim_{n\to\infty} M\left([u_n]_{s,p(x)}^{p(x)}\right)\langle Q(u), u_n - u\rangle = 0. \tag{23}$$

Similarly,

$$\lim_{n\to\infty} M\left([u]_{s,p(x)}^{p(x)}\right)\langle B(u), u_n - u\rangle = 0. \tag{24}$$

It follows from (13) that

$$\lim_{n\to\infty} \left\langle M\left([u_n]_{s,p(x)}^{p(x)}\right) Q(u_n) - M\left([u]_{s,p(x)}^{p(x)}\right) Q(u), u_n - u\right\rangle = 0.$$

Hence,

$$\lim_{n\to\infty} M\left([u_n]_{s,p(x)}^{p(x)}\right)\langle Q(u_n) - Q(u), u_n - u\rangle = 0. \tag{25}$$

By (23) and (24), the assumption (M) implies at once that

$$\lim_{n\to\infty} \langle Q(u_n) - Q(u), u_n - u\rangle = 0.$$

According to (18), (22), (25) and the Brezis-Lieb lemma [9], our result is proved.
□

Proposition 3.3. *([11, Proposition 1]) For any fixed $x \in \Omega$, the functions $\overline{\phi}(x,s)$ and $\underline{\phi}(x,s)$ are upper semicontinuous (u.s.c.) functions on $\mathbb{R}^N$.*

Lemma 3.4. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded open set with smooth boundary. The operator $A : W_0^{s,p(x,y)}(\Omega) \to \left(W_0^{s,p(x,y)}(\Omega)\right)^*$ setting by*

$$\langle Au, v\rangle = \int_\Omega (|u(x)|^{q(x)-2}u(x) + \lambda H(x,u))v dx, \forall u, v \in W_0$$

is compact.

Proof. The proof was broken down into three sections.
Step 1 Let $\mathcal{B}: W_0 \to L^{q'(x)}(\Omega)$ be the operator setting by

$$\mathcal{B}u(x) := -|u(x)|^{q(x)-2}u(x) \quad \text{for} \quad u \in W_0 \quad \text{and} \quad x \in \Omega.$$

It is obvious that $\mathcal{B}$ is continuous. Next we show that $\mathcal{B}$ is bounded. For every $u \in W_0$, we have by the inequalities (9) and (10) that

$$\begin{aligned} \|\mathcal{B}u\|_{q'(x)} &\le \rho_{q'(\cdot)}(\mathcal{B}u) + 1 \\ &= \int_\Omega \left| |u|^{q(x)-1} \right|^{q'(x)} dx + 1 \\ &= \rho_{q(\cdot)}(u) + 1 \\ &\le \|u\|_{q(x)}^{q^-} + \|u\|_{q(x)}^{q^+} + 1. \end{aligned}$$

By the compact embedding $W_0 \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$ we have

$$\|\mathcal{B}u\|_{q'(x)} \le \text{const}\left(\|u\|_{W_0}^{q^-} + \|u\|_{W_0}^{q^+}\right) + 1.$$

This implies that $\mathcal{B}$ is bounded on W_0.
Step 2 We show that the operator $\mathcal{A}$ defined from W_0 into $L^{p'(x)}(\Omega)$ by

$$\mathcal{A}u(x) := -\lambda H(x,u) \quad \text{for} \quad u \in W_0 \quad \text{and} \quad x \in \Omega$$

is bounded and continuous. Let $u \in W_0$, by using the growth condition (H_0) we obtain

$$\begin{aligned} \|\mathcal{A}u\|_{p'(x)}^{p'(x)} &\le \int_\Omega |\lambda H(x,u)|^{p'(x)} dx \\ &\le (\varrho\lambda)^{p'(x)} \int_\Omega \left(|e(x)|^{p'(x)} + |u|^{(q(x)-1)p'(x)}\right) dx \\ &\le (\varrho\lambda)^{p'(x)} \int_\Omega \left(|e(x)|^{p'(x)} + |u|^{(p(x)-1)p'(x)}\right) dx \\ &\le (\varrho\lambda)^{p'(x)} \int_\Omega |e(x)|^{p'(x)} dx + (\varrho\lambda)^{p'(x)} \int_\Omega |u|^{p(x)} dx \\ &\le (\varrho\lambda)^{p'(x)} (\|e\|_{p'(x)}^{p'+} + \|e\|_{p'(x)}^{p'-}) + (\varrho\lambda)^{p'(x)} (\|u\|_{p(x)}^{p+} + \|u\|_{p(x)}^{p-}) \\ &\le C_m (\|u\|_{W_0}^{p+} + \|u\|_{W_0}^{p-} + 1), \end{aligned} \tag{26}$$

where $C_m = \max\left((\varrho\lambda)^{p'(x)}(\|e\|_{p'(x)}^{p'+} + \|e\|_{p'(x)}^{p'-}), (\varrho\lambda)^{p'(x)}\right)$. (Due to $e(x)$ is a positive function in $L^{p'(x)}(\Omega)$).
Therefore $\mathcal{A}$ is bounded on $W^{s,q(x),p(x,y)}(\Omega)$.
Next, we show that $\mathcal{A}$ is continuous, let $u_n \to u$ in $W^{s,q(x),p(x,y)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$. Thus there exist a subsequence still denoted by (u_n) and measurable function α in $L^{p(x)}(\Omega)$ such that

$$u_n(x) \to u(x),$$

$$|u_n(x)| \le \alpha(x),$$

for a.e. $x \in \Omega$ and all $n \in \mathbb{N}$. Since H satisfies the Carathéodory condition, we obtain

$$H(x,u_n(x)) \to H(x,u(x)) \quad \text{a.e. } x \in \Omega. \tag{27}$$

Thanks to (H_0) we obtain

$$|H(x,u_n(x))| \le \varrho\left(e(x) + |\alpha(x)|^{q(x)-1}\right)$$

for a.e. $x \in \Omega$ and for all $k \in \mathbb{N}$.
Since

$$e(x) + |\alpha(x)|^{p(x)-1} \in L^{p'(x)}(\Omega),$$

and from (27), we get

$$\int_\Omega |H(x, u_k(x)) - H(x, u(x))|^{p'(x)} dx \longrightarrow 0,$$

by using the dominated convergence theorem we have

$$\mathcal{A}u_k \to \mathcal{A}u \quad \text{in} \quad L^{p'(x)}(\Omega).$$

Thus the entire sequence $(\mathcal{A}u_n)$ converges to $\mathcal{A}u$ in $L^{p'(x)}(\Omega)$ and then $\mathcal{A}$ is continuous.

Step 3

Since the embedding $I : W_0 \to L^{q(x)}(\Omega)$ is compact, it is known that the adjoint operator $I^* : L^{q'(x)}(\Omega) \to W_0^*$ is also compact. Therefore, the compositions $I^* \circ \mathcal{B}$ and $I^* \circ \mathcal{A} : W_0 \to W_0^*$ are compact. We conclude that $S = I^* \circ \mathcal{B} + I^* \circ \mathcal{A}$ is compact. □

Lemma 3.5. *Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded open set with smooth boundary. If the hypotheses (H_1) and (H_2) hold, then the set-valued operator $\mathcal{N}$ defined above is bounded, upper semicontinuous (u.s.c.) and compact.*

Proof. Let $C : L^{p(x)}(\Omega) \to 2^{L^{p'(x)}(\Omega)}$ be a set-valued operator defined as follows

$$Cu = \Big\{ h \in L^{p'(x)}(\Omega) | \quad \underline{\phi}(x, u(x)) \leq h(x) \leq \overline{\phi}(x, u(x)) \quad a.e.\ x \in \Omega \Big\}.$$

Let $u \in W_0$, by the assumption (H_2) we obtain

$$\max \Big\{ |\underline{\phi}(x,s)| \,;\, |\overline{\phi}(x,s)| \Big\} \leq d(x) + c(x)|s|^{\zeta(x)-1},$$

for all $(x,t) \in \Omega \times \mathbb{R}$ where $1 < \zeta(x) < p(x)$ for all $x \in \overline{\mathbb{R}}$.
As a result

$$\int_\Omega |\overline{\phi}(x, u(x))|^{p'(x)} dx \leq 2^{p'^+ + 1} \Big(\int_\Omega |d(x)|^{p'(x)} dx + \int_\Omega |c|^{p'(x)} |u(x)|^{p(x)} dx \Big).$$

A same inequality is shown for $\underline{\phi}(x,s)$, it follows that the set-valued operator C is bounded on W_0. It remains to prove that C is upper semi-continuous (u.s.c.) of C, i.e.,

$$\forall \varepsilon > 0, \ \exists \delta > 0, \quad ||u - u_0||_p < \delta \ \Rightarrow \ Cu \subset Cu_0 + B_\varepsilon,$$

such that B_ε is the ε-ball in $L^{p'(x)}(\Omega)$.
Come to an end, given $u_0 \in L^{p(x)}(\Omega)$, let us consider the sets

$$G_{m,\varepsilon} = \bigcap_{t \in \mathbb{R}^N} K_t,$$

where

$$K_t = \Big\{ x \in \Omega, \text{ if } |t - u_0(x)| < \frac{1}{n}, \text{ then } [\underline{\phi}(x,t), \overline{\phi}(x,t)] \subset \Big]\underline{\phi}(x, u_0(x)) - \frac{\varepsilon}{R}, \overline{\phi}(x, u_0(x)) + \frac{\varepsilon}{R}\Big[\Big\},$$

n being an integer, $|t| = \max\limits_{1 \leq i \leq N} |t_i|$ and R is a constant to be determined in the following pages. In view of Proposition 3.3, we define the sets of points as follows

$$G_{n,\varepsilon} = \bigcap_{r \in \mathbb{R}_a^N} K_r,$$

where $\mathbb{R}_a^N$ denotes the set of all rational grids in $\mathbb{R}^N$. For any $r = (r_1, \cdots, r_N) \in \mathbb{R}_a^N$,

$$K_r = \Big\{x \in \Omega \mid u_0(x) \in C\prod_{i=1}^{N}\Big]r_i - \frac{1}{n}, r_i + \frac{1}{n}\Big[\Big\} \cup \Big\{x \in \Omega \mid u_0(x) \in \prod_{i=1}^{N}\Big]r_i - \frac{1}{n}, r_i + \frac{1}{n}\Big[\Big\}$$
$$\cap \Big\{x \in \Omega \mid \overline{\phi}(x, r) < \overline{\phi}(x, u_0(x)) + \frac{\varepsilon}{R} \text{ and } \underline{\phi}(x, r) > \underline{\phi}(x, u_0(x)) - \frac{\varepsilon}{R}\Big\},$$

so that K_r and therefore $G_{n,\varepsilon}$ are measurable. It is obvious that

$$G_{1,\varepsilon} \subset G_{2,\varepsilon} \subset \cdots$$

In light of the Proposition 3.3, we have

$$\bigcup_{m=1}^{\infty} G_{n,\varepsilon} = \Omega,$$

therefore there exists $m_0 \in \mathbb{N}$ such that

$$m(G_{n_0,\varepsilon}) > n(\Omega) - \frac{\varepsilon}{R}. \tag{28}$$

But for each $\varepsilon > 0$, there is $\eta = \eta(\varepsilon) > 0$, such that $n(T) < \eta$ yield

$$2^{p'^+-1}\int_T 2|b(x)|^{p'(x)} + c^{p'(x)}(x)(2^{p'^+-1} + 1)|u_0(x)|^{p(x)}dx < \Big(\frac{\varepsilon}{3}\Big)^{p'^+}, \tag{29}$$

because of $b \in L^{p'(x)}(\Omega)$ and $u_0 \in L^{p(x)}(\Omega)$.
Let now

$$0 < \delta < \min\Big\{\frac{1}{n_0}\Big(\frac{\eta}{2}\Big)^{\frac{1}{p^-}}, \frac{1}{2^{p^+-2}}\Big(\frac{\varepsilon}{6C}\Big)^{\frac{p'^+}{\theta}}\Big\}, \tag{30}$$

$$R > \max\Big\{\frac{2\varepsilon}{\eta}, 3\Big(n(\Omega)\Big)^{\frac{1}{p'^-}}\Big\}. \tag{31}$$

where

$$\theta = \begin{cases} p^+ & \text{if} \quad \|u - u_0\|_{p(x)} \leq 1 \\ p^- & \text{if} \quad \|u - u_0\|_{p(x)} \geq 1. \end{cases}$$

Suppose that $\|u - u_0\|_{p(x)} < \delta$ and define the set $G = \{x \in \Omega \setminus |u(x) - u_0(x)| \geq \frac{1}{n_0}\}$, we get

$$n(G) < (n_0\delta)^{p(x)} < \frac{\eta}{2}. \tag{32}$$

If $x \in G_{n_0,\varepsilon}\backslash G$, then, for any $h \in Cu$,

$$|u(x) - u_0(x)| < \frac{1}{n_0}$$

and

$$h(x) \in \Big]\underline{\phi}(x, u_0(x)) - \frac{\varepsilon}{R}, \overline{\phi}(x, u_0(x)) + \frac{\varepsilon}{R}\Big[.$$

Let

$$K^0 = \Big\{x \in \Omega; \quad h(x) \in \Big[\underline{\phi}(x, u_0(x)), \overline{\phi}(x, u_0(x))\Big]\Big\},$$
$$K^- = \Big\{x \in \Omega; \quad h(x) < \underline{\phi}(x, u_0(x))\Big\},$$
$$K^+ = \Big\{x \in \Omega; \quad h(x) > \overline{\phi}(x, u_0(x))\Big\},$$

and

$$w(x)=\begin{cases}\overline{\phi}(x,u_0(x)), & \text{for} \quad x\in K^+;\\ h(x) & \text{for} \quad x\in K^0;\\ \underline{\phi}(x,u_0(x)), & \text{for} \quad x\in K^-.\end{cases}$$

Hence $w\in Cu_0$ and

$$|w(x)-h(x)|<\frac{\varepsilon}{R} \quad \text{for all} \quad x\in G_{n_0,\varepsilon}\backslash G. \tag{33}$$

From (31) and (33), we have

$$\int_{G_{n_0,\varepsilon}\backslash G}|w(x)-h(x)|^{p'(x)}dx<\Big(\frac{\varepsilon}{R}\Big)^{p'^+}n(\Omega)<\Big(\frac{\varepsilon}{3}\Big)^{p'^+}. \tag{34}$$

Assume that V is a coset in Ω of $G_{n_0,\varepsilon}\backslash G$, then $V=(\Omega\backslash G_{n_0,\varepsilon})\cup(G_{n_0,\varepsilon}\cap G)$ and

$$n(V)\le n(\Omega\backslash G_{n_0,\varepsilon})+n(G_{n_0,\varepsilon}\cap G)<\frac{\varepsilon}{R}+n(G)<\eta.$$

According to (28), (31) and (32). From (H_2), (29) and (30), we obtain

$$\begin{aligned}\int_V|w(x)-h(x)|^{p'(x)}dx&\le\int_V|w(x)|^{p'(x)}+|h(x)|^{p'(x)}dx\\ &\le 2^{p'^+-1}\Big(\int_V|d(x)|^{p'(x)}+c^{p'(x)}(x)|u_0(x)|^{p(x)}+|d(x)|^{p'(x)}+c^{p'}(x)|u(x)|^{p(x)}dx\Big)\\ &\le 2^{p'^+-1}\Big(\int_V 2|d(x)|^{p'(x)}+c^{p'(x)}(x)(2^{p^+-1}+1)|u_0(x)|^{p(x)}dx\Big)\\ &\quad+2^{p'^+-1}\Big(\int_V 2^{p^+-1}c^{p'(x)}(x)|u(x)-u_0(x)|^{p(x)}dx\Big)\\ &\le 2^{p'^+-1}\int_V 2|d(x)|^{p'(x)}+c^{p'(x)}(x)(2^{p^+-1}+1)|u_0(x)|^{p(x)}dx\\ &\quad+2^{p^++p'^+-2}\,\|c^{p'^+}\|_{L^\infty(\Omega)}\int_V|u(x)-u_0(x)|^{p(x)}dx\\ &\le\Big(\frac{\varepsilon}{3}\Big)^{p'^+}+2^{p^++p'^+-2}\,\|c^{p'^+}\|_{L^\infty(\Omega)}\delta^\theta\le 2\Big(\frac{\varepsilon}{3}\Big)^{p'^+}\le\varepsilon^{p'^+}.\end{aligned} \tag{35}$$

Thanks to (34), (35) and (9), we get $\|w-h\|_{p'(x)}\le\int_\Omega|w(x)-h(x)|^{p'(x)}dx+1<\varepsilon$.
Hence C is upper semicontinuous (u.s.c.). Hence $\mathcal{N}=I^*\circ C\circ I$ is clearly bounded, upper semicontinuous (u.s.c.) and compact. □

Next, we give the proof of Theorem 3.1.

Let $S:=A+\mathcal{N}:\ W_0^{s,p(x,y)}(\Omega)\to 2^{\left(W_0^{s,p(x,y)}(\Omega)\right)^*}$, where A and $\mathcal{N}$ be defined in Lemma 3.4 and in Section 2 respectively. This means that a point $u\in W_0^{s,p(x,y)}(\Omega)$ is a weak solution of (1) if and only if

$$Fu\in -Su, \tag{36}$$

with F is setting in (8). By the properties of the operator F given in Lemma 3.2 and the Minty-Browder's Theorem on monotone operators in [34, Theorem 26 A], we guarantee that the inverse operator $E:=F^{-1}:\left(W_0^{s,p(x,y)}(\Omega)\right)^*\to W_0^{s,p(x,y)}(\Omega)$ is continuous, of type (S_+) and bounded. Moreover, thanks to Lemma 3.4 the operator S is quasi-monotone, upper semicontinuous (u.s.c.) and bounded. As a result, the equation (36) is equivalent to the abstract Hammerstein equation

$$u=Ev \quad \text{and} \quad v\in -S\circ Ev. \tag{37}$$

To solve the equations (37), we will use the notion of degrees discussed in Section 3. We begin by displaying the following. Lemma

Lemma 3.6. *The set*

$$B := \Big\{ v \in \Big(W_0\Big)^* \text{ such that } v \in -tS \circ Ev \quad \text{for some} \quad t \in [0,1]\Big\}$$

is bounded.

Proof. Let $v \in B$, so, $v + ta = 0$ for every $t \in [0,1]$, with $a \in S \circ Ev$. setting $u := Ev$, we can write $a = Au + \varphi \in Su$, where $\vartheta \in \mathcal{N}u$, namely,

$$\langle \vartheta, u\rangle = \int_\Omega h(x)u(x)dx,$$

for each $h \in L^{p'(x)}(\Omega)$ with $\underline{\phi}(x, u(x)) \leq h(x) \leq \overline{\phi}(x, u(x))$ for almost all $x \in \Omega$.
If $||u||_{W_0} \leq 1$, then $||Ev||_{W_0}$ is bounded.
If $||u||_{W_0} > 1$, then we get by the implication (i) in Proposition 2.1 and the inequality (10) and using (H_0), the Young inequality, the compact embedding $W_0 \hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$, the estimate

$$\begin{aligned}
||Ev||^{p^-}_{W_0} &= ||u||^{p^-}_{W_0}\\
&\leq \rho_{p(\cdot,\cdot)}(u)\\
&\leq t|\langle a, Ev\rangle|\\
&\leq t\int_\Omega |u|^{q(x)}\,dx + t\int_\Omega C|H(x,u)|udx + t\int_\Omega |hu|dx\\
&\leq t\int_\Omega |u|^{q(x)} + tC_{p'}\int_\Omega |CH(x,u)|^{q'(x)}dx + tC_p\int_\Omega |u|^{q(x)}\,dx\\
&\qquad\qquad + C_\zeta t\Big(\int_\Omega |u|^{\zeta(x)}dx\Big) + C_{\zeta'}t\Big(\int_\Omega |h|^{\zeta'(x)}dx\Big)\\
&\leq \mathrm{Const}\Big(||u||^{q^-}_{q(x)} + ||u||^{q^+}_{q(x)} + ||u||^{\zeta-}_{\zeta(x)} + ||u||^{\zeta+}_{\zeta(x)} + 1\Big)\\
&\leq \mathrm{Const}\Big(||u||^{q^-}_{W_0} + ||u||^{q^+}_{W_0} + ||u||^{\zeta-}_{W_0} + ||u||^{\zeta+}_{W_0} + 1\Big)\\
&\leq \mathrm{Const}\Big(||Ev||^{q^+}_{W_0} + ||Ev||^{\zeta+}_{W_0} + 1\Big).
\end{aligned}$$

Hence it is obvious that $\Big\{ Ev \ \backslash \ v \in B \Big\}$ is bounded.
As the operator S is bounded and from (37), we deduce the set B is bounded in $\Big(W_0\Big)^*$. □

Thanks to Lemma 3.6, we can find a positive constant R such that

$$||v||_{\left(W_0\right)^*} < R \quad \text{for any} \quad v \in B.$$

This says that

$$v \in -tS \circ Ev \quad \text{for each} \quad v \in \partial B_R(0) \quad \text{and each} \quad t \in [0,1].$$

Under the Lemma 2.9, we get

$$I + S \circ E \in \mathcal{F}_E(\overline{B_R(0)}) \quad \text{and} \quad I = F \circ E \in \mathcal{F}_E(\overline{B_R(0)}).$$

Now, we are in a position to consider the affine homotopy $H : [0,1] \times \overline{B_R(0)} \to 2^{\left(W_0\right)^*}$ setting by

$$H(t, v) := (1-t)Iv + t(I + S \circ E)v \quad \text{for} \quad (t, v) \in [0,1] \times \overline{B_R(0)}.$$

By applying the normalization and homotopy invariance property of the degree d fixed in Theorem 2.12, we have

$$d(I + S \circ E, B_R(0), 0) = d(I, B_R(0), 0) = 1.$$

It follows that, we can get a function $v \in B_R(0)$ such that

$$v \in -S \circ Ev.$$

Which implies that $u = Ev$ is a weak solution of (1). This completes the proof.

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