Filomat 38:6 (2024), 2127–2142
https://doi.org/10.2298/FIL2406127E

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Sufficient conditions for existence of mild solutions for nondensely defined conformable fractional evolution equations in Banach spaces

Hiba El Asraoui[a], Ali El Mfadel[a,b,*], Khalid Hilal[a], Mhamed Elomari[a]

[a]*Laboratory of Applied Mathematics and Scientific Computing,*
Sultan Moulay Slimane University, Beni Mellal, Morocco
[b]*Superior School of Technology,*
Sultan Moulay Slimane University, Khénifra, Morocco

Abstract. The main crux of this paper is to give some new sufficient conditions for the existence and uniqueness of solutions to a class of conformable fractional evolution equations with nondense domain in a Banach space. The proofs of our main results are based on some basic tools of conformable fractional calculus, conformable semigroup and Hile-Yosida theorem. As an application, a nontrivial example is given to illustrate the theoretical results.

1. Introduction

The main crux of this work is to study the existence and uniqueness of solutions for the following nonlinear conformable fractional evolution equation:

$$\begin{cases} D^{\alpha}u(t) = A_{\alpha}u(t) + f(t, u(t)),\ t \in [0, T], \\ \\ u(0) = u_0 + g(u), \end{cases} \tag{1}$$

where D^{α} is the conformable fractional derivative of order $\alpha \in (0, 1)$ and A_{α} is a nondensely α-infinitesimal generator of $\alpha - C_0$-semigroup $T_{\alpha}(t)$ on a given Banach space X which will be defined in the sequal.
The case when A_{α} is densely defined operator has been widely studied in in [4] by G.D. Prato and E. Sinestrari. They showed that the density is not necessary to solve their problem. A great number of researchers considered evolution problems with nondensly defined operators. On the other hand, fractional calculus has become the point of interest of a lot of mathematiciens. In particular, the conformable derivative, which was introduced in [11], has been extensively used in applied mathematics. Its main charateristics are its natural form, (i.e. its forme is very close to the form of the usual derivative), and the fact that it satisfies most of the properties of the usual derivative, like the derivative of the product and the quotient of two functions. In [5], the authors studied a class of nondensely defined fractional semilinear differential

2020 *Mathematics Subject Classification*. 34A08; 26A33; 34K37.
Keywords. Conformable fractional derivative, conformable semigroup, Hile-Yosida theorem.
Received: 12 April 2023; Accepted: 10 August 2023
Communicated by Maria Alessandra Ragusa
* Corresponding author: Ali El Mfadel
Email address: a.elmfadel@usms.ma (Ali El Mfadel)

equation, they proved that under some assumptions, it has a unique integral solution. In [6], Y. Zhou et al considered a nonhomogeneous fractional order evolution equation with nondensely defined operator, they proved that the proposed problem is equivalent to an integral solution, by using the Laplace transform and they showed that this problem has at least one solution by using the method of noncompact measure. Motivated by the works mentioned above, we establish the existence and uniqueness results for the following nonlinear conformable fractional evolution problem (1) by using the conformable Laplace transform, fractional conformable semigroup and some suitable assumptions. The reader is advised to consult the articles [1, 2, 7–10] and the references therein for more details on the existence and uniqueness results for fractional differential equations.

Our paper is organized as follows. in Section 2, we will recall some basic definitions ans properties concerning fractional conformable derivative, conformable semigroup, fractional Laplace transform and the Hille-Yosida theorem associated to the conformable semigroup. In Section 3, we establish the existence of solutions for the conformable fractional problem (1). As application, an illustrative example is presented in Section 4 followed by conclusion in Section 5.

2. Auxiliary results

The aim of this section, is to introduce some basic definitions and properties concerning the conformable derivative, the fractional Laplace transform, the conformable semigroup, and the Hille-Yosida theorem associated with the conformable semigroup.

2.1. Conformable derivatives

In this subsection, we introduce the definition of the fractional conformable derivative adopted in this work, but also the associated fractional integral.

Definition 2.1. *[11] Let $\alpha \in (n, n+1]$ and $f : [0,\infty) \to \mathbb{R}$ be n-differentiable at $t > 0$, then the conformable fractional derivative of g of order α is defined by*

$$g^{(\alpha)}(t) = \lim_{\epsilon \to 0} \frac{g^{(n)}(t + \epsilon t^{n+1-\alpha}) - g^{(n)}(t)}{\epsilon},$$
$$g^{(\alpha)}(0) = \lim_{t \to 0} g^{(\alpha)}(t).$$

One of the main properties of the conformable derivative is the following result.

Remark 2.2. *[11] Using the previous definition one can easily show that*

$$g^{(\alpha)}(t) = t^{n+1-\alpha} g^{(n+1)}(t).$$

where $\alpha \in (n, n+1]$, and g is $(n+1)$-differentiable at $t > 0$.

The fractional integral is defined as follows.

Definition 2.3. *[11] Let $\alpha \in (0, 1]$, the fractional integral is giving by*

$$(I^\alpha g)(t) = \int_0^t s^{\alpha-1} g(s) ds.$$

The composition of the fractional conformable derivative and the fractional integral is given in the following theorem.

Theorem 2.4. *[11] The fractional integral satisfies the following property,*

$$(I^\alpha g)^{(\alpha)}(t) = g(t),$$

for $t \geq 0$.

Example 2.5. *We have,*

$$I^{\alpha}t^{p} = \frac{t^{\alpha+p}}{\alpha+p},$$

and,

$$I^{\alpha}(sin(t)) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{t^{\alpha+2n+1}}{\alpha+2n+1}.$$

If we take $\alpha = \frac{1}{2}$*, we get*

$$I^{\frac{1}{2}}(\sin(t)) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n+\frac{3}{2}}}{(2n+\frac{3}{2})(2n+1)!}.$$

2.2. Conformable fractional Laplace transform

The aim of this paragraph, is to give the definition of the fractional Laplace transform that will be used in this work.

Definition 2.6. *[7] Let* $a \in \mathbb{R}$, $0 < \alpha \leq 1$ *and* $f : [0, \infty) \to X$ *be an X-valued Bochner function. The fractional Laplace transform of order* α *is given by*

$$\mathcal{L}_a^{\alpha} f(\lambda) = \int_a^{\infty} e^{-s\frac{(t-a)^{\alpha}}{\alpha}} f(t)(t-a)^{\alpha-1}dt.$$

Now, we give the definition of the fractional convolution.

Definition 2.7. *[7] Let* $0 < \alpha \leq 1$, *and* u, v *be twoX-valued Bochner functions. We define the fractional convolution of* u *and* v *of order* α *by*

$$(u *_{\alpha} v)(t) = \int_0^t u((t^{\alpha} - \tau^{\alpha})^{\frac{1}{\alpha}})v(\tau)\tau^{\alpha-1}d\tau.$$

As in the case of the classical Laplace transform, the fractional Laplace transform satisfies the following property.

Proposition 2.8. *[7] Let* $0 < \alpha \leq 1$ *and* u, v *two Bochner functions, then we have*

$$\mathcal{L}_0^{\alpha}(u *_{\alpha} v)(\lambda) = \mathcal{L}_0^{\alpha}(u)(\lambda)\mathcal{L}_0^{\alpha}(v)(\lambda).$$

2.3. Conformable semigroup and its associated Hille-Yosida theorem

The purpuse of this subsection is to introduce the definition of the conformable semigroup, the conformable α-resolvent and the Hille-Yosida theorem for the α-semigroup. All the results that we are going to present in this paragraph were first introduced in [9], [8], and [5]. So for more details see these references. We begin by defining the notion of α-semigroup.

Definition 2.9. *[9] Let* $\alpha > 0$. *For a Banach space X, a family* $\{T_{\alpha}(t)\}_{t\geq 0} \subset \mathcal{L}(X, X)$ *is called a fractional* α*-semigroup if:*

1. $T_{\alpha}(0) = I$,
2. $T_{\alpha}\left((s+t)^{\frac{1}{\alpha}}\right) = T_{\alpha}\left(s^{\frac{1}{\alpha}}\right)T_{\alpha}\left(t^{\frac{1}{\alpha}}\right)$, *for all* $s, t \in [0, \infty)$.

Example 2.10. *[9] Let A be a bounded linear operator on X. Define* $T_{\alpha}(t) = e^{2\sqrt{t}A}$. *Then* $T_{\alpha}(t)_{t\geq 0}$ *is a* $\frac{1}{2}$ *semigroup. Indeed:*

1. $T_{\alpha}(0) = e^{0A} = I$.

2. $\forall s, t \in [0,\infty)$, $T_\alpha\left((s+t)^2\right) = e^{2(t+s)A} = e^{2tA}e^{2sA} = T_\alpha(s^2)T_\alpha(t^2)$.

As in the case of the classical semigroup, we define the notion of an $\alpha - C_0$-semigroup.

Definition 2.11. *[9] An α-semigroup $T_\alpha(t)$ is called a α-c_0-semigroup if, for each fixed $x \in X$, $T_\alpha(t)x \to x$ as $t \to 0^+$.*

The conformable α-derivative of an α-semigroup $T_\alpha(t)$ at $t = 0$ is called the infinitesimal generator of $T_\alpha(t)$ which we denote A_α and its domain is giving by

$$D(A_\alpha) = \left\{x \in X,\ \lim_{t\to 0} T_\alpha^{(\alpha)}(t)x \text{ exists}\right\}.$$

A C_0-α-semigroup, and its α-infinitesimal generator satisfy the following results.

Theorem 2.12. *[9] Let $T_\alpha(t)$ be a C_0-α-semigroup with the infinitesimal generator A_α and $x \in D(A_\alpha)$, then*

$$T_\alpha^{(\alpha)}(t)x = A_\alpha T_\alpha(t)x = T_\alpha(t)A_\alpha x.$$

Theorem 2.13. *[8] Let $T_\alpha(t)$ be a C_0 - α-semigroup where $\alpha \in (0,1]$. There exist constants $\omega \geq 0$ and $M \geq 1$ such that*

$$\|T_\alpha(t)\| \leq Me^{\omega t^\alpha}, \quad \text{for } 0 \leq t \leq \infty.$$

Corollary 2.14. *[8] If $T_\alpha(t)$ is a C_0 - α-semigroup, then for every $x \in X, t \to T_\alpha(t)x$ is a continuous function from $\mathbb{R}_0^+$ (the nonnegative real line) into X.*

Theorem 2.15. *[8] Let $T_\alpha(t)$ be aC_0 - α-semigroup where $\alpha \in (0,1]$ and let A_α be its α-infinitesimal generator. Then*

- *For $x \in X$*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_t^{t+\varepsilon t^{1-\alpha}} \frac{1}{s^{1-\alpha}}T_\alpha(s)xds = T(t)x, \quad \text{for every } t > 0.$$

- *For $x \in X$, $\int_0^t \frac{1}{s^{1-\alpha}}T_\alpha(s)xds \in D(A_\alpha)$ and*

$$A_\alpha\left(\int_0^t \frac{1}{s^{1-\alpha}}T_\alpha(s)xds\right) = T(t)x - x.$$

- *For $x \in D(A_\alpha)$, $T_\alpha(t)x \in D(A_\alpha)$ and*

$$\frac{d^\alpha}{dt^\alpha}T_\alpha(t)x = A_\alpha T_\alpha(t)x = T_\alpha(t)A_\alpha x.$$

- *For $x \in D(A_\alpha)$*

$$T_\alpha(t)x - T_\alpha(s)x = \int_s^t \frac{1}{u^{1-\alpha}}T_\alpha(u)A_\alpha x\,du = \int_s^t \frac{1}{u^{1-\alpha}}A_\alpha T_\alpha(u)x du.$$

Corollary 2.16. *[8] Let A_α be an α-infinitesimal generator of a C_0 - α-semigroup $T_\alpha(t)$. Then A_α is closed and densely defined linear operator.*

Theorem 2.17. *[8] Let $T_\alpha(t)$ and $S_\alpha(t)$ be fractional C_0 - α-semigroups of bounded linear operators, where A_α and B_α are their infinitesimal generators, respectively. If $A_\alpha = B_\alpha$, then $T_\alpha(t) = S_\alpha(t)$ for $t \geq 0$.*

Before announcing the Hille-Yosida theorem for the α-semigroup, we define the notion of an α-resolvent.

Definition 2.18. *[5, 8] The resolvent of A_α is set of all $\lambda \in \mathbb{R}$ satisfying*

$$\begin{aligned} R_\alpha(\lambda, A_\alpha)x &= (\lambda I - A_\alpha)^{-1} x \\ &= \int_0^\infty \frac{e^{-\frac{\lambda t^\alpha}{\alpha}} T_\alpha(t)}{t^{1-\alpha}} x dt \end{aligned}$$

$\forall x \in D(A_\alpha)$.

Theorem 2.19. *[5] A linear operator A_α is an α-infinitesimal generator of a α-c_0-semigroup if only if*

1. *A_α is closed.*
2. *$\mathbb{R}^+ \subset \rho(A_\alpha)$ and for every $\lambda > 0$*

$$\| R_\alpha(\lambda, A_\alpha) \| \leq \frac{1}{\lambda}.$$

3. Main results

We will use the above results to define the integral solution to the problem given in the following lemma.

Lemma 3.1. *The problem*

$$\begin{cases} D^\alpha u(t) = A_\alpha u(t) + h(t), \ t \in [0, T], \\ u(0) = u_0 + g(u), \end{cases} \tag{2}$$

is equivalent to the following integral equation

$$u(t) = u_0 + g(u) + I^\alpha (A_\alpha u(t)) + I^\alpha (h(t)). \tag{3}$$

Proof. By applying D^α to (3), we obtain the problem (2). conversaly, if we apply I^α to the problem (2), we get

$$u(t) - u_0 - g(u) = I^\alpha(A_\alpha u(t)) + I^\alpha(h(t)).$$

Which completes the proof. □

We begin by introducing some notations which we will use throughout this paper. We denote by I the following time interval $I = [0, T]$, by X a real Banach space, and by C the set of all continuous functions from I into X, with the norm $\|x\| = \sup_{t\in I} |x(t)|$. For $r > 0$, we set $C_r(I, X) = \{f \in C, \ t^r f \in C\}$. Notice that $C_r(I, X)$ endowed with the norm $\| f \|_{C_r} = \sup_{t\in I} \| t^r f(t) \|$ is a Banach space.

Now, let A_0^α the part of A_α in $\overline{D(A_\alpha)}$ defined by

$$\begin{cases} D(A_0^\alpha) = \left\{ x \in D(A_\alpha), \ A_\alpha x \in \overline{D(A_\alpha)} \right\}, \\ A_0^\alpha x = A_\alpha x. \end{cases} \tag{4}$$

We need the following assumption to ensure that A_0^α generates a C_0-α-semigroup.

- (H_1) the operator $A_\alpha : D(A_\alpha) \subset X \to X$ satisfies the Hille-Yosida condition, i.e, there is $M > 0$ and $\omega > 0$ such that $(\omega, \infty) \subset \rho(A_\alpha)$, and

$$\sup \left\{ (\lambda - \omega)^n \| R_\alpha(\lambda, A_\alpha) \| : \ n \in \mathbb{N}, \ \lambda > \omega \right\} \leq M. \tag{5}$$

Then A_0^α generates a c_0-α-semigroup $\{T_\alpha(t)\}_{t\geq 0}$ on $\overline{D(A_\alpha)}$. The following assumption will also be useful to prove the existence of the integral solution.

- (H_2) The operator $T_\alpha(t)$ generated by A_0^α is compact in $\overline{D(A_\alpha)}$ when $t > 0$, continuous in the uniform topology, and

$$\sup_{t\in I} \| T_\alpha(t) \| \leq M_T.$$

Let us first introduce the definition of the integral solution to the problem (2).

Definition 3.2. *We say that a function $u : I \to X$ is an integral solution of* (2) *on I if the following conditions are satisfied:*

1. $u \in C_r,\ r < \alpha.$
2. $(I^\alpha u)(t) \in D(A_\alpha).$
3.

$$u(t) = u_0 + g(u) + A_\alpha \int_0^t \frac{u(s)}{s^{1-\alpha}} ds + I^\alpha h(t),\ t \in I. \tag{6}$$

Lemma 3.3. *If u is an integral solution of* (2) *on I, then $u(t) \in \overline{D(A_\alpha)}$, in particular $u(0) \in \overline{D(A_\alpha)}$.*

Proof. For $h > 0$ satisfying $t + h \in I$. We have

$$\frac{1}{h} \int_t^{t+ht^{1-\alpha}} u(s) s^{\alpha-1} ds \in D(A_\alpha),$$

sinc $I^\alpha u(t) \in D(A_\alpha)$. Thus,

$$u(t) = \lim_{h\to 0} \frac{1}{h} \int_t^{t+ht^{1-\alpha}} u(s) s^{\alpha-1} ds \in \overline{D(A_\alpha)},$$

and in particular, $u(0) \in \overline{D(A_\alpha)}$. □

Definition 3.4. *[6] The following function,*

$$M_q(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\gamma(1-qn)},$$

is called the Wright function, and satisfies

$$\int_0^\infty \theta^\delta M_q(\theta) d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+q\delta)},\ \ for\ \delta \geq 0.$$

Let us consider the following notation,

$$B_\lambda^\alpha = \lambda R_\alpha(\lambda, A_\alpha).$$

Remark 3.5. *The operator B_λ^α satisfies,*

$$\lim_{\lambda\to\infty} \| B_\lambda^\alpha \| \leq M.$$

Proof. Under the assumption (H_0) we have

$$\|B_\lambda^\alpha\| = \|\lambda R_\alpha(\lambda, A_\alpha)\| \leq \frac{\lambda M}{\lambda - \omega},$$

which implies the result. □

Let us now consider the following auxiliary problem

$$\begin{cases} u^{(\alpha)}(t) = A_0^\alpha u(t) + h(t), \quad t \in I \\ u(0) = u_0 + g(u). \end{cases} \tag{7}$$

By Definition 3.2, the integral solution of (7) is given by

$$u(t) = u_0 + g(u) + A_0^\alpha \int_0^t \frac{u(s)}{s^{1-\alpha}} ds + I^\alpha h(t), \tag{8}$$

for $u(0) \in \overline{D(A_\alpha)}$ and $t \in I$. The fractional Laplace transform allows us to introduce an equivalent form of (8).

Lemma 3.6. *If h is $\overline{D(A_\alpha)}$-valued, then* (8) *can be rewritten in the following way*

$$u(t) = Q_\alpha(t^\alpha)(u_0 + g(u)) + \alpha \int_0^t Q_\alpha(t^\alpha - \tau^\alpha) h(\tau) \tau^{\alpha-1} d\tau, \tag{9}$$

where,

$$Q_\alpha(t) = \int_0^\infty \frac{1}{\theta\alpha} \psi_1(\theta) T_\alpha\Big(\Big(\frac{t}{\theta}\Big)^{\frac{1}{\alpha}}\Big) d\theta. \tag{10}$$

Proof. We begin by applying the fractional Laplace transform to (8), we obtain

$$\mathcal{L}_\alpha(u(t))(\lambda) = \frac{1}{\lambda}(u_0 + g(u)) + \frac{1}{\lambda} A_0^\alpha \mathcal{L}_\alpha(u(t))(\lambda) + \frac{1}{\lambda} \mathcal{L}_\alpha(h(t)(\lambda).$$

Thus,

$$\begin{aligned} \mathcal{L}_\alpha(u(t))(\lambda) &= (\lambda I - A_0^\alpha)^{-1}(u_0 + g(u)) + (\lambda I - A_0^\alpha)^{-1} \mathcal{L}_\alpha(h(t))(\lambda) \\ &= I_1 + I_2. \end{aligned}$$

We have

$$\begin{aligned} I_1 &= (\lambda I - A_0^\alpha)^{-1}(u_0 + g(u)) \\ &= \int_0^\infty e^{-\lambda \frac{t^\alpha}{\alpha}} T_\alpha(t) t^{\alpha-1}(u_0 + g(u)) dt \\ &= \frac{1}{\alpha} \int_0^\infty e^{-\lambda \frac{s}{\alpha}} T_\alpha(s^{\frac{1}{\alpha}})(u_0 + g(u)) dt \end{aligned}$$

By using the fact that,

$$\int_0^\infty e^{-\lambda\theta} \psi_q(\theta) d\theta = e^{-\lambda q}, \tag{11}$$

we get,

$$\begin{aligned} I_1 &= \frac{1}{\alpha} \int_0^\infty \int_0^\infty e^{-\frac{\lambda s \theta}{\alpha}} \psi_1(\theta) T_\alpha(s^{\frac{1}{\alpha}})(u_0 + g(u)) d\theta ds \\ &= \int_0^\infty e^{\frac{-\lambda t}{\alpha}} \int_0^\infty \frac{1}{\alpha\theta} \psi_1(\theta) T_\alpha\Big(\Big(\frac{t}{\theta}\Big)^{\frac{1}{\alpha}}\Big)(u_0 + g(u)) d\theta dt \\ &= Q_\alpha(t^\alpha)(u_0 + g(u)). \end{aligned}$$

In the other hand, we have

$$\begin{aligned}
I_2 &= (\lambda I - A_0^\alpha)^{-1}\mathcal{L}_\alpha(f(t,u(t))(\lambda) \\
&= \int_0^\infty e^{-\lambda\frac{t^\alpha}{\alpha}} T_\alpha(t)t^{\alpha-1}\mathcal{L}_\alpha(h(t))(\lambda)dt \\
&= \frac{1}{\alpha}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\frac{\lambda\tau}{\alpha}}\frac{1}{\theta}\psi_1(\theta)T_\alpha\Big(\Big(\frac{\tau}{\theta}\Big)^{\frac{1}{\alpha}}\Big)e^{-\frac{\lambda t^\alpha}{\alpha}}h(t)t^{\alpha-1}d\theta dt d\tau \\
&= \alpha\mathcal{L}_\alpha(Q_\alpha(t^\alpha))(\lambda)\mathcal{L}_\alpha(h(t))(\lambda).
\end{aligned}$$

By applying the inverse Laplace transform as well as its properties we obtain the result. □

Proposition 3.7. *[13] Under the assumption (H_2), $Q_\alpha(t)$ is continuous in the uniform topology.*

Remark 3.8. *According to [13],*

$$\int_0^\infty \frac{1}{\theta^q}\psi_q(\theta)d\theta = \frac{1}{\Gamma(1+q)},$$

thus, under the assumption (H_2) we have,

$$|Q_\alpha(t)x| \leq M_T|x|.$$

If we assume that h is $\overline{D(A_\alpha)}$-valued, then (9) can have the following form,

$$u(t) = Q_\alpha(t^\alpha)(u_0 + g(u)) + \alpha\int_0^t Q_\alpha(t^\alpha - \tau^\alpha)\lim_{\lambda\to\infty} B_\lambda^\alpha h(\tau)\tau^{\alpha-1}d\tau. \tag{12}$$

Or,

$$u(t) = Q_\alpha(t^\alpha)(u_0 + g(u)) + \lim_{\lambda\to\infty}\alpha\int_0^t Q_\alpha(t^\alpha - \tau^\alpha)B_\lambda^\alpha h(\tau)\tau^{\alpha-1}d\tau. \tag{13}$$

Due to the fact that

$$\lim_{\lambda\to+\infty} B_\lambda^\alpha x = x,$$

for $x \in \overline{D(A_\alpha)}$.

Remark 3.9. *If the values of h are in X and not in $\overline{D(A_\alpha)}$, then the limit in* (13) *exists, however the limit in* (12) *does not exist.*

Lemma 3.10. *The solutions of* (6) *which are $\overline{D(A_\alpha)}$-valued can be represented by* (13).

Proof. Let us first consider the following notations,

$$u_\lambda^\alpha(t) = B_\lambda^\alpha u(t), \;\; h_\lambda^\alpha(t,u(t)) = B_\lambda^\alpha h(t), \;\; u_\lambda^\alpha = B_\lambda^\alpha(u_0 + g(u)).$$

We begin by applying B_λ^α to (6), we obtain

$$u(t) = u_0 + g(u) + A_0^\alpha I^\alpha u(t) + I^\alpha h_\lambda^\alpha(t).$$

Using the Lemma 3.6, we get

$$u_\lambda^\alpha(t) = Q_\alpha(t^\alpha)u_\lambda + \alpha\int_0^t Q_\alpha(t^\alpha - \tau^\alpha)h_\lambda^\alpha(\tau)d\tau,$$

since $u(t)$, $u_0 + g(u) \in \overline{D(A_\alpha)}$, then

$$u_\lambda^\alpha(t) \to u(t), \; u_\lambda^\alpha \to u_0 + g(u), \; Q_\alpha(t)u_\lambda \to Q_\alpha(u_0 + g(u)), \text{ as } \lambda \to \infty.$$

Which implies the result. □

We define the following operator,

$$\mathcal{X}_\alpha(t)x = \lim_{\lambda\to\infty} \alpha \int_0^t Q_\alpha(t^\alpha - \tau^\alpha)B_\lambda^\alpha x\tau^{\alpha-1}d\tau = \lim_{\lambda\to\infty} \alpha \int_0^t Q_\alpha(\tau^\alpha)B_\lambda^\alpha x\tau^{\alpha-1}d\tau, \tag{14}$$

for $x \in X$, and $t \geq 0$.

Proposition 3.11. *If $x \in X$ and $t \geq 0$, then the limit in* (14) *exists. Furthermore, $\mathcal{X}_\alpha(t)$ is a bounded linear operator.*

Proof. We begin by defining the operator

$$\mathcal{X}_\alpha^0(t)x = \alpha \int_0^t s^{\alpha-1}Q_\alpha(t^\alpha - s^\alpha)xds = \alpha \int_0^t s^{\alpha-1}Q_\alpha(s^\alpha)xds, \tag{15}$$

for $x \in \overline{D(A_\alpha)}$ and $t \geq 0$. Then

$$\mathcal{X}_\alpha = (\lambda I - A)\mathcal{X}_\alpha^0(t)(\lambda I - A)^{-1}.$$

for $\lambda > \omega$, which implies that $\mathcal{X}_\alpha(t)$ extends $\mathcal{X}_\alpha^0(t)$ from $\overline{D(A_\alpha)}$ to X. Since $\mathcal{X}_\alpha(t)$ maps X into $\overline{D(A_\alpha)}$, then

$$\mathcal{X}_\alpha(t)x = \lim_{\lambda\to\infty} B_\lambda^\alpha \mathcal{X}_\alpha(t)x = \lim_{\lambda\to\infty} \mathcal{X}_\alpha^0(t)B_\lambda^\alpha x.$$

Which completes the proof. □

Proposition 3.12. *Let $x \in \overline{D(A_\alpha)}$ and $t \geq 0$, then $D^\alpha \mathcal{X}_\alpha^0(t)x = I^{1-\alpha}(\alpha Q_\alpha(t^\alpha))x$ and $Q_\alpha(t^\alpha)x = A\mathcal{X}_\alpha^0(t)x + x$.*

Proof. We have

$$\begin{aligned} D^\alpha \mathcal{X}_\alpha^0(t)x &= D^\alpha \alpha \int_0^t s^{\alpha-1}Q_\alpha(t^\alpha - s^\alpha)xds \\ &= D^\alpha \alpha \int_0^t s^{\alpha-1}Q_\alpha(s^\alpha)xds \\ &= I^{1-\alpha}(\alpha Q_\alpha(t^\alpha))x. \end{aligned}$$

Let us now prove the second point of the proposition,

$$\begin{aligned} A\mathcal{X}_\alpha^0(t)x &= A\Big(\int_0^t \int_0^\infty \frac{1}{\theta}\psi_1(\theta)T_\alpha\Big(\frac{s}{\theta^{\frac{1}{\alpha}}}\Big)d\theta s^{\alpha-1}xds\Big) \\ &= \int_0^\infty \frac{1}{\theta}\psi_1(\theta)A\Big(\int_0^t T_\alpha\Big(\frac{s}{\theta^{\frac{1}{\alpha}}}\Big)s^{\alpha-1}ds\Big)d\theta \\ &= Q_\alpha(t^\alpha)x - x. \end{aligned}$$

□

Lemma 3.13. 1. *Let $x \in X$ and $t \geq 0$, then $I^\alpha(W)_\alpha(t) \in D(A_\alpha)$, and*

$$\mathcal{X}_\alpha(t)x = A_\alpha(I^\alpha \mathcal{X}_\alpha(t)x) + \frac{t^\alpha}{\alpha}x.$$

2. *Let $x \in D(A_\alpha)$, then*

$$\mathcal{X}(t)A_\alpha x + x = Q_\alpha(t^\alpha)x.$$

Proof. 1. Let $x \in X$ and $t \geq 0$, we consider the following function

$$W(t) = \lambda I^{\alpha}\mathcal{X}_{\alpha}^{0}(t)(\lambda I - A)^{-1}x + \frac{1}{\alpha}t^{\alpha}(\lambda I - A)^{-1}x - \mathcal{X}_{\alpha}^{0}(t)(\lambda I - A)^{-1}x.$$

Obviously, $W(0) = 0$. And we have

$$\begin{aligned}
D^{\alpha}W(t)x &= \lambda\mathcal{X}_{\alpha}^{0}(t)(\lambda I - A)^{-1}x + (\lambda I - A)^{-1}x - D^{\alpha}\mathcal{X}_{\alpha}^{0}(t)(\lambda I - A)^{-1}x \\
&= \lambda\mathcal{X}_{\alpha}^{0}(t)(\lambda I - A)^{-1}x + (\lambda I - A)^{-1}x - Q_{\alpha}(t^{\alpha})(\lambda I - A)^{-1}x \\
&= \lambda\mathcal{X}_{\alpha}^{0}(t)(\lambda I - A)^{-1}x + (\lambda I - A)^{-1}x - A_{\alpha}\mathcal{X}_{\alpha}^{0}(t)(\lambda I - A)^{-1}x - (\lambda I - A)^{-1}x \\
&= \lambda\mathcal{X}_{\alpha}^{0}(t)(\lambda I - A)^{-1}x - A_{\alpha}\mathcal{X}_{\alpha}^{0}(t)(\lambda I - A)^{-1}x \\
&= (\lambda I - A)\mathcal{X}_{\alpha}^{0}(t)(\lambda I - A)^{-1}x \\
&= \mathcal{X}_{\alpha}(t)x.
\end{aligned}$$

Then,

$$W(t) = I^{\alpha}\mathcal{X}_{\alpha}(t)x + W(0) = I^{\alpha}\mathcal{X}_{\alpha}(t)x,$$

and,

$$\begin{aligned}
(\lambda I - A)W(t) &= (\lambda I - A)T^{\alpha}\mathcal{X}_{\alpha}(t)x \\
&= \lambda I^{\alpha}\mathcal{X}_{\alpha}(t)x - AI^{\alpha}\mathcal{X}_{\alpha}(t)x \\
&= \lambda I^{\alpha}\mathcal{X}_{\alpha}(t)x - I^{\alpha}Q_{\alpha}(t^{\alpha})x + I^{\alpha}x \\
&= \lambda I^{\alpha}\mathcal{X}_{\alpha}(t)x - \mathcal{X}_{\alpha}^{0}(t)x + \frac{1}{\alpha}t^{\alpha}x
\end{aligned}$$

2. Let $x \in D(A_{\alpha})$, we have

$$\begin{aligned}
\mathcal{X}_{\alpha}(t)Ax &= \lim_{\lambda\to\infty}\alpha\int_{0}^{t}Q_{\alpha}(\tau^{\alpha})B_{\lambda}^{\alpha}Ax\tau^{\alpha-1}d\tau \\
&= A\mathcal{X}_{\alpha}^{0}(t)x \\
&= Q_{\alpha}(t^{\alpha})x - x.
\end{aligned}$$

Which completes the proof.
□

Theorem 3.14. *$u(t)$ is an integral solution of* (2) *if and only if*

$$u(t) = Q_{\alpha}(t^{\alpha})(u_0 + g(u)) + \lim_{\lambda\to\infty}\alpha\int_{0}^{t}Q_{\alpha}(t^{\alpha} - \tau^{\alpha})B_{\lambda}^{\alpha}h(\tau)\tau^{\alpha-1}d\tau, \tag{16}$$

for $t \in I$ and $u(0) \in \overline{D(A_{\alpha})}$.

Proof. It is sufficient to show that (16) is an integral solution to problem (2). We only need to prove that it is true for $u(0) = 0$. To this end, we will proceed as follows

Step 1

If f is continuously differentiable, then for $t \in I$, we have

$$\begin{aligned}
u_{\lambda}(t) &= \alpha\int_{0}^{t}Q_{\alpha}(s^{\alpha})B_{\lambda}^{\alpha}h(s)d_{\alpha}s \\
&= \alpha\int_{0}^{t}Q_{\alpha}(s^{\alpha})B_{\lambda}^{\alpha}(h(0) + \int_{0}^{s}h'(\tau)d\tau)d_{\alpha}s \\
&= \alpha\int_{0}^{t}Q_{\alpha}(s^{\alpha})B_{\lambda}^{\alpha}h(0)d_{\alpha}s + \alpha\int_{0}^{t}Q_{\alpha}(s^{\alpha})B_{\lambda}^{\alpha}(\int_{0}^{s}h'(\tau)d\tau)d_{\alpha}s \\
&= \mathcal{X}_{\alpha}^{0}(t)B_{\lambda}^{\alpha}h(0) + \int_{0}^{t}\mathcal{X}_{\alpha}^{0}(t - \tau)B_{\lambda}^{\alpha}h'(\tau)d\tau.
\end{aligned}$$

And we have

$$
\begin{aligned}
u(t) &= \lim_{\lambda\to+\infty} u_\lambda(t) \\
&= \mathcal{X}_\alpha(t)h(0) + \int_0^t \mathcal{X}_\alpha(t-s)h'(s)ds \\
&= A_\alpha(I^\alpha(t)h(0)) + \frac{t^\alpha}{\alpha}h(0) \\
&+ \int_0^t \Big[A_\alpha(I^\alpha \mathcal{X}_\alpha(t-s)h'(s)) + \frac{(t-s)^\alpha}{\alpha}h'(s)\Big]ds \\
&= A_\alpha\Big(I^\alpha \mathcal{X}_\alpha(t)h(0) + \int_0^t I^\alpha \mathcal{X}_\alpha(t-s)h'(s)ds\Big) \\
&+ \frac{t^\alpha}{\alpha}h(0) + \int_0^t \frac{(t-s)^\alpha}{\alpha}h'(s)ds \\
&= A_\alpha I_\alpha u(t) + I^\alpha h(t).
\end{aligned}
$$

Step 2

In this step, we consider an approximation of h by continuously differentiable functions h_n such that

$$\sup_{t\in I} |h(t) - h_n(t)| \longrightarrow 0, \text{ as } n \longrightarrow +\infty.$$

We set

$$u_n(t) = \lim_{\lambda\to+\infty} \alpha \int_0^t Q_\alpha(t^\alpha)B_\lambda^\alpha h_n(s)ds.$$

According to the first step, we have

$$u_n(t) = A_\alpha(I^\alpha u_n(t)) + I^\alpha h_n(t). \tag{17}$$

Then

$$
\begin{aligned}
|u_n(t) - u_m(t)| &= |\lim_{\lambda\to+\infty} \int_0^t \alpha Q_\alpha(t^\alpha)B_\lambda^\alpha(h_n(s) - h_m(s))ds| \\
&\le \alpha M_T M \int_0^t |h_n(s) - h_m(s)|ds \\
&\le \alpha M_T M T \|h_n - h_m\|,
\end{aligned}
$$

hence, $\{u_n\}$ is a Cauchy sequence, and it has a limit which we denote by $u(t)$.
By passing to the limit in (17), we get

$$u(t) = A_\alpha(I^\alpha u(t)) + I^\alpha h(t).$$

Consequently, (16) is the integral solution of (2). □

The following part of this paper is dedicated to study the existence and uniqueness of solutions for the problem (1).
We will need the following hypotheses:

- (H_3) $f : I \times X \to X$ is continuous and for any $k > 0$ there exists a positive function $\mu_k \in L^\infty(I, \mathbb{R}^+)$ such that

$$\sup_{\|x\|\le k} \| f(t,x) \| \le \mu_k(t)$$

- (H_4) $g : C \to \overline{D(A)}$ is continuous, and there exists a constant η such that

$$\| g(x) - g(y) \| \le \eta \| x - y \| \ \ \forall x, y \in X$$

According to Theorem 3.14, the integral solution to problem (1) is equal to the solution of

$$u(t) = Q_\alpha(t^\alpha)(u_0 + g(u)) + \lim_{\lambda\to\infty} \alpha \int_0^t Q_\alpha(t^\alpha - \tau^\alpha)B_\lambda^\alpha f(\tau, u(\tau))\tau^{\alpha-1} d\tau. \tag{18}$$

Theorem 3.15. *[5] Let B be a closed convex and nonempty subset of a Banach space X. Let L_1 and L_2 be two operators such that*

1. *$L_1x + L_2y \in B$ whenever $x, y \in B$.*
2. *L_1 is contracting mapping.*
3. *L_2 is compact and continuous.*
   Then there exists $z \in B$ such that $z = L_1z + L_2z$

In this section we set

- $(L_1u)(t) = Q_\alpha(t^\alpha)(u_0 + g(u))$,
- $(L_2u)(t) = \lim_{\lambda\to\infty} \alpha \int_0^t \tau^{\alpha-1} Q_\alpha(t^\alpha - \tau^\alpha)B_\lambda^\alpha f(\tau, u(\tau))d\tau$.

Theorem 3.16. *Assume that $H_1 - H_4$ hold. If $M_T\eta < 1$ then* (1) *has at least one integral solution on I.*

Proof. **Step 1.**
Choose $r \geq \frac{M_T}{1-M_T\eta}\Big(\|u_0\| + \|g(0)\| + \frac{MT^\alpha}{\alpha}\|\mu\|_{L^\infty(I,\mathbb{R}^+)}\Big)$, we set B_r the unit ball of $(C, \|.\|_C)$. And, let $u, v \in B_r$.

$$\begin{aligned}
\| (L_1u)(t) + (L_2v)(t) \| &\leq M_T\Big(\|u_0 + g(u)\|\Big) + \lim_{\lambda\to\infty} \alpha \int_0^t s^{\alpha-1}\|Q_\alpha(t^\alpha - s^\alpha)B_\lambda^\alpha f(s, u(s))\|ds \\
&\leq M_T\Big(\|u_0\| + \|g(0)\| + \eta r\Big) + MM_TT^\alpha\|\mu_r\|_{L^\infty(I,\mathbb{R}^+)} \\
&\leq r
\end{aligned}$$

And for $t \in I$, $u, v \in C$ we have

$$\| (L_1u)(t) - (L_1v)(t) \| \leq M_T\eta\|u - v\|.$$

However $M_T\eta < 1$, then L_1 is a contraction.
Step 2.
Let (U_n) be a sequence in B_r, such that $u_n \to u$ in B_r.
As f is continuous,

$$f(\tau, u_n(\tau)) \to f(\tau, u(\tau)), \text{ as } n \to \infty.$$

In other hand, we have

$$\forall t \in I, \ \ \|\tau^{\alpha-1}(f(\tau, u_n(\tau)) - f(\tau, u(\tau)))\| \leq 2\tau^{\alpha-1}\|\eta_r\|_{L^1} \in L^1(I, \mathbb{R}^+).$$

Then, by the Lebesgue dominated convergence theorem we obtain that,

$$\|\tau^{\alpha-1}(f(\tau, u_n(\tau)) - f(\tau, u(\tau)))\| \to 0 \ \text{ as} n \to \infty,$$

which implies that L_2 is continuous.
Now, we will show that $\{(L_2u)(t), u \in B_r\}$, is compact. To this end, we will show that $\{(L_2u)(t), u \in B_r\}$ is equicontinuous and uniformly bounded and for any $t \in I$, $\{(L_2u)(t), u \in B_r\}$ is relatively compact in X.
Let $0 < t_1 < t_2 < T$, we have

$$\begin{aligned}
\|(L_2u)(t_2) - (L_2u)(t_1)\| &\leq \lim_{\lambda\to\infty} \Big\|\alpha \int_{t_1}^{t_2} Q_\alpha(t_2^\alpha - \tau_\alpha)B_\lambda^\alpha f(\tau, u(\tau))\tau^{\alpha-1}d\tau\Big\| \\
&\quad + \lim_{\lambda\to\infty} \Big\|\alpha \int_0^{t_1} (Q_\alpha(t_2^\alpha - \tau^\alpha) - Q_\alpha(t_1^\alpha - \tau^\alpha))B_\lambda^\alpha f(\tau, u(\tau))\tau^{\alpha-1}d\tau\Big\| \\
&\leq M_TM\|\mu\|_{L^\infty(I,\mathbb{R})} \\
&\qquad + T^\alpha\|\mu\|_{L^\infty(I,\mathbb{R})} \sup_{s\in[0,t_1]} |Q_\alpha(t_2^\alpha - s^\alpha) - Q_\alpha(t_1^\alpha - s^\alpha)|
\end{aligned}$$

According to Proposition 3.7, the second term in the right hand side of the last inequality tends to 0 as t_1 tends to t_2.
Hence, $\{(L_2u)(t), u \in B_r\}$ is equicontinuous.
Further, we have (L_2u) is uniformly bounded.
Let us prove that $\forall t \in I,\ V(t) = \{(L_2u)(t), u \in B_r\}$ is relatively compact in X.
Let $t \in (0, T]$ be fixed, for each $h \in (0, t)$, for each $\delta > 0$ and $u \in B_r$, we define the operator

$$\begin{aligned}(L_{h,\delta}u)(t) &= \lim_{\lambda\to\infty} \alpha \int_0^{t-h} \int_\delta^\infty \frac{1}{\theta}\psi_1(\theta) T_\alpha\Big(\Big(\frac{t^\alpha - \tau^\alpha}{\theta}\Big)^{\frac{1}{\alpha}}\Big)\tau^{\alpha-1} f(\tau, u(\tau))d\theta d\tau \\ &= T_\alpha\Big(\Big(\frac{h}{\delta}\Big)^{\frac{1}{\alpha}}\Big) \lim_{\lambda\to\infty} \alpha \int_0^{t-h} \int_\delta^\infty \frac{1}{\theta}\psi_1(\theta) T_\alpha\Big(\Big(\frac{t^\alpha - \tau^\alpha}{\theta}\Big)^{\frac{1}{\alpha}} - \Big(\frac{h}{\delta}\Big)^{\frac{1}{\alpha}}\Big)\tau^{\alpha-1} f(\tau, u(\tau))d\theta d\tau\end{aligned}$$

where $u \in B_r$. The compactness of $T_\alpha\Big(\Big(\frac{h}{\delta}\Big)^{\frac{1}{\alpha}}\Big)$ implies that the set $\{(L_{h,\delta}u)(t), u \in B_r\}$ is relatively compact in X, $\forall h \in (0, t)$ and $\delta > 0$. Furthermore, we have

$$\begin{aligned}\|(L_2u)(t) - (L_{h,\delta}u)(t)\| &= \alpha\Big\| \int_0^t \int_0^\delta \frac{1}{\theta}\psi_1(\theta)\tau^{\alpha-1} T_\alpha\Big(\Big(\frac{t^\alpha - \tau^\alpha}{\delta}\Big)^{\alpha-1}\Big)\tau^{\alpha-1} f(\tau, u(\tau))d\theta d\tau \\ &+ \int_0^t \int_\delta^\infty \frac{1}{\theta}\tau^{\alpha-1}\psi_1(\theta) T_\alpha\Big(\Big(\frac{t^\alpha - \tau^\alpha}{\delta}\Big)^{\alpha-1}\Big) f(\tau, u(\tau))d\theta d\tau \\ &- \int_0^{t-h} \int_\delta^\infty \frac{1}{\theta}\tau^{\alpha-1}\psi_1(\theta) T_\alpha\Big(\Big(\frac{t^\alpha - \tau^\alpha}{\delta}\Big)^{\alpha-1}\Big) f(\tau, u(\tau))d\theta d\tau\Big\| \\ &\leq \alpha\Big\| \int_0^t \int_0^t \int_0^\delta \frac{1}{\theta}\tau^{\alpha-1}\psi_1(\theta) T_\alpha\Big(\Big(\frac{t^\alpha - \tau^\alpha}{\delta}\Big)^{\alpha-1}\Big) f(\tau, u(\tau))d\theta d\tau\Big\| \\ &+ \Big\| \int_{t-\delta}^t \int_\delta^\infty \frac{1}{\theta}\tau^{\alpha-1}\psi_1(\theta) T_\alpha\Big(\Big(\frac{t^\alpha - \tau^\alpha}{\delta}\Big)^{\alpha-1}\Big) f(\tau, u(\tau))d\theta d\tau\Big\| \\ &\leq M_T\|\mu_t\|_{L^\infty(I,\mathbf{R}^+)} + M_T\|\mu_t\|_{L^\infty(I,\mathbf{R}^+)}T^\alpha + (T-\delta)^\alpha.\end{aligned}$$

We conclude that $V(t), t \in (0, T]$ is relatively compact, and $V(0)$ is also relatively compact, then according to Arzela-Ascoli theorem, $\{(L_2u)(t), u \in B_r\}$ is completely continuous $\forall t \in I$.
Hence, Krasnoselskii's theorem implies that $L_1 + L_2$ has at least one fixed point on B_r. Therefore, our nonlocal Cauchy problem (1) has at least one mild solution. □

In what follows, we are going to give additional assumptions under which the existence of the mild solution of problem (1) is unique.

Theorem 3.17. *We suppose that $f : I \times X \to X$ is continuous and there exists a functions $\mu \in L^\infty(I, \mathbb{R}^+)$ such that*

$$\| f(t,x) - f(t,y) \| \leq \mu_1(t) \| x - y \|,\ \forall t \in I,\ x, y \in X$$

and the function $\varphi : t \to M_T\left(\eta + t^\alpha M_T \| \mu_1 \|_{L^1(I,\mathbb{R}^+)}\right) : I \to \mathbb{R}^+$, satisfies

$$0 < \varphi(t) < \tau < 1,\ \forall t \in I$$

Then under $H_1 - H_4$ and if $u(0) \in \overline{D(A_\alpha)}$ the problem (1) has a unique integral solution in C.

Proof. Define $P : C \to C$ by

$$(Pu)(t) = Q_\alpha(t^\alpha)(u_0 + g(u)) + \lim_{\lambda\to\infty} \alpha \int_0^t s^{\alpha-1} Q_\alpha(t^\alpha - s^\alpha) B^\alpha f_\lambda f(s, u(s))ds$$

Observing that P is well defined on C.
Now take $t \in I$ and $x, y \in C$, we have

$$\begin{aligned} \| (Pu)(t) - (Pv)(t) \| &\leq \eta \| Q_\alpha(t) \| \| u - v \| \\ &+ \lim_{\lambda \to \infty} \int_0^t s^{\alpha-1} \| Q_\alpha(t^\alpha f - s^\alpha f) \| \| B_\lambda^\alpha \| f(s, u(s)) - f(s, v(s)) \| ds \\ &\leq M_T \left(\eta + t^\alpha M_T \| \mu_1 \|_{L^1(I, \mathbb{R}^+)} \right) \|u - v\| \\ &< \tau \| u - v \| \end{aligned}$$

□

4. An illustrative example

In this section, we give an example to illustrate the above results. Consider the following conformable fractional equation

$$\begin{cases} \frac{\partial^\alpha u(t,x)}{\partial t^\alpha} = \frac{\partial u(t,x)}{\partial x} + f(t, u(t,x)),\ 0 \leq t \leq T,\ 0 \leq x \leq \pi, \\ u(t,0) = u(t,\pi) = 0,\ t \in [0,T], \\ u(x,0) = g(u),\ 0 \leq x \leq \pi. \end{cases} \tag{19}$$

Where

$$f(t, u(t,x)) = e^{-t} u(t,x) + e^{-t}$$

and g is a continuous $\overline{D(A_\alpha)}$-valued function defined by

$$g(u)(x) = \sum_{s}^{j=1} p_j u(t_j, x),\ 0 < t_1 < ... < t_s < T,\ x \in [0,\pi],$$

with

$$\sum_{j=1}^{s} |p_j| \leq \frac{1}{2}.$$

Let $m(t) = e^{-t}$, we have

$$\sup_{|u(t,x)| \leq k} |f(t, u(t,x)| \leq \mu_k(t) = (k+1)m(t),\ \text{for } k > 0.$$

which implies that (H_3) is satisfied.
On the other hand, we have

$$|g(u_1) - g(u_2)| \leq \sum_{j=1}^{s} |p_j| \|u_1 - u_2\|_X.$$

Hence, (H_4) is satisfied. Let $X = C[0,\pi]$ and consider the following operator

$$A_\alpha : D(A_\alpha) \subset X \longrightarrow X,$$

defined by

$$D(A_\alpha) = \{u \in C^1[0,\pi],\ u(0) = u(\pi) = 0\}.$$

and

$$A_\alpha u = \frac{\partial}{\partial x} u,\ \ \forall u \in D(A_\alpha).$$

Note that

$$\overline{D(A_\alpha)} = \{u \in C[0,\pi],\ u(0) = u(\pi) = 0\} \neq X.$$

In the other hand, we have

$$(\lambda - A_\alpha)^{-1}u(t) = \int_0^{+\infty} e^{-\frac{\lambda t^\alpha}{\alpha}} t^{\alpha-1}u(t)dt,$$

it is easy to see that

$$\|(\lambda - A_\alpha)^{-1}u\|_X \leq \frac{1}{\alpha}\|u\|_X.$$

Thus our assumption (H_1) is satisfied, and the part A_0^α of A_α generates a $C_0 - \alpha$-semigroup $T_\alpha(t)$. And according to [9], A_0^α generates the following $C_0 - \alpha$-semigroup

$$T_\alpha(t)u(s) = u\Big(s + \frac{t^\alpha}{\alpha}\Big),\ \forall t \geq 0, \forall s \in [0,T].$$

Observe that T_α maps any bounded set to a bounded set, then the assumption (H_2) is also satisfied. Then the conditions of the Theorem 3.16 are satisfied, consequently our problem has at least one integral solution given by

$$u(t) = Q_\alpha(t^\alpha)(u_0 + g(u)) + \lim_{\lambda\to\infty} \alpha \int_0^t Q_\alpha(t^\alpha - \tau^\alpha)B_\lambda^\alpha f(\tau, u(\tau))\tau^{\alpha-1}d\tau.$$

Where

$$Q_\alpha(t^\alpha)u(s) = \int_0^{+\infty} \frac{1}{\theta\alpha}\psi_1(\theta)u\Big(s + \frac{t}{\theta\alpha}\Big)d\theta.$$

5. Conclusion

The existence and uniqueness of solutions for conformable fractional evolution equations with nondense domain in Banach spaces is demonstrated in this article. As a preliminary step, we construct a generic structure of solutions associated with our proposed model utilizing conformable fractional calculus tools and some basic properties of conformable fractional derivative and conformable fractional integral. Our main results are established by using Hile-Yosida theorem associated with conformable fractional semigroup. Finally, by using an appropriate example, the investigation of our theoritical result has been illustrated.

Acknowledgment

The authors would like to express their thanks to the editor and anonymous referees for their suggestions and comments that improved the quality of the paper.

Conflict of interest

The authors declare that they have no conflict of interest.

Data Availability

The data used to support the findings of this study are included in the references within the article.

References

[1] M. I. Abbas, M. A. Ragusa, Solvability of Langevin equations with two Hadamard fractional derivatives via Mittag-Leffler functions, Appl. Anal., 101 (2022), 3231–3245. https://doi.org/10.1080/00036811.2020.1839645

[2] M. I. Abbas, M. A. Ragusa, On the hybrid fractional differential equations with fractional proportional derivatives of a function with respect to a certain function, Symmetry, 13 (2021). https://doi.org/10.3390/sym13020264

[3] T. Abdeljawad, On conformable fractional calculus, Journal of computational and Applied Mathematics. 279(2015), 57-66.

[4] R.P. Agarwal, H. Zhou, Y. He, Existence of fractional neutral functional differential equations, Comput. Math. Appl. 59(3)(2010), 1095-1100.

[5] S. Al-Sharif, R. Al-Shorman. New Results On The Genarator Of Conformable Semigroups, Progr. Fract. Differ. Appl. 8(1)(2022), 77-89.

[6] A. Balakrishnan, Fractional Powers Of Closed Operators And The Semigroups Generated By Them, Pacific Journal of Mathematics 10(1960), 419-439.

[7] A. El Mfadel, S. Melliani, A. Kassidi and M. Elomari, Existence of mild solutions for nonlocal $\psi-$Caputo-type fractional evolution equations with nondense domain. Nonautonomous Dynamical Systems, 9(1), 272-289.

[8] A. El Mfadel, S. Melliani, M. Elomari, Existence and uniqueness results for Caputo fractional boundary value problems involving the p-Laplacian operator. U.P.B. Sci. Bull. Series A. **84**(1)(2022) , 37–46

[9] A. El Mfadel, S. Melliani, M. Elomari, New existence results for nonlinear functional hybrid differential equations involving the $\Psi-$ Caputo fractional derivative. Results in Nonlinear Analysis. 5(1)(2022), 78-86

[10] A. El Mfadel, S. Melliani, M. Elomari, Existence results for nonlocal Cauchy problem of nonlinear $\Psi-$Caputo type fractional differential equations via topological degree methods. Advances in the Theory of Nonlinear Analysis and its Application. **6**(2)(2022), 270–279

[6] H. Gu, Y. Zhou, B. Ahmad, A. Alsaedi, Integral Solutions Of Fractional Evolution Equations With Nondense Domain, Electronic Journal of Differential Equations. 145 (2017),1-15.

[9] M.A. Horani, R. Khalil, T. Abdeljawad, Conformable fractional semigroups of operators,(2014), arXiv preprint arXiv:1502.06014.

[11] R. Khalil, M. Al Horani, A. Yousef, M.Sababheh, A new Definition Of Fractional Derivative, J. Comput. Appl. Math. (264)(2014), 65-80.

[5] G.M. Mophou, G.M. N'Guérékata, On integral solutions of some nonlocal fractional differential equations with nondense domain, Nonlinear analysis. 71(2009), 4668-4675.

[4] G. Prato, E. Sinestrari, Differential operators with non dense domain. Annali della scuola normale superiore di pisa-classe di scienze, 14(2)(1987), 285-344.

[7] L. Rabhi, M. Al Horani, R. Khalil, Existence results of mild solutions for nonlocal fractional delay integro-differential evolution equations via Caputo conformable fractional derivative, AIMS Mathematics. 7(7)(2022),11614-11634.

[8] L. Rabhi, M. Al Horani, R. Khalil. homogeneous Conformable Abstract Cauchy Problem, Open Mathematics. 19 (2021), 690–705.

[10] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differntial Equations, Springer-Verlag, (1983).

[13] Y. Zhou, F. Jiao, Nonlocal Cauchy problem for fractional evolution equations, Nonlinear Analysis: Real World Applications. 11 (2010), 4465–4475.