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Operational matrix approach for solving fractional vibration equation of large membranes with error estimation

Sima Aghchi[a], HongGunag Sun[a], Hossein Fazli[a,*]

[a] *State Key Laboratory of Hydrology-Water Resources and Hydraulic Engineering, College of Mechanics and Materials, Hohai University, Nanjing 210098, Jiangsu, China*

Abstract. The principal purpose of this work is to present a numerical technique for the fractional vibration equation of large membranes. This method uses the Chebyshev cardinal functions and the required approximate solution as the elements of Chebyshev cardinal functions. Using the operational matrix of derivative, the time fractional vibration equation is reduced to a set of algebraic equations. Meanwhile, an estimation of the error bound for this algorithm is given on the basis of some theorems. Two numerical examples are included by taking different initial conditions to demonstrate the efficiency and applicability of this approach. To examine the accuracy of the suggested method, the numerical results are compared with the existing analytical methods.

1. Introduction

Dynamical behaviors of physical phenomena and processes are typically characterized by differential equations. Differential equations of fractional order provide an excellent instrument in deriving an accurate model for the complex physical processes with memory and hereditary properties, for instance, viscoelastic deformation process [11], chemotactic model of bacteria colonies [22], anomalous transport in complex systems [29], relaxation in filled polymer networks [17], reaction kinetics and relaxation dynamics of proteins [9], description of damping behaviour of material [8], the constitutive modeling of human brain tissue [31]. The existence of memory term in such models not only takes into account the history of the involved physical system, but also carries its impact to present and even future physical behaviors. The fractional operators depending on their nonlocal kernel can be used effectively as an powerful instrument for the description of memory and hereditary properties. Therefore, fractional calculus has been usefully employed as an admissible candidate to provide an effective mathematical framework to accurately characterize complex physical systems and processes. For details in this connection, the reader can refer to the monographs of Diethelm [6], Hilfer [10], Kilbas et al. [13], Podlubny [21] and the references therein. In order to find more about the fractional calculus, we refer the readers to the recently published works of Srivastava [25–27]. It

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* Corresponding author: Hossein Fazli
Email addresses: sima.aghchi@gmail.com (Sima Aghchi), shg@hhu.edu.cn (HongGunag Sun), h_fazli@tabrizu.ac.ir (Hossein Fazli)

should be noted here that the bibliographies of these survey-cum-expository articles include a number of recently published journal articles which have dealt with the extensively investigated subject of fractional calculus and its widespread applications.

Vibration is a natural phenomenon that is virtually omnipresent in almost all branches of natural sciences and engineering. It has an appreciable effect on the nature of engineering design, especially the design of complex systems [18, 32]. The most important types of vibration is the vibration of large membranes which plays a dominant role in various fields such as analysing the propagation of two-dimensional waves [20], biomedical engineering [12], and structural acoustics analysis [30]. Vibration equation provides a powerful approach of describing the vibration of a membrane.

Our goal of this paper is to investigate the fractional derivative vibration equation model of circular membranes as follows:

$$\frac{1}{c^2}\frac{\partial^{\varrho} u(t,r)}{\partial t^{\varrho}} = \frac{\partial^2 u(t,r)}{\partial r^2} + \frac{1}{r}\frac{\partial u(t,r)}{\partial r}, \quad 1 < \varrho \le 2, \tag{1}$$

with the initial conditions

$$u(0,r) = \varsigma(r), \ \frac{\partial u(0,r)}{\partial t} = c\zeta(r), \quad 0 \le r, t \le 1, \tag{2}$$

where the component $u(r,t)$ represents the displacement of particle at the position r at time t, and c is the wave velocity of free vibration. From the physical and mathematical point of view it will be felicitous to specify two initial conditions, on the displacement u and the velocity u_t, at time $t = 0$ [7].

A variety of efficient numerical techniques for solving vibration equation of both fractional and integer orders are available in literature such as homotopy perturbation method [19], variational iteration method and modified decomposition method [3], modified decomposition method [4], homotopy perturbation method and homotopy analysis method [5]. Srivastava et al. [28] presented an efficient analytical approach based on the q-homotopy analysis transform technique to analyze a fractional model of the vibration equation for large membranes. In [14], Kumar et al. have reported a novel extension of fractional vibration equation using Atangana-Baleanu fractional derivative operator and proposed a numerical algorithm based on homotopic technique to examine the fractional vibration equation for very large membranes with distinct special cases. Operational matrices of orthonormal polynomials such as the Legendre polynomials and Jacobi polynomials are used to obtain approximate numerical solutions of the fractional vibration equation [23, 24]. However, it should be emphasized that efficient numerical method for time fractional derivative vibration equation has not been well established until now.

In the present paper, we make an attempt to use a computational method which is based on the operational matrices of Chebyshev cardinal functions. The advantage and importance of this numerical algorithm arises from the fact that it is a straightforward to implement and computationally very striking. The outline of this paper is organized as follows. In Section 2, we give some fractional calculus preliminaries and some properties of two-dimensional shifted orthonormal Chebyshev cardinal functions with a view to constructing some operational matrices based on Chebyshev cardinal functions. In Section 3, the proposed approach is used to numerically solve the fractional vibration equation (1)-(2), and in Section 4, an estimation of the error bound is derived for the proposed numerical technique. Finally in the last section, some test problems and comparisons with other numerical approaches are provided.

2. Preliminaries

2.1. Fractional operators

The purpose of this subsection is to collect a number of definitions and lemmas concerning the fractional derivatives and integrals.

Definition 2.1. *[6] Let $\varrho > 0$. The Riemann-Liouville fractional integral of order $\varrho > 0$ of a function $\varphi : [0,1] \to \mathbb{R}$ is defined as*

$$J_t^{\varrho}\varphi(t) = \frac{1}{\Gamma(\varrho)}\int_0^t (t-s)^{\varrho-1}\varphi(s)ds.$$

provided that the right-hand side integral exists and is finite.

Definition 2.2. *[6] Let $n-1<\varrho\leq n$ and $n\in\mathbb{N}$. The Liouville-Caputo fractional derivative of order $\varrho>0$ of a function $\varphi:[0,1]\to\mathbb{R}$ is defined as*

$$\frac{d^{\varrho}\varphi(t)}{dt^{\varrho}} = J_t^{n-\varrho}\frac{d^n}{dt^n}\varphi(t),$$

provided that the right-hand side integral exists and is finite.

2.2. Chebyshev cardinal functions

Here, we review some basic definitions and results related to the Chebyshev polynomials [1, 2].

Let $T_{N+1}(r)$ be the first kind Chebyshev polynomials of order $N+1$ on the interval $[-1,1]$. The Chebyshev cardinal functions of order N on the interval $[-1,1]$ are given as

$$\psi_j(r)=\frac{T_{N+1}(r)}{T_{N+1,r}(r_j)(r-r_j)},\quad j=1,2,\cdots,N+1, \tag{3}$$

where $r_j,\ j=1,2,\cdots,N+1$ are the zeros of $T_{N+1}(r)$ given by $\cos\left((2j-1)\pi/(2N+2)\right),\ j=1,2,\cdots,N+1$ and the subscript r stands for r-differentiation.

Lemma 2.3. *The Chebyshev cardinal functions $\psi_j(r), j=1,2,\cdots,N+1$ are orthogonal with respect to weight function $(1-r^2)^{-\frac{1}{2}}$ on the interval $[-1,1]$, i.e.,*

$$\int_{-1}^{1}\psi_i(r)\psi_j(r)(1-r^2)^{-\frac{1}{2}}dr=\frac{\pi}{N+1}\delta_{ij}, \tag{4}$$

where δ_{ij} denotes the usual Kronecker delta function.

In a similar manner, two-dimensional shifted Chebyshev cardinal functions of order N in variables r and r' are defined on $[-1,1]\times[-1,1]$ as follows

$$P_{i,j}(r,r')=\psi_i(r)\psi_j(r'),\quad i,j=0,1,2,\cdots,N+1. \tag{5}$$

We consider the space $L^2([-1,1]\times[-1,1])$ equipped with the following weighted norm

$$\|u(r,r')\|=\left(\int_{-1}^{1}\int_{-1}^{1}|u(r,r')|^2\omega(r,r')\,drdr'\right)^{\frac{1}{2}}, \tag{6}$$

where $\omega(r,r')=(1-r^2)^{-\frac{1}{2}}(1-r'^2)^{-\frac{1}{2}}$.

The set of two-dimensional shifted Chebyshev cardinal functions forms a complete orthogonal system in the $L^2([-1,1]\times[-1,1])$ sense such that the orthogonality condition is

$$\int_{-1}^{1}\int_{-1}^{1}P_{i,j}(r,r')P_{m,n}(r,r')\omega(r,r')drdr'=\begin{cases}\frac{\pi^2}{(N+1)^2} & \text{for } i=m, j=n,\\ 0 & \text{otherwise.}\end{cases} \tag{7}$$

For using these functions on $[0,1]$, we make a simple linear change of variable $x=(r+1)/2$. Now we can approximate any function g defined on the interval $[0,1]$ as follows

$$g(x)\approx\sum_{j=1}^{N+1}g(x_j)\psi_j(x)=G^T\Psi_N(x), \tag{8}$$

where $x_j,\ j=1,2,\cdots,N+1$ are the shifted points of $r_j,\ j=1,2,\cdots,N+1$ by using the transformation $x=(r+1)/2$,

$$G=\left[g(x_1),g(x_2),\cdots,g(x_{N+1})\right]^T, \tag{9}$$

and

$$\Psi_N(x) = [\psi_1(x), \psi_2(x), \cdots, \psi_{N+1}(x)]^T. \tag{10}$$

Also, the unknown function $u(t, x)$ of two independent variables $(t, x) \in [0, 1] \times [0, 1]$ can be expanded in terms of double Chebyshev cardinal functions as

$$u(t, x) = \sum_{i=1}^{N+1} \sum_{j=1}^{N+1} u(t_i, x_j)\psi_i(t)\psi_j(x) = \Psi_N^T(t)U\Psi_N(x), \tag{11}$$

where the unknown matrix U is $(N + 1) \times (N + 1)$ as

$$U = \begin{bmatrix} u_{1,1} & \cdots & u_{1,N+1} \\ \vdots & & \vdots \\ u_{N+1,1} & \cdots & u_{N+1,N+1} \end{bmatrix},$$

and

$$u_{i,j} := u(t_i, x_j), \quad i, j = 1, 2, \cdots, N + 1.$$

2.3. The operational matrices

Our object in this subsection is to introduce and construct some operational matrices based on Chebyshev cardinal functions.

Theorem 2.4. *Let $\Psi_N(x) = [\psi_1(x), \psi_2(x), \cdots, \psi_{N+1}(x)]^T$ be Chebyshev cardinal vector. Then the differentiation of Ψ_N can be characterized by*

$$\Psi_N' = D\Psi_N, \tag{12}$$

where $D = [D(i, j)]$ is $(N + 1) \times (N + 1)$ operational matrix of derivative based on Chebyshev cardinal functions and the entries of D can be given by

$$D(i, i) = \sum_{\substack{k=1 \\ k\neq i}}^{N+1} \frac{1}{x_i - x_k}, \quad i = 1, 2, \cdots, N + 1, \tag{13}$$

and

$$D(i, j) = \frac{2^{2N+1}}{T_{N+1,x}(x_i)} \prod_{\substack{k=1 \\ k\neq i,j}}^{N+1} (x_j - x_k), \quad i, j = 1, 2, \cdots, N + 1,\ i \neq j. \tag{14}$$

Proof. See [15]. □

Theorem 2.5. *Let $\Psi_N(x) = [\psi_1(x), \psi_2(x), \cdots, \psi_{N+1}(x)]^T$ be Chebyshev cardinal vector. Then the fractional integration of order $\varrho > 0$ of Ψ_N can be characterized by*

$$J_x^{\varrho}\Psi_N = J_{\varrho}\Psi_N, \tag{15}$$

where $J_{\varrho} = \left[J_{\varrho}(i, j)\right]$ is $(N+1)\times(N+1)$ operational matrix of fractional integration of order $\varrho > 0$ based on Chebyshev cardinal functions and the entries of J_{ϱ} can be given by

$$J_{\varrho}(i, j) = \frac{2^{2N+1}}{T_{N+1,x}(x_i)} \left(\sum_{k=0}^{N} a_k \frac{\Gamma(k + 1)}{\Gamma(k + 1 + \varrho)} x_j^{k+\varrho} \right), \quad i, j = 1, 2, \cdots, N + 1, \tag{16}$$

where

$$a_{N-k} = (-1)^k \left(\sum_{\substack{1\le i_1<i_2<\cdots<i_k\le N+1 \\ i_1,i_2,\cdots,i_k\neq j}} x_{i_1}x_{i_2}\cdots x_{i_k} \right) \quad ,k = 1,2,\cdots,N, \tag{17}$$

and $a_N = 1$.

Proof. Owing to the representation $T_{N+1}(x) = 2^{2N+1}\prod_{k=1}^{N+1}(x-x_k)$, we have

$$J_x^{\varrho}\psi_j(x) = \frac{2^{2N+1}}{T_{N+1,x}(x_j)} J_x^{\varrho}\left(\prod_{\substack{k=1\\k\neq j}}^{N+1}(x-x_k) \right), \quad j = 1,2,\cdots,N+1. \tag{18}$$

Now by expanding the product $\prod_{\substack{k=1\\k\neq j}}^{N+1}(x-x_k)$ as

$$\prod_{\substack{k=1\\k\neq j}}^{N+1}(x-x_k) = a_N x^N + a_{N-1}x^{N-1} + \cdots + a_0, \tag{19}$$

where the coefficients $a_{N-k}, k = 0,1,2,\cdots,N$ can be given via Vieta's formula as

$$a_{N-k} = (-1)^k \left(\sum_{\substack{1\le i_1<i_2<\cdots<i_k\le N+1 \\ i_1,i_2,\cdots,i_k\neq j}} x_{i_1}x_{i_2}\cdots x_{i_k} \right) \quad ,k = 1,2,\cdots,N, \tag{20}$$

and $a_N = 1$. Therefore, we have

$$\begin{aligned} J_x^{\varrho}\psi_j(x) &= \frac{2^{2N+1}}{T_{N+1,x}(x_j)} J_x^{\varrho}\left(x^N + a_{N-1}x^{N-1} + \cdots + a_0\right) \\ &= \frac{2^{2N+1}}{T_{N+1,x}(x_j)} \left(\sum_{k=0}^{N} a_k \frac{\Gamma(k+1)}{\Gamma(k+1+\varrho)} x^{k+\varrho} \right), \end{aligned} \tag{21}$$

for $j = 1,2,\cdots,N+1$. Now using (8), any function $J^{\varrho}\psi_j(x)$ can be approximated by a finite sum of the form

$$J_x^{\varrho}\psi_j(x) = \sum_{k=1}^{N+1} J_x^{\varrho}\psi_j(x_k)\psi_k(x). \tag{22}$$

Comparing (15) and (22), we have

$$J_{\varrho} = \begin{bmatrix} J_x^{\varrho}\psi_1(x_1) & \cdots & J^{\varrho}\psi_1(x_{N+1}) \\ \vdots & & \vdots \\ J_x^{\varrho}\psi_{N+1}(x_1) & \cdots & J^{\varrho}\psi_{N+1}(x_{N+1}) \end{bmatrix},$$

where the entries of matrix J_{ϱ} can be calculated by (21). □

Theorem 2.6. *Let* $\Psi_N(x) = [\psi_1(x), \psi_2(x), \cdots, \psi_{N+1}(x)]^T$ *be Chebyshev cardinal vector. Then*

$$x\Psi_N(x) = E\Psi_N(x), \tag{23}$$

where $E = [E(i,j)]$ *is* $(N+1)\times(N+1)$ *operational matrix based on Chebyshev cardinal functions and the entries of* E *can be given by*

$$E(i,j) = \frac{2^{2N+1}}{T_{N+1,x}(x_i)}\sum_{k=0}^{N} a_k x_j^{k+1}, \quad i,j = 1,2,\cdots,N+1, \tag{24}$$

where

$$a_{N-k} = (-1)^k \left(\sum_{\substack{1\le i_1<i_2<\cdots<i_k\le N+1\\ i_1,i_2,\cdots,i_k\neq j}} x_{i_1}x_{i_2}\cdots x_{i_k} \right) \quad ,k = 1,2,\cdots,N, \tag{25}$$

and $a_N = 1$.

Proof. Owing to the representation $T_{N+1}(x) = 2^{2N+1}\prod_{k=1}^{N+1}(x - x_k)$, we have

$$x\psi_j(x) = \frac{2^{2N+1}x}{T_{N+1,x}(x_j)}\left(\prod_{\substack{k=1\\k\neq j}}^{N+1}(x - x_k)\right), \quad j = 1,2,\cdots,N+1. \tag{26}$$

By calculations similar to those for the proof of 2.5, we have

$$x\psi_j(x) = \frac{2^{2N+1}}{T_{N+1,x}(x_j)}\left(\sum_{k=0}^{N} a_k x^{k+1}\right), \tag{27}$$

for $j = 1,2,\cdots,N+1$. Now using (8), any function $x\psi_j(x)$ can be approximated by a finite sum of the form

$$x\psi_j(x) = \sum_{k=1}^{N+1} x_k\psi_j(x_k)\psi_k(x). \tag{28}$$

Comparing (23), (27) and (28), we obtain the desired result. □

3. Description of numerical method

To use the Chebyshev cardinal functions for solving the main problem (1)-(2), we use (11) to approximate $u(t,x)$ as

$$u(t,x) = \Psi_N^T(t)U\Psi_N(x), \tag{29}$$

where U is the unknown $(N+1)\times(N+1)$ matrix and must be found.

Now, by applying the fractional integral operator J_t^α to both sides of (1) and using the initial condition (2), we have

$$\frac{1}{c^2}u(t,x) = J_t^\alpha\frac{\partial^2 u(t,x)}{\partial x^2} + \frac{1}{x}J_t^\alpha\frac{\partial u(t,x)}{\partial x} + \varsigma(x) + ct\zeta(x), \tag{30}$$

and then

$$xu(t,x) = c^2xJ_t^\alpha\frac{\partial^2 u(t,x)}{\partial x^2} + c^2J_t^\alpha\frac{\partial u(t,x)}{\partial x} + f(t,x), \tag{31}$$

where $f(t,x) = c^2\left(x\varsigma(x) + ctx\zeta(x)\right)$ which approximated as

$$f(t,x) = \sum_{i=1}^{N+1}\sum_{j=1}^{N+1} f(t_i,x_j)\psi_i(t)\psi_j(x) = \Psi_N^T(t)F\Psi_N(x), \tag{32}$$

where F is an $(N+1)\times(N+1)$ matrix as

$$F = \begin{bmatrix} F_{1,1} & \cdots & F_{1,N+1} \\ \vdots & \ddots & \vdots \\ F_{N+1,1} & \cdots & F_{N+1,N+1} \end{bmatrix},$$

and

$$F_{i,j} = f(t_i,x_j), \quad i,j = 1,2,\cdots,N+1.$$

Then, employing the operational matrices as indicated in the last section, we get

$$\Psi_N^T(t)UE\Psi_N(x) = c^2\Psi_N^T(t)J_\varrho^T UED^2\Psi_N(x) + c^2\Psi_N^T(t)J_\varrho^T UD\Psi_N(x) + \Psi_N^T(t)F\Psi_N(x), \tag{33}$$

or equivalently,

$$\Psi_N^T(t)\left\{UE - c^2 J_\varrho^T UED^2 - c^2 J_\varrho^T UD - F\right\}\Psi_N(x) = 0. \tag{34}$$

The entries of vectors $\Psi_N(t)$ and $\Psi_N(x)$ are independent, so we get

$$H := UE - c^2 J_\varrho^T UED^2 - c^2 J_\varrho^T UD - F = 0. \tag{35}$$

The final task is solving the Sylvester equation (35) to get the unknown matrix U. Using the value of U in (29), we can obtain an approximate solution $u(t,x)$ for the main problem (1)-(2).

4. Error estimation

In this section, we will derive an estimation of the error bound for the approximate solution to the main problem (1)-(2). To do this, let us consider the Banach space $C[0,1]\times C[0,1]$ endowed with the mentioned norm in Section 2.

Definition 4.1. *[16] Let X be a normed linear space, let Y be a subspace of X and let $u(x)$ be a given function in X.*

- *An approximation $u^*(x)$ in Y is said to be a good approximation if $\|u(x) - u^*(x)\| \le \varepsilon$, where ε is a prescribed level of absolute accuracy.*
- *An approximation $u_B^*(x)$ in Y is a best approximation if, for any other approximation $u^*(x)$ in Y, $\|u(x) - u_B^*(x)\| \le \|u(x) - u^*(x)\|$.*

Lemma 4.2. *Let $u_B^*(t,x)$ be the best approximation of sufficiently smooth function $u(t,x)$. Then,*

$$\left\|u(t,x) - u_B^*(t,x)\right\| \le \frac{\pi}{2^{2N+1}(N+1)!}\left(M_1 + M_2 + \frac{M_3}{2^N(N+1)!}\right),$$

where $M_1 = \max_{(t,x)\in\Delta}\left|\frac{\partial^{N+1}u(t,x)}{\partial t^{N+1}}\right|$, $M_2 = \max_{(t,x)\in\Delta}\left|\frac{\partial^{N+1}u(t,x)}{\partial x^{N+1}}\right|$, $M_3 = \max_{(t,x)\in\Delta}\left|\frac{\partial^{2N+2}u(t,x)}{\partial t^{N+1}\partial x^{N+1}}\right|$, *and* $\Delta = [0,1]\times[0,1]$.

Proof. Let $\Theta_{N,N}(t,x)$ be an arbitrary polynomial of degree $\leq N$ in variables t and x, then from the definition of the best approximation, we have

$$\|u(t,x) - u_B^*(t,x)\| \leq \|u(t,x) - \Theta_{N,N}(t,x)\|. \tag{36}$$

The estimate (36) holds if $\Theta_{N,N}(t,x)$ is the interpolating polynomial of u on the nodes (t_i, x_j) with $t_i = \frac{r_i+1}{2}$ and $x_j = \frac{r_j+1}{2}$ where $r_i, r_j,\ i,j = 1,2,\cdots,N+1$ are the zeros of $T_{N+1}(r)$. Then, one can write

$$\begin{aligned} u(t,x) - \Theta_{N,N}(t,x) &= \frac{\partial^{N+1}u(\tau,x)}{(N+1)!\partial t^{N+1}}\prod_{i=0}^{N}(t-t_i) + \frac{\partial^{N+1}u(t,\chi)}{(N+1)!\partial x^{N+1}}\prod_{j=0}^{N}(x-x_j) \\ &\quad - \frac{\partial^{2N+2}u(\tau',\chi')}{(N+1)!^2\partial t^{N+1}\partial x^{N+1}}\prod_{i=0}^{N}(t-t_i)\prod_{j=0}^{N}(x-x_j) \end{aligned}$$

for $\tau, \chi, \tau', \chi' \in [0,1]$. Therefore, we obtain

$$\begin{aligned} \left|u(t,x) - \Theta_{N,N}(t,x)\right| &\leq \max_{(t,x)\in\Delta}\left|\frac{\partial^{N+1}u(t,x)}{\partial t^{N+1}}\right|\Omega(t)| + \max_{(t,x)\in\Delta}\left|\frac{\partial^{N+1}u(t,x)}{\partial x^{N+1}}\right||\Omega(x)| \\ &\quad + \max_{(t,x)\in\Delta}\left|\frac{\partial^{2N+2}u(t,x)}{\partial t^{N+1}\partial x^{N+1}}\right||\Omega(t)\Omega(x)|, \end{aligned}$$

where $\Omega(z) = \prod_{j=0}^{N}\frac{(z-z_i)}{(N+1)!}$, $z_i = \frac{r_i+1}{2}$, and $r_i,\ i = 1,2,\cdots,N+1$ are the zeros of $T_{N+1}(r)$. Let us therefore attempt to find a bound for $\Omega(z)$. Using the change of variable $z = \frac{r+1}{2}$, we find

$$\Omega(z) = \frac{1}{2^{N+1}(N+1)!}\prod_{i=0}^{N}(r-r_i).$$

Now since the leading term of the Chebyshev polynomial $T_{N+1}(r)$ is 2^N, we have

$$\prod_{i=0}^{N}(r-r_i) = \frac{1}{2^N}T_{N+1}(r).$$

On the other hand, we know that $\max_{r\in[-1,1]}|T_{N+1}(r)| = 1$. Therefore,

$$\left|u(t,x) - \Theta_{N,N}(t,x)\right| \leq \frac{1}{2^{2N+1}(N+1)!}\left(M_1 + M_2 + \frac{M_3}{2^N(N+1)!}\right).$$

Consequently (36) implies

$$\left\|u(t,x) - u_B^*(t,x)\right\| \leq \frac{\pi}{2^{2N+1}(N+1)!}\left(M_1 + M_2 + \frac{M_3}{2^N(N+1)!}\right), \tag{37}$$

which is the desired result. $\square$

Theorem 4.3. *Let $u(t,x)$, $u_B^*(t,x) = \Psi_N^T(t)U^*\Psi_N(x)$, and $\bar{u}(t,x) = \Psi_N^T(t)\bar{U}\Psi_N(x)$ be the exact solution, the best approximation and the approximate solution produced by the proposed method, respectively. Then*

$$\|u(t,x) - \bar{u}(t,x)\| \leq \frac{\pi}{2^{2N+1}(N+1)!}\left(M_1 + M_2 + \frac{M_3}{2^N(N+1)!}\right) + \frac{\pi}{2\sqrt{N+1}}\left\|U^* - \bar{U}\right\|_2, \tag{38}$$

where the norm $\|\cdot\|_2$ on the right hand side is the usual Euclidean norm for matrices.

Proof. Let $u_B^*(t,x) = \Psi_N^T(t)U^*\Psi_N(x)$ be the best approximation. Then, we have

$$\left\|u_B^*(t,x) - \bar{u}(t,x)\right\| = \left(\int_0^1\int_0^1 \left|\sum_{i=1}^{N+1}\sum_{j=1}^{N+1}(u_{i,j}^* - \bar{u}_{i,j})\psi_i(t)\psi_j(x)\right|^2 \bar{\omega}(t,x)dtdx\right)^{\frac{1}{2}},$$

where $\bar{\omega}(t,x) = \omega(2t-1, 2x-1)$. Using Hölder's inequality, we then have

$$\begin{aligned}
&\int_0^1\int_0^1 \left|\sum_{i=1}^{N+1}\sum_{j=1}^{N+1}(u_{i,j}^* - \bar{u}_{i,j})\psi_i(t)\psi_j(x)\right|^2 \bar{\omega}(t,x)dtdx \\
&\leq \int_0^1\int_0^1 \left(\sum_{i=1}^{N+1}\sum_{j=1}^{N+1}\left|u_{i,j}^* - \bar{u}_{i,j}\right|^2\right)\left(\sum_{i=1}^{N+1}\sum_{j=1}^{N+1}\left|\psi_i(t)\psi_j(x)\right|^2\right)\bar{\omega}(t,x)dtdx \\
&= \sum_{i=1}^{N+1}\sum_{j=1}^{N+1}\left|u_{i,j}^* - \bar{u}_{i,j}\right|^2 \left(\sum_{i=1}^{N+1}\sum_{j=1}^{N+1}\int_0^1\int_0^1 \left|\psi_i(t)\psi_j(x)\right|^2 \bar{\omega}(t,x)dtdx\right) \\
&= \sum_{i=1}^{N+1}\sum_{j=1}^{N+1}\left|u_{i,j}^* - \bar{u}_{i,j}\right|^2 \left(\sum_{i=1}^{N+1}\sum_{j=1}^{N+1}\int_0^1\int_0^1 \left|\psi_i(t)\psi_j(x)\right|^2 \bar{\omega}(t,x)dtdx\right) \\
&= \frac{\pi^2}{4(N+1)}\left(\sum_{i=1}^{N+1}\sum_{j=1}^{N+1}\left|u_{i,j}^* - \bar{u}_{i,j}\right|^2\right).
\end{aligned}$$

The last equality follows by Lemma 2.3. From this we conclude that

$$\begin{aligned}
\left\|u_B^*(t,x) - \bar{u}(t,x)\right\| &\leq \frac{\pi}{2\sqrt{N+1}}\left(\sum_{i=1}^{N+1}\sum_{j=1}^{N+1}\left|u_{i,j}^* - \bar{u}_{i,j}\right|^2\right)^{\frac{1}{2}} \\
&= \frac{\pi}{2\sqrt{N+1}}\left\|U^* - \bar{U}\right\|_2. \qquad (39)
\end{aligned}$$

Finally, the desired estimate is a trivial consequence of the triangle inequality together with (37) and (39). □

5. Numerical experiments and discussion

In the section, we examine the reliability and approximate capability of the proposed method by testing our algorithm to the some special cases of the main problem which arise by taking different initial conditions. For each case we have shown results graphically by taking different velocities and fractional order in FVE. The following are some particular cases of the main problem with initial conditions:

Case 1: $\varsigma(r) = r^2$ and $\zeta(r) = r$.

Case 2: $\varsigma(r) = r^2$ and $\zeta(r) = r^2$.

We now carry out the proposed method to compute the displacement profile $u(t,r)$ for the two distinct cases as stated above for the different values of time fractional derivative order and radii of membrane and explain in Tables 1 and 2 and Figures 1-4. From the Tables 1 and 2, we observe that the results derived by expanding the displacement profile $u(t,r)$ as the elements of Chebyshev cardinal functions are in close consonance with the results derived in [3–5, 14, 19, 23, 24, 28]. The approximate solutions for distinct values of ϱ are shown in Figures 1, 3 and 5 for the two distinct cases 1 and 2, respectively. These figures show that the displacement profile of membranes will increase with an increase of both time and radii at the velocity of wave $c = 0.1$. Figures 2(a) and 4(a) show the behaviour of the displacement profile of membranes with time for distinct value of $\varrho = 1.5, 1.7, 1.9$ and $\varrho = 2$, respectively, at fix value of radii of membranes. It is clear from Figures 2(b) and 4(b) that the displacement profile of membranes increases with increasing of wave speed.

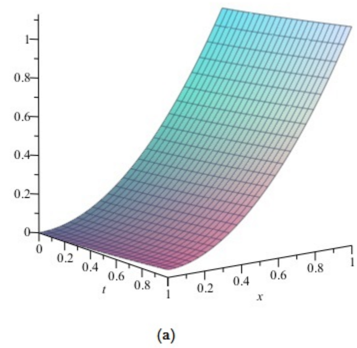

Figure 1: (a) Approximate solution for $\varrho = 1.5$ and (b) Approximate solution for $\varrho = 2$ for Case 1.

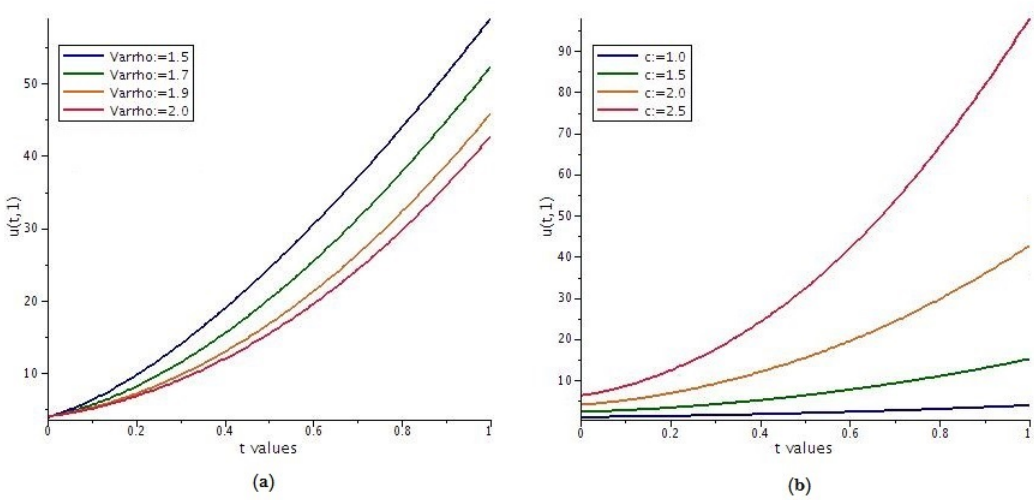

Figure 2: (a) Approximate solution for different values of ϱ at $c = 2$ and (b) Approximate solution for different values of wave velocities c at $\varrho = 2$ for Case 1.

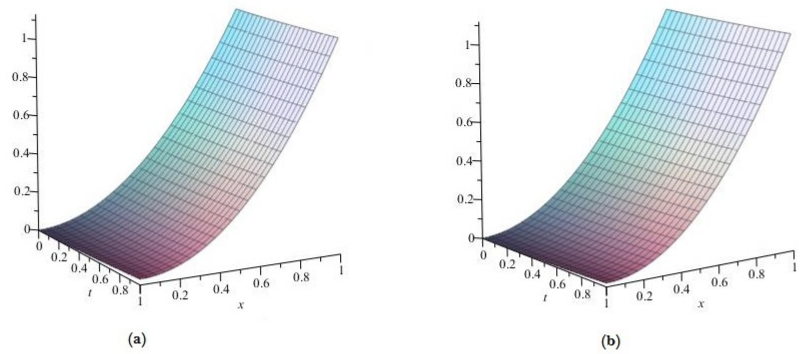

Figure 3: (a) Approximate solution for $\varrho = 1.5$ and (b) Approximate solution for $\varrho = 2$ for Case 1.

Table 1: The comparison of results for Case 1, $\varsigma(r) = r^2, \zeta(r) = r$ at $c = 0.1$ and different values of ϱ.

(t, r)	ϱ	Present method	Methods in [3–5, 14, 19, 23, 24, 28]
$(0.2, 0.2)$	1.5	0.0466	0.0467
	2.0	0.0448	0.0448
$(0.4, 0.4)$	1.5	0.1837	0.1837
	2.0	0.1792	0.1792
$(0.6, 0.6)$	1.5	0.4101	0.4101
	2.0	0.4032	0.4032
$(0.8, 0.8)$	1.5	0.7256	0.7257
	2.0	0.7168	0.7169
$(1.0, 1.0)$	1.5	1.1299	1.1303
	2.0	1.1199	1.1202

Table 2: The comparison of results for Case 3, $\varsigma(r) = r^2, \zeta(r) = r^2$ at $c = 0.1$ and different values of ϱ.

(t, r)	ϱ	Present method	Methods in [3–5, 14, 19, 23, 24, 28]
$(0.2, 0.2)$	1.5	0.0435	0.0435
	2.0	0.0416	0.0416
$(0.4, 0.4)$	1.5	0.1741	0.1741
	2.0	0.1696	0.1696
$(0.6, 0.6)$	1.5	0.3959	0.3959
	2.0	0.3889	0.3889
$(0.8, 0.8)$	1.5	0.7134	0.7134
	2.0	0.7043	0.7043
$(1.0, 1.0)$	1.5	1.1312	1.1313
	2.0	1.1206	1.1207

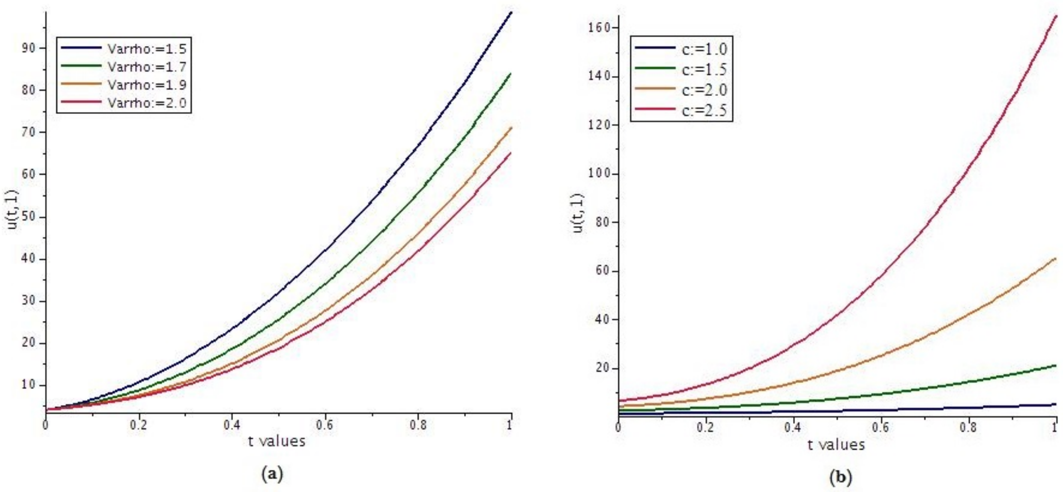

Figure 4: (a) Approximate solution for different values of ϱ at $c = 2$ and (b) Approximate solution for different values of wave velocities c at $\varrho = 2$ for Case 1.

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