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A nonlinear fuzzy contraction principle via control functions

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Abstract. In this work, we develop a new type of nonlinear fuzzy contraction, namely fuzzy $\mathcal{L}_\lambda$-weak contraction, based on fuzzy $\mathcal{L}_\lambda$-simulation function and fuzzy Θ_f-contractive mappings. Then, using the specifically developed contraction, we show the existence and uniqueness of a fixed point for a self-mapping in complete fuzzy metric spaces. We provide an example, together with some illustrative corollaries and remarks, to further prove the validity of our findings. The presented findings combine, enhance, and extend a number of earlier research findings.

1. Introduction and Preliminaries

Since Zadeh's famous 1965 paper [1], fuzzy sets have become ever more appealing. With numerous applications in the fields of computing and engineering, this has resulted a significant development in both theory and application in the fields of logic, topology, and analysis. Fuzzy metric spaces were first established by Kramosil and Michaelek [2], and were further refined by George and Veeramani [4], who also showed that each fuzzy metric results Hausdorff topology. The technique for developing contraction mapping in fuzzy metric spaces is a relevant theoretical progress at the present. In 1988, Grabiec [3] introduced the Banach and Edelstein principles to fuzzy metric spaces for the first time. The idea of fuzzy contractive mappings was pioneered by Gregori and Sapena [6]. By modifying and adjusting the contraction assumptions, several authors have recently aimed to apply the Banach contraction concept more generally (see [8, 11, 13, 15, 16, 20, 21, 23–28]).

A generalization of the Banach contraction theorem in generalized metric spaces was offered by Jleli and Samet [14] and initiated the idea of θ-contractions. Influenced by these concepts, Cho [13] created a novel contraction known as L-contraction and demonstrated certain fixed point theorems in generalized metric space for these kind of contraction. Following a similar general approaches, multiple research studies introduced numerous contraction forms emerging with an auxiliary function fulfilling the appropriate requirements and yielded interesting fuzzy fixed point results. Mihet [10] proposed the type of fuzzy ψ-contractive mappings. Fuzzy $\mathcal{H}$-contractive mappings are a concept that Wardowski [12] developed and studied. Abdelhamid Moussaoui *et al*. [18, 19] created the concept of $\mathcal{FZ}$-contraction and offered the

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simulation function method to fuzzy metric context. This idea was further enhanced in [23] by establishing the family of extended $\mathcal{FZ}$-simulation functions.

Utilizing the ideas of fuzzy Θ_f-contractive mappings and fuzzy $\mathcal{L}_\lambda$-simulation function, we construct a new type of fuzzy contraction principle. We then employ the specially developed contraction to demonstrate the existence and uniqueness of a fixed point for a self-mapping in complete fuzzy metric spaces. An example and some illustrative corollaries and remarks serve to further highlight the validity of our results. The provided findings integrate, extend, and improve a number of previously released research results.

2. Preliminaries

In this section, we discuss some essential concepts in order to render our study self-contained.

Definition 2.1. *[9] An operation $\curlyvee : [0,1]\times[0,1] \longrightarrow [0,1]$ is called a continuous t-norm if $([0,1],\curlyvee)$ is an Abelien topological monoid such that $c \curlyvee 1 = c$ for all $c \in [0,1]$ and $c \curlyvee v \leq w \curlyvee p$ whenever $c \leq w$ and $v \leq p$, for all $c, v, w, p \in [0,1]$.*

Example 2.2. *The following are some typical continuous t-norm instances:*

i) $c \curlyvee_m a = \min\{c, a\}$.

ii) $c \curlyvee_P a = c.a$,

iii) $c \curlyvee_L a = \max\{0, c + a - 1\}$,

Definition 2.3. *[4] The triple $(\mathcal{V}, \mathcal{R}, \curlyvee)$ is called a fuzzy metric space if $\mathcal{V}$ is a nonempty set, $\curlyvee$ is a continuous t-norm and $\mathcal{R}$ is a fuzzy set on $\mathcal{V}^2 \times (0,+\infty)$ satisfying:*

($\mathcal{M}$1) $\mathcal{R}(c, a, \Im) > 0$,

($\mathcal{M}$2) $\mathcal{R}(c, a, \Im) = 1$ *if and only if* $c = a$,

($\mathcal{M}$3) $\mathcal{R}(c, a, \Im) = \mathcal{R}(a, c, \Im)$,

($\mathcal{M}$4) $\mathcal{R}(c, v, \Im) \curlyvee \mathcal{R}(v, a, \gamma) \leq \mathcal{R}(c, a, \Im + \gamma)$,

($\mathcal{M}$5) $\mathcal{R}(c, a, .) : (0,+\infty) \to [0,1]$ *is continuous.*

for all $c, a, v \in \mathcal{V}$ and $\Im, \gamma > 0$.

$\mathcal{R}(c, a, \Im)$ can be regarded as the degree of nearness of c and a with respect to the parameter $\Im$.

Example 2.4. *([5]) Let $\mathfrak{R} : \Lambda \to \mathbb{R}^+$ be a one-to-one mapping, $\kappa : \mathbb{R}^+ \to [0,\infty)$ be a continuous increasing function and $\tau, \sigma > 0$. Let*

$$\zeta(c, a, \Im) = \left(\frac{(\min\{\mathfrak{R}(c), \mathfrak{R}(a)\})^\tau + \kappa(\Im)}{(\max\{\mathfrak{R}(c), \mathfrak{R}(a)\})^\tau + \kappa(\Im)}\right)^\sigma$$

Then $(\mathcal{V}, \curlyvee_P)$ is a fuzzy metric.

Example 2.5. *[4] Let $(\mathcal{V}, \Xi)$ be a metric space, $c \curlyvee a = c \curlyvee_m a$ and*

$$\mathcal{R}(c, a, \Im) = \frac{\theta \Im^\rho}{\theta \Im^\rho + q\Xi(c, a)}\ , \theta, q, \rho \in \mathbb{R}^+$$

Then $(\mathcal{V}, \mathcal{R}, \curlyvee)$ is a fuzzy metric space.
Taking $\theta = q = \rho = 1$, we get

$$\mathcal{R}(c, a, \Im) = \frac{\Im}{\Im + \Xi(c, a)}.$$

Lemma 2.6. *[3] $\mathcal{R}(c,a,.)$ is nondecreasing function for all c,a in $\mathcal{V}$.*

Definition 2.7. *[4] Let $(\mathcal{V},\mathcal{R},\vee)$ be a fuzzy metric space, let $\{c_s\} \subseteq \mathcal{V}$ be a sequence in $\mathcal{V}$ and $c \in \mathcal{V}$. Then we say that*

(1) *Convergence: $\{c_s\}$ is convergent or converges to $c \in \mathcal{V}$ if $\lim_{s\to+\infty} \mathcal{R}(c_s,c,\mathfrak{I}) = 1$ for all $\mathfrak{I} > 0$.*

(2) *Cauchy sequence: $\{c_s\}$ is a Cauchy if for all $\hbar \in (0,1)$ and $\mathfrak{I} > 0$, there exists $s_0 \in \mathbb{N}$ such that $\mathcal{R}(c_r,c_t,\mathfrak{I}) > 1-\hbar$ for all $r,t \geq s_0$.*

(3) *Completeness: $(\mathcal{V},\mathcal{R},\vee)$ is complete if each Cauchy sequence is convergent in $\mathcal{V}$.*

Gregori and Sapena [6] coined the concept of fuzzy contractive mappings, which is as follows: Let $(\mathcal{V},\mathcal{R},\vee)$ be a fuzzy metric space. A mapping $Q : \mathcal{V} \to \mathcal{V}$ is said to be a fuzzy contractive mapping, if there exists $k \in (0,1)$ such that

$$\frac{1}{\mathcal{R}(Qc,Qa,\mathfrak{I})} - 1 \leq k\left(\frac{1}{\mathcal{R}(c,a,\mathfrak{I})} - 1\right), \tag{1}$$

for all $c,a \in \mathcal{V}$ and $\mathfrak{I} > 0$. The authors proved various significant fixed point results for such class of contractions. The study of Tirado [7] led to the establishment of the following theorem.

Theorem 2.8. *[7] Let $(\mathcal{V},\mathcal{R},\vee_L)$ be a complete fuzzy metric space and $Q : \mathcal{V} \to \mathcal{V}$ be a mapping such that*

$$1 - \mathcal{R}(Qc,Qa,\mathfrak{I}) \leq k\,(1 - \mathcal{R}(c,a,\mathfrak{I}))\,.$$

for all $c,a \in \mathcal{V}$, $\mathfrak{I} > 0$ and for some $k \in (0,1)$. Then Q has a unique fixed point.

In order to develop a new type of fuzzy contractions known as $\mathcal{FZ}$-contractions, the following class of fuzzy simulation functions was suggested in [18], and further extended by Moussaoui *et al.* [23].

Definition 2.9. *([18])*
The function $\gamma : (0,1] \times (0,1] \longrightarrow \mathbb{R}$ is called an $\mathcal{FZ}$-simulation function, if the following conditions hold:

(γ1) $\gamma(1,1) = 1$,

(γ2) $\gamma(\tau,\sigma) < \frac{1}{\sigma} - \frac{1}{\tau}$ *for all $\tau,\sigma \in (0,1)$,*

(γ3) *if $\{\tau_n\},\{\sigma_n\}$ are sequences in $(0,1]$ such that $\lim_{n\to+\infty} \tau_n = \lim_{n\to+\infty} \sigma_n < 1$ then $\lim_{n\to+\infty} \sup \gamma(\tau_n,\sigma_n) < 0$.*

The class of all $\mathcal{FZ}$-simulation functions is denoted by $\mathcal{FZ}$.

Definition 2.10. *([18],[26]) Let $(\mathcal{V},\mathcal{R},\vee)$ be a fuzzy metric space, $Q : \mathcal{V} \longrightarrow \mathcal{V}$ a mapping and $\gamma \in \mathcal{FZ}$. Then Q is called a $\mathcal{FZ}$-contraction w.r.t $\zeta \in \mathcal{FZ}$ if:*

$$\gamma(\mathcal{R}(Qc,Qa,\mathfrak{I}),\mathcal{R}(c,a,\mathfrak{I})) \geq 0 \text{ for all } c,a \in \Lambda, \mathfrak{I} > 0.$$

In 2020, motivated by the investigation of Jleli *et al.* [14], Saleh *et al.* [29] proposed the notion of fuzzy θ_f-contractive mappings with the help of the class Ω of the functions $\theta_f : (0,1) \to (0,1)$ fulfilling the following conditions:

(Θ_1) θ_f is non-decreasing and continuous,

(Θ_2) $\lim_{s\to+\infty} \theta_f(c_s) = 1$ if and only if $\lim_{s\to+\infty} c_s = 1$, where $\{c_s\}$ is a sequence in $(0,1)$.

Definition 2.11. *[29] Let $(\mathcal{V},\mathcal{R},\vee)$ be a fuzzy metric space. A mapping $Q : \mathcal{V} \to \mathcal{V}$ is called a fuzzy Θ_f-contractive mapping w.r.t $\theta_f \in \Omega$ if there exists $k \in (0,1)$ such that*

$$\mathcal{R}(Qc,Qa,\mathfrak{I}) < 1 \Rightarrow \theta_f(\mathcal{R}(Qc,Qa,\mathfrak{I})) \geq \left[\theta_f(\mathcal{R}(c,a,\mathfrak{I}))\right]^k,$$

for all $c,a \in \mathcal{V}$ and $\mathfrak{I} > 0$.

This class has further been weakened by the following contractive condition:

Definition 2.12. *[29] Let $(\mathcal{V},\mathcal{R},\curlyvee)$ be a fuzzy metric space. A mapping $Q:\mathcal{V}\to\mathcal{V}$ is called a fuzzy Θ_f-weak contractive mapping w.r.t $\theta_f\in\Omega$ if there exists $k\in(0,1)$ such that*

$$\begin{aligned}&\mathcal{R}(Qc,Qa,\mathfrak{I})<1\Rightarrow\\&\qquad\Big[\theta_f(\min\{\mathcal{R}(c,a,\mathfrak{I}),\mathcal{R}(c,Qc,\mathfrak{I}),\mathcal{R}(a,Qa,\mathfrak{I})\})\Big]^k\le\theta_f(\mathcal{R}(Qc,Qa,\mathfrak{I})),\end{aligned}$$

for all $c,a\in\mathcal{V}$ and $\mathfrak{I}>0$.

The authors then proved the following theorem involving fuzzy Θ_f-contractive mapping.

Theorem 2.13. *[29] Let $(\mathcal{V},\mathcal{R},\curlyvee)$ be a complete fuzzy metric space and $Q:\mathcal{V}\longrightarrow\mathcal{V}$ be a fuzzy Θ_f- weak contractive mapping, then Q has a unique fixed point.*

3. Main results

In the setting of complete fuzzy metric spaces, we initiate a novel kind of fuzzy contraction and we prove some fixed point results for this class of operators.

Definition 3.1. *We say that the function $\zeta:(0,1]\times(0,1]\longrightarrow\mathbb{R}$ is a fuzzy $\mathcal{L}_\lambda$-simulation function, if:*

$(\mathcal{FL}1):$ $\zeta(1,1)=1$,

$(\mathcal{FL}2):$ $\zeta(\nu,\mu)<\frac{\lambda(\nu)}{\lambda(\mu)}$ *for all $\nu,\mu\in(0,1)$, where λ is a non decreasing function on $(0,1]$ such that $\lambda^{-1}(\{1\})=1$.*

$(\mathcal{FL}3):$ *if $\{\nu_s\},\{\nu_s\}$ are sequences in $(0,1]$ such that $\lim_{\to+\infty}t_n=\lim_{s\to+\infty}\mu_n<1$ then $\lim_{s\to+\infty}\sup\zeta(\nu_s,\mu_s)<1$.*

We denote by $\mathcal{FL}_\lambda$ the class of all fuzzy $\mathcal{L}_\lambda$-simulation functions.

Example 3.2. *Let $\zeta:(0,1]\times(0,1]\longrightarrow\mathbb{R}$ be the function given by*

$$\zeta(\nu,\mu)=\frac{\lambda(\nu)}{[\lambda(\mu)]^k}\text{ for all } s,t\in(0,1],$$

where $k\in(0,1)$. Then $\zeta\in\mathcal{FL}_\lambda$.

Example 3.3. *Let $\zeta:(0,1]\times(0,1]\longrightarrow\mathbb{R}$ be the function given by*

$$\zeta(\nu,\mu)=\frac{\lambda(\nu)}{\lambda(\beta(\mu))}\text{ for all }\nu,\mu\in(0,1],$$

where $\beta:(0,1]\to(0,1]$ such that β is continuous, increasing and $\beta(c)>c$, for all $c\in(0,1)$. Then $\zeta\in\mathcal{FL}_\lambda$.

Now, we define the concept of fuzzy $\mathcal{L}_\lambda$-weak contraction mapping as follows:

Definition 3.4. *Let $(\mathcal{V},\mathcal{R},\curlyvee)$ be a fuzzy metric space and let $Q:\Lambda\longrightarrow\Lambda$ be a mapping. Then Q is said to be a fuzzy $\mathcal{L}_\lambda$-weak contraction with respect to $\zeta\in\mathcal{FL}_\lambda$ and $\theta_f\in\Omega$, if for all $c,a\in\mathcal{V}$, $\mathfrak{I}>0$ with $\mathcal{R}(Qc,Qa,\mathfrak{I})<1$, we have*

$$\zeta(\theta_f(\mathcal{R}(Qc,Qa,\mathfrak{I})),\theta_f(\min\{\mathcal{R}(c,a,\mathfrak{I}),\mathcal{R}(c,Qc,\mathfrak{I}),\mathcal{R}(a,Qa,\mathfrak{I})\}))\ge1\quad,\tag{2}$$

for all $c,a\in\mathcal{V}$, $\mathfrak{I}>0$.

Remark 3.5. *If $\zeta(\nu,\mu)=\frac{\lambda(\nu)}{[\lambda(\mu)]^k}$ for all $\nu,\mu\in(0,1]$, where $k\in(0,1)$ and λ as the identity function, then Definition 3.4 yields to the notion of fuzzy θ_f-weak contractive mappings. Indeed, for all $c,a\in\mathcal{V}$, $\mathfrak{I}>0$ such that $\mathcal{R}(Qc,Qa,\mathfrak{I})<1$, we have*

$$\begin{aligned}1&\le\zeta(\theta_f(\mathcal{R}(Qc,Qa,\mathfrak{I})),\theta_f(\min\{\mathcal{R}(c,a,\mathfrak{I}),\mathcal{R}(c,Qc,\mathfrak{I}),\mathcal{R}(a,Qa,\mathfrak{I})\}))\\&=\frac{\lambda(\theta_f(\mathcal{R}(Qc,Qa,\mathfrak{I})))}{[\lambda(\theta_f(\min\{\mathcal{R}(c,a,\mathfrak{I}),\mathcal{R}(c,Qc,\mathfrak{I}),\mathcal{R}(a,Qa,\mathfrak{I})\})))]^k}\\&=\frac{\theta_f(\mathcal{R}(Qc,Qa,\mathfrak{I}))}{[\theta_f(\min\{\mathcal{R}(c,a,\mathfrak{I}),\mathcal{R}(c,Qc,\mathfrak{I}),\mathcal{R}(a,Qa,\mathfrak{I})\}))]^k}\end{aligned}$$

Hence,

$$\Big[\theta_f(\min\{\mathcal{R}(c,a,\mathfrak{I}),\mathcal{R}(c,Qc,\mathfrak{I}),\mathcal{R}(a,Qa,\mathfrak{I})\})\Big]^k\le\theta_f(\mathcal{R}(Qc,Qa,\mathfrak{I})).$$

The following theorem is our first main finding.

Theorem 3.6. *Let $(\mathcal{V},\mathcal{R},\vee)$ be a complete fuzzy metric space and $Q:\Lambda\longrightarrow\Lambda$ be a fuzzy $\mathcal{L}$-weak contraction with respect to $\zeta\in\mathcal{FL}$. Then Q has a unique fixed point.*

Proof. In the beginning, we prove that fixed point is unique if it exists. By contradiction, suppose that $c,a\in\mathcal{V}$ are two distinct fixed points. Therefore, $\mathcal{R}(c,a,\mathfrak{I})<1$ for all $\mathfrak{I}>0$. Applying (2), we obtain

$$\begin{aligned}1&\le\zeta\Big(\theta_f(\mathcal{R}(Qc,Qa,\mathfrak{I})),\theta_f(\min\{\mathcal{R}(c,a,\mathfrak{I}),\mathcal{R}(c,Qc,\mathfrak{I}),\mathcal{R}(a,Qa,\mathfrak{I})\})\Big)\\&=\zeta\Big(\theta_f(\mathcal{R}(c,a,\mathfrak{I})),\theta_f(\min\{\mathcal{R}(c,a,\mathfrak{I}),\mathcal{R}(c,c,\mathfrak{I}),\mathcal{R}(a,a,\mathfrak{I})\})\Big)\\&=\zeta\Big(\theta_f(\mathcal{R}(c,a,\mathfrak{I})),\theta_f(\min\{\mathcal{R}(c,a,\mathfrak{I}),1,1\})\Big)\\&=\zeta\Big(\theta_f(\mathcal{R}(c,a,\mathfrak{I})),\theta_f(\mathcal{R}(c,a,\mathfrak{I}))\Big)\\&<\frac{\lambda(\theta_f(\mathcal{R}(c,a,\mathfrak{I})))}{\lambda(\theta_f(\mathcal{R}(c,a,\mathfrak{I})))}.\end{aligned}\tag{3}$$

Hence,

$$\lambda(\theta_f(\mathcal{R}(c,a,\mathfrak{I})))<\lambda(\theta_f(\mathcal{R}(c,a,\mathfrak{I}))).$$

A contradiction. Then $c=a$, which means that the fixed point of Q is unique. Next, we demonstrate the existence of the fixed point. Define the sequence $\{c_s\}$ in $\mathcal{V}$ by $Qc_s=c_{s+1}$ for all $s\ge0$. If there exists $l_0\in\mathbb{N}$ such that $c_{l_0}=c_{l_0+1}$, it follows that c_{l_0} is a fixed point of Q. Assume that $c_s\ne c_{s+1}$ for all $s\in\mathbb{N}$, then $\mathcal{R}(c_s,c_{s+1},\mathfrak{I})<1$ for all $s\in\mathbb{N}$ and $\mathfrak{I}>0$. From (2), we get

$$\begin{aligned}1&\le\zeta\Big(\theta_f(\mathcal{R}(Qc_{s-1},Qc_s,\mathfrak{I})),\theta_f(\min\{\mathcal{R}(c_{s-1},c_s,\mathfrak{I}),\\&\qquad\mathcal{R}(c_{s-1},Qc_{s-1},\mathfrak{I}),\mathcal{R}(c_s,Qc_s,\mathfrak{I})\})\Big)\\&=\zeta\Big(\theta_f(\mathcal{R}(c_s,c_{s+1},\mathfrak{I})),\theta_f(\min\{\mathcal{R}(c_{s-1},c_s,\mathfrak{I}),\\&\qquad\mathcal{R}(c_{s-1},c_s,\mathfrak{I}),\mathcal{R}(c_s,c_{s+1},\mathfrak{I})\})\Big)\\&<\frac{\lambda(\theta_f(\mathcal{R}(c_s,c_{s+1},\mathfrak{I})))}{\lambda(\theta_f(\min\{\mathcal{R}(c_{s-1},c_s,\mathfrak{I}),\mathcal{R}(c_{s-1},c_s,\mathfrak{I}),\mathcal{R}(c_s,c_{s+1},\mathfrak{I})\}))}.\end{aligned}\tag{4}$$

Thus,

$$\lambda(\theta_f(\min\{\mathcal{R}(c_{s-1}, c_s, \mathfrak{I}), \mathcal{R}(c_{s-1}, c_s, \mathfrak{I}), \mathcal{R}(c_s, c_{s+1}, \mathfrak{I})\})) \\ < \lambda(\theta_f(\mathcal{R}(c_s, c_{s+1}, \mathfrak{I}))).$$

Which yields

$$\theta_f(\min\{\mathcal{R}(c_{s-1}, c_s, \mathfrak{I}), \mathcal{R}(c_{s-1}, c_s, \mathfrak{I}), \mathcal{R}(c_s, c_{s+1}, \mathfrak{I})\}) \\ < \theta_f(\mathcal{R}(c_s, c_{s+1}, \mathfrak{I})). \tag{5}$$

If $\min\{\mathcal{R}(c_{s-1}, c_s, \mathfrak{I}), \mathcal{R}(c_{s-1}, c_s, \mathfrak{I}), \mathcal{R}(c_s, c_{s+1}, \mathfrak{I})\} = \mathcal{R}(c_s, c_{s+1}, \mathfrak{I})$ for some $s \in \mathbb{N}$, then from (5), we get

$$\theta_f(\mathcal{R}(c_s, c_{s+1}, \mathfrak{I})) < \theta_f(\mathcal{R}(c_s, c_{s+1}, \mathfrak{I})).$$

As θ_f is non-decreasing, we obtain

$$\mathcal{R}(c_s, c_{s+1}, \mathfrak{I}) < \mathcal{R}(c_s, c_{s+1}, \mathfrak{I}).$$

Which is a contradiction. Therefore,

$$\min\{\mathcal{R}(c_{s-1}, c_s, \mathfrak{I}), \mathcal{R}(c_{s-1}, c_s, \mathfrak{I}), \mathcal{R}(c_s, c_{s+1}, \mathfrak{I})\} = \mathcal{R}(c_{s-1}, c_s, \mathfrak{I})$$

for all $s \in \mathbb{N}$. Then, using (5), we derive

$$\mathcal{R}(c_{s-1}, c_s, \mathfrak{I}) < \mathcal{R}(c_s, c_{s+1}, \mathfrak{I}).$$

Thus $\{\mathcal{R}(c_s, c_{s+1}, \mathfrak{I})\}$ is a nondecreasing sequence of positive real numbers in $[0, 1]$. Thus, there exists $x(\mathfrak{I}) \leq 1$ such that $\lim_{s\to+\infty} \mathcal{R}(c_s, c_{s-1}, \mathfrak{I}) = x(\mathfrak{I}) \geq 1$ for all $\mathfrak{I} > 0$. We prove that

$$\lim_{s\to+\infty} \mathcal{R}(c_s, c_{s-1}, \mathfrak{I}) = 1.$$

On contrary assume that $x(\mathfrak{I}_0) < 1$ for some $\mathfrak{I}_0 > 0$. Now, if we consider the sequences $\{\alpha_s = \mathcal{R}(c_s, c_{s+1}, \mathfrak{I}_0)\}$ and $\{\beta_s = \mathcal{R}(c_{s-1}, c_s, \mathfrak{I}_0)\}$ and taking into account (($\mathcal{FL}$3), we derive

$$1 \leq \lim_{s\to+\infty} \sup \zeta(\alpha_s, \beta_s) < 1.$$

A contradiction, hence

$$\lim_{s\to+\infty} \mathcal{R}(c_s, c_{s-1}, \mathfrak{I}) = 1 \tag{6}$$

Next, we show that the sequence $\{c_s\}$ is Cauchy. Reasoning by contradiction, suppose that $\{c_s\}$ is not a Cauchy sequence. Then, there exists $\hbar \in (0, 1)$, $\mathfrak{I}_0 > 0$ and two subsequences $\{c_{s_k}\}$ and $\{c_{r_k}\}$ of $\{c_s\}$ with $r_k > s_k \geq k$ for all $k \in \mathbb{N}$ such that

$$\mathcal{R}(c_{r_k}, c_{s_k}, \mathfrak{I}_0) \leq 1 - \hbar. \tag{7}$$

Taking in account Lemma 2.6, we have

$$\mathcal{R}(c_{r_k}, c_{s_k}, \frac{\mathfrak{I}_0}{2}) \leq 1 - \hbar. \tag{8}$$

By choosing s_k as the smallest index satisfying (8), we have

$$\mathcal{R}(c_{r_k-1}, c_{s_k}, \frac{\mathfrak{I}_0}{2}) > 1 - \hbar. \tag{9}$$

Applying (2) for c_{r_k-1} and c_{s_k-1}, we obtain

$$
\begin{aligned}
1 &\leq \zeta\Big(\theta_f(\mathcal{R}(Qc_{r_k-1}, Qc_{s_k-1}, \mathfrak{I}_0)), \theta_f(\mathcal{R}(c_{r_k-1}, c_{s_k-1}, \mathfrak{I}_0),\\
&\qquad \mathcal{R}(c_{r_k-1}, Qc_{r_k-1}, \mathfrak{I}_0), \mathcal{R}(c_{s_k-1}, Qc_{s_k-1}, \mathfrak{I}_0))\Big)\\
&= \zeta\Big(\theta_f(\mathcal{R}(c_{r_k}, c_{s_k}, \mathfrak{I}_0)), \theta_f(\mathcal{R}(c_{r_k-1}, c_{s_k-1}, \mathfrak{I}_0),\\
&\qquad \mathcal{R}(c_{r_k-1}, c_{r_k}, \mathfrak{I}_0), \mathcal{R}(c_{s_k-1}, c_{s_k}, \mathfrak{I}_0))\Big)\\
&< \frac{\lambda(\theta_f(\mathcal{R}(c_{r_k}, c_{s_k}, \mathfrak{I}_0)))}{\lambda(\theta_f(\mathcal{R}(c_{r_k-1}, c_{s_k-1}, \mathfrak{I}_0), \mathcal{R}(c_{r_k-1}, c_{r_k}, \mathfrak{I}_0), \mathcal{R}(c_{s_k-1}, c_{s_k}, \mathfrak{I}_0)))}. \qquad (10)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\lambda(\theta_f(\min\{\mathcal{R}(c_{r_k-1}, c_{s_k-1}, \mathfrak{I}_0), \mathcal{R}(c_{r_k-1}, c_{r_k}, \mathfrak{I}_0), \mathcal{R}(c_{s_k-1}, c_{s_k}, \mathfrak{I}_0)\}))\\
&\qquad < \lambda(\theta_f(\mathcal{R}(c_{r_k}, c_{s_k}, \mathfrak{I}_0)))
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\theta_f(\min\{\mathcal{R}(c_{r_k-1}, c_{s_k-1}, \mathfrak{I}_0), \mathcal{R}(c_{r_k-1}, c_{r_k}, \mathfrak{I}_0), \mathcal{R}(c_{s_k-1}, c_{s_k}, \mathfrak{I}_0)\}) \qquad (11)\\
&\qquad < \theta_f(\mathcal{R}(c_{r_k}, c_{s_k}, \mathfrak{I}_0))
\end{aligned}
$$

As θ_f is nondecreasing, we derive

$$
\begin{aligned}
&\min\{\mathcal{R}(c_{r_k-1}, c_{s_k-1}, \mathfrak{I}_0), \mathcal{R}(c_{r_k-1}, c_{r_k}, \mathfrak{I}_0), \mathcal{R}(c_{s_k-1}, c_{s_k}, \mathfrak{I}_0)\} \qquad (12)\\
&\qquad < \mathcal{R}(c_{r_k}, c_{s_k}, \mathfrak{I}_0).
\end{aligned}
$$

Denote,

$$
\begin{aligned}
\hat{\phi}_k(c_{r_k-1}, c_{r_k-1}, \mathfrak{I}_0) = \min\{&\mathcal{R}(c_{r_k-1}, c_{s_k-1}, \mathfrak{I}_0), \mathcal{R}(c_{r_k-1}, c_{r_k}, \mathfrak{I}_0),\\
&\mathcal{R}(c_{s_k-1}, c_{s_k}, \mathfrak{I}_0)\}.
\end{aligned}
$$

On the other hand, by (6), we have

$$
\begin{aligned}
\lim_{k\to+\infty} \hat{\phi}_k(c_{r_k-1}, c_{s_k-1}, \mathfrak{I}_0) &= \lim_{k\to+\infty} \min\{\mathcal{R}(c_{r_k-1}, c_{s_k-1}, \mathfrak{I}_0),\\
&\qquad \mathcal{R}(c_{r_k-1}, c_{r_k}, \mathfrak{I}_0), \mathcal{R}(c_{s_k-1}, c_{s_k}, \mathfrak{I}_0)\}\\
&= \min\{\lim_{k\to+\infty} \mathcal{R}(c_{r_k-1}, c_{s_k-1}, \mathfrak{I}_0), 1, 1\}\\
&= \lim_{k\to+\infty} \mathcal{R}(c_{r_k-1}, c_{s_k-1}, \mathfrak{I}_0).
\end{aligned}
$$

On account of (7), (9) and the triangular inequality, we obtain

$$
\begin{aligned}
1 - \hbar &\geq \mathcal{R}(c_{r_k}, c_{s_k}, \mathfrak{I}_0)\\
&> \mathcal{R}(c_{r_k-1}, c_{s_k-1}, \mathfrak{I}_0)\\
&\geq \mathcal{R}(c_{r_k-1}, c_{s_k}, \frac{\mathfrak{I}_0}{2}) \vee \mathcal{R}(c_{s_k}, c_{s_k-1}, \frac{\mathfrak{I}_0}{2})\\
&> (1 - \hbar) \vee \mathcal{R}(c_{s_k-1}, c_{s_k}, \frac{\mathfrak{I}_0}{2})
\end{aligned}
$$

Taking limit as $k \to +\infty$ in both sides of the above inequality and using (6), we derive that

$$
\lim_{k\to+\infty} \mathcal{R}(c_{r_k}, c_{s_k}, \mathfrak{I}_0) = \lim_{k\to+\infty} \mathcal{R}(c_{r_k-1}, c_{s_k-1}, \mathfrak{I}_0) = 1 - \hbar \qquad (13)
$$

Now, we consider the sequences $\hat{\beta}_k = \theta_f(\hat{\phi}_k(c_{s_k-1}, c_{r_k-1}, \mathfrak{I}_0))$ and $\hat{\alpha}_k = \theta_f(\mathcal{R}(c_{r_k}, c_{s_k}, \mathfrak{I}_0))$, then $\lim_{k\to+\infty} \hat{\beta}_k = \lim_{k\to+\infty} \hat{\alpha}_k = \theta_f(1-\hbar) < 1$. Applying (($\mathcal{FL}3$), we get

$$1 \leq \lim_{k\to+\infty} \sup \zeta(\hat{\alpha}_k, \hat{\beta}_k) < 1$$

which is a contradiction. Hence, $\{c_s\}$ is a Cauchy sequence. Since $(\mathcal{V}, \mathcal{R}, \curlyvee)$ a complete fuzzy metric space, there exists $c \in \mathcal{V}$ such that $c_s \to c$. Hence

$$\lim_{s\to+\infty} \mathcal{R}(c_s, c, \mathfrak{I}) = 1, \tag{14}$$

Applying 3.4, we get

$$1 \leq \zeta\Big(\theta_f(\mathcal{R}(c_{s+1}, Qc, \mathfrak{I})), \theta_f(\min\{\mathcal{R}(c_s, c, \mathfrak{I}), \mathcal{R}(c_s, c_{s+1}, \mathfrak{I}), \mathcal{R}(c, Qc, \mathfrak{I})\})\Big)$$

Where,

$$\begin{aligned}
&\lim_{s\to+\infty} \min\{\mathcal{R}(c_s, c, \mathfrak{I}), \mathcal{R}(c_s, c_{s+1}, \mathfrak{I}), \mathcal{R}(c, Qc, \mathfrak{I})\} \\
&\qquad = \min\{1, 1, \mathcal{R}(c, Qc, \mathfrak{I})\} \\
&\qquad = \mathcal{R}(c, Qc, \mathfrak{I})
\end{aligned}$$

If $Qc \neq c$, that is, $\mathcal{R}(c, Qc, \mathfrak{I}) < 1$. Taking the sequences $\hat{\nu}_s = \mathcal{R}(c_{s+1}, Qc, \mathfrak{I})$ and $\hat{\mu}_s = \min\{\mathcal{R}(c_s, c, \mathfrak{I}), \mathcal{R}(c_s, c_{s+1}, \mathfrak{I}), \mathcal{R}(c, Qc, \mathfrak{I})\}$, we have $\lim_{s\to+\infty} \hat{\nu}_s = \lim_{s\to+\infty} \hat{\mu}_s$ and by ($\mathcal{FL}3$), we get

$$1 \leq \lim_{s\to+\infty} \sup \zeta(\hat{\nu}_s, \hat{\mu}_s) < 1$$

Which is a contradiction. Therefore, $\mathcal{R}(c, Qc, \mathfrak{I}) = 1$, That is $Qc = c$. □ □

Example 3.7. *Let $\mathcal{V} = [0,1]$ be equipped with the fuzzy metric $\mathcal{R}$ given by*

$$\mathcal{R}(Qc, Qa, \mathfrak{I})) = \frac{\mathfrak{I}}{\mathfrak{I} + \Xi(c, a)}$$

for all $c, a \in \mathcal{V}$, $\mathfrak{I} > 0$, where Ξ is the usual metric. Then, $(\mathcal{V}, \mathcal{R}, \curlyvee_p)$ is a fuzzy metric space. Consider the mapping $Q : \mathcal{V} \to \mathcal{V}$ defined by

$$Qc = \begin{cases} \frac{1}{4} & \text{if } c = 1, \\ \\ \frac{1}{2} & \text{if } 0 \leq c < 1. \end{cases}$$

Notice that, for $c \in [0,1)$ and $a = 1$ and $\mathfrak{I} > 0$, we get

$$\begin{aligned}
1 - \frac{1}{\mathcal{R}(Qc, Qa, \mathfrak{I})} &= 1 - \frac{\mathfrak{I} + d(Qc, Qa)}{\mathfrak{I}} \\
&= 1 - \frac{\mathfrak{I} + \frac{1}{4}}{\mathfrak{I}} = -\frac{1}{4\mathfrak{I}}.
\end{aligned}$$

We consider two cases:

. *If $c \in [0, \frac{1}{4})$ and $a = 1$, then*

$$\begin{aligned}
&1 - \frac{1}{\min\{\mathcal{R}(c, 1, \mathfrak{I}), \mathcal{R}(c, Qc, \mathfrak{I}), \mathcal{R}(1, Q1, \mathfrak{I})\}} \\
&= 1 - \frac{1}{\mathcal{R}(c, 1, \mathfrak{I})} \\
&= 1 - \frac{\mathfrak{I} + \mid c - 1 \mid}{\mathfrak{I}} \\
&= -\frac{\mid c - 1 \mid}{\mathfrak{I}}.
\end{aligned}$$

. *If $c \in [\frac{1}{4}, 1)$ and $a = 1$, then*

$$\begin{aligned}
&1 - \frac{1}{\min\{\mathcal{R}(c, 1, \mathfrak{I}), \mathcal{R}(c, Qc, \mathfrak{I}), \mathcal{R}(1, Q1, \mathfrak{I})\}} \\
&= 1 - \frac{1}{\mathcal{R}(1, Q1, \mathfrak{I})} \\
&= 1 - \frac{\mathfrak{I} + \frac{3}{4}}{\mathfrak{I}} \\
&= -\frac{3}{4\mathfrak{I}}.
\end{aligned}$$

Now, define $\zeta : (0, 1] \times (0, 1] \longrightarrow \mathbb{R}$ by

$$\zeta(\nu, \mu) = \frac{\lambda(\nu)}{[\lambda(\mu)]^k}$$

for all $\nu, \mu \in (0, 1]$, where $k = \frac{1}{3}$, λ is the identity function and the mapping $\theta_f \in \Omega$ given by $\theta_f(\omega) = e^{1-\frac{1}{\omega}}$ for all $\omega \in (0, 1)$. we get in both cases that

$$\begin{aligned}
\zeta\Big(\theta_f(\mathcal{R}(Qc, Qa, \mathfrak{I})), (\theta_f(Q(c, a, \mathfrak{I})))\Big) &= \frac{\theta_f(\mathcal{R}(Qc, Qa, \mathfrak{I}))}{\Big(\theta_f(\mathcal{R}(c, a, \mathfrak{I}))\Big)^k} \\
&= \frac{e^{1-\frac{1}{\mathcal{R}(Qc,Qa,\mathfrak{I})}}}{\Big(e^{1-\frac{1}{\mathcal{R}(c,a,\mathfrak{I})}}\Big)^k} \\
&\geq 1.
\end{aligned}$$

Therefore, Q is a fuzzy $\mathcal{L}_\lambda$-weak contraction w.r.t $\zeta \in \mathcal{FL}_\lambda$. Thus, by Theorem 3.6, Q has a unique fixed point, namely $c = \frac{1}{2}$.

Corollary 3.8. *Let $(\mathcal{V}, \mathcal{R}, \vee)$ be a complete fuzzy metric space, $\zeta \in \mathcal{FL}$ and $Q : \mathcal{V} \to \mathcal{V}$ be a self mapping with $\mathcal{R}(Qc, Qa, \mathfrak{I}) < 1$, such that*

$$\zeta(\mathcal{R}(Qc, Qa, \mathfrak{I})), \min\{\mathcal{R}(c, a, \mathfrak{I}), \mathcal{R}(c, Qc, \mathfrak{I}), \mathcal{R}(a, Qa, \mathfrak{I})\}) \geq 1,$$

for all $c, a \in \mathcal{V}$, $\mathfrak{I} > 0$. Then Q has a unique fixed point.

Proof. The result can be achieved from Theorem 3.6 by setting $\theta_f(\eta) = \eta$ for all $\eta \in (0, 1)$. □ □

Corollary 3.9. *[29] Let $(\mathcal{V}, \mathcal{R}, \vee)$ be a complete fuzzy metric space and $Q : \mathcal{V} \to \mathcal{V}$ be a self mapping such that for all $c, a \in \mathcal{V}$ with $\mathcal{R}(Qc, Qa, \mathfrak{I}) < 1$ we have*

$$\theta_f(\mathcal{R}(Qc, Qa, \mathfrak{I})) \geq \Big[\theta_f(\min\{\mathcal{R}(c, a, \mathfrak{I}), \mathcal{R}(c, Qc, \mathfrak{I}), \mathcal{R}(a, Qa, \mathfrak{I})\})\Big]^k.$$

Then Q has a unique fixed point.

Proof. The result can be drawn from Theorem 3.6 by taking $\zeta(\nu, \mu) = \frac{\lambda(\nu)}{[\lambda(\mu)]^k}$ for all $\nu, \mu \in (0, 1]$, λ as the identity function. □ □

Corollary 3.10. *[29] Let $(\mathcal{V}, \mathcal{R}, \vee)$ be a complete fuzzy metric space and $Q : \mathcal{V} \to \mathcal{V}$ be a self mapping such that $\mathcal{R}(Qc, Qa, \mathfrak{I}) < 1$ for all $c, a \in \mathcal{V}$ and*

$$\begin{aligned}
&\Big[1 - \cos\Big(\frac{\pi}{2}\min\{\mathcal{R}(c, a, \mathfrak{I}), \mathcal{R}(c, Qc, \mathfrak{I}), \mathcal{R}(a, Qa, \mathfrak{I})\}\Big)\Big]^k \\
&\qquad \leq 1 - \cos\Big(\frac{\pi}{2}\mathcal{R}(Qc, Qa, \mathfrak{I})\Big),
\end{aligned}$$

Then Q has a unique fixed point.

Proof. Follows from Theorem 3.6 by setting $\zeta(\nu, \mu) = \frac{\lambda(\nu)}{[\lambda(\mu)]^k}$ for all $\nu, \mu \in (0, 1]$, λ as the identity function and $\theta_f(\eta) = 1 - \cos\Big(\frac{\pi}{2}(\eta)\Big)$ for all $\eta \in (0, 1)$. □ □

Conclusion

With the help of fuzzy $\mathcal{L}_\lambda$-simulation functions and fuzzy θ_f-contractive mappings, we developed a novel fuzzy contraction principle in this study. We then demonstrated the existence and uniqueness of a fixed point for a self-mapping in complete fuzzy metric spaces using the specially developed contraction. It is vital to highlight that we may specifically and deduce a wide range of potential outcomes from our main results by appropriately presenting varied instances of the control functions ζ and θ_f. Our approach may open the path for new studies in fuzzy fixed point theory.

References

[1] L. A. Zadeh, *Fuzzy sets,* Inform. Control., 8 (**1965**), 338-353.
[2] I. Kramosil and J. Michalek, *Fuzzy metrics and statistical metric spaces,* Kybernetika 11(5) (**1975**), 336-344.
[3] M. Grabiec, *Fixed points in fuzzy metric spaces,* Fuzzy Sets Syst. 27 (3) (**1988**), 385-389.
[4] A. George and P. Veeramani, *On some results in fuzzy metric spaces*, Fuzzy Sets Syst. 64 (3) (**1994**), 395-399.
[5] V. Gregori, S. Morillas, and A. Sapena, "Examples of fuzzy metrics and applications," Fuzzy Sets Syst. 170 (**2011**), no. 1, 95-111.
[6] V. Gregori and A. Sapena, *On fixed-point theorems in fuzzy metric spaces,* Fuzzy Sets Syst. 125 (**2002**), 245-252.
[7] P. Tirado, *Contraction mappings in fuzzy quasi-metric spaces and* $[0,1]$*-fuzzy posets,* Fixed Point Theory. 13, (**2012**), no. 1, 273-283.
[8] F. Khojasteh, S. Shukla and S. Radenović, *A new approach to the study of fixed point theory for simulation functions*, Filomat 29(6) (**2015**), 1189-1194.
[9] B. Schweizer and Sklar, *Statistical metric spaces,* Pacific J. Math 10 (1) (**1960**), 313-334.
[10] D. Mihet, *Fuzzy* ψ*-contractive mappings in non-archimedean fuzzy metric spaces,* Fuzzy Sets Syst. 159, (**2008**), no. 6, 739-744.
[11] E. Karapınar, *Fixed points results via simulation functions, Filomat*, 30(8), 2343-2350 (**2016**).
[12] D. Wardowki, *Fuzzy contractive mappings and fixed points in fuzzy metric spaces,* Fuzzy Sets Syst. 222, (**2013**), 108-114.
[13] Seong-Hoon Cho, *Fixed Point Theorems for* $\mathcal{L}$*-Contractions in Generalized Metric Spaces*, Abstract and Applied Analysis, Volume **2018**, Article ID 1327691, 6 pages.
[14] M. Jleli and B. Samet, *A new generalization of the Banach contraction principle,* J. Inequal. Appl. **2014** (2014), Paper No. 38.
[15] M. Demma, R. Saadati, P. Vetro, *Fixed point results on b- metric space via Picard sequences and b-simulations, Iran.J. Math.Sci. Inform.*, 11(1), 123-136 (**2016**).
[16] U.D.Patel, S. Radenović, *An application to nonlinear fractional differential equation via* $\alpha - \Gamma F-$*fuzzy contractive mappings in a fuzzy metric space,* Mathematics, **2022**,
[17] D. Rakić, A. Mukheimer, T. Došenović, Z. D. Mitrović, S. Radenović**,** *Some new fixed point results in b-fuzzy metric spaces,* J. Inequalities Appl., (**2020**) 2020:99.
[18] S. Melliani and A. Moussaoui, *Fixed point theorem using a new class of fuzzy contractive mappings,* J. Universal Math. 1, (**2018**), no. 2, 148-154.
[19] A. Moussaoui, N. Saleem, S. Melliani and M. Zhou, *Fixed point results for new types of fuzzy contractions via admissible functions and FZ-simulation functions,* Axioms, 11 (**2022**), Paper No. 87.
[20] D. Rakić, A. Mukheimer, T. Došenović, Z. D. Mitrović, S. Radenović**,** *Some new fixed point results in b-fuzzy metric spaces,* J. Inequalities Appl., (**2020**) 2020:99.
[21] S. Melliani, A. Moussaoui and L. S. Chadli, *Admissible almost type Z-contractions and fixed point results,* Int. J. Math. Math. Sci. (**2020**), Article ID 9104909.
[22] E. Karapinar, C. Heidary, F. Khojateh and S. Radenović, *Study of* Γ*-Simulation Functions and Revisting* $\mathcal{L}-$*contraction and a characterization of* $\mathcal{Z}-$*contraction,* Filomat 35:1 (**2021)**, 201-224
[23] A. Moussaoui, N. Hussain, S. Melliani, N. Hayel and M. Imdad, *Fixed point results via extended* $\mathcal{FZ}$*-simulation functions in fuzzy metric spaces,* J. Inequal. Appl. 2022 (**2022**), Paper No. 69.
[24] N. Mlaiki, N. Dedović, H. Aydi, M. Gardašević-Filipović, B. Bin-Mohsin, S. Radenović, *Some new observations on Geraghty and Ćirić type results in b-metric spaces,* Mathematics, 7 (**2019**), 643.
[25] N. S. Hayel, I.A. Khan, M. Imdad and W.M. Alfaqih, *New fuzzy* φ*-fixed point results employing a new class of fuzzy contractive mappings.*, J. Intell. Fuzzy Syst. 37(4) (**2019**), 5391-5402.
[26] A. Moussaoui, N. Hussain and S. Melliani, *Global optimal solutions for proximal fuzzy contractions involving control functions,* J. Math. **2021** (2021), Article ID 6269304.
[27] P. Debnath et al., *Metric Fixed Point Theory, Applications in Science, Engineering and Behavioural Sciences, Springer 2021.*
[28] Cho, S.-H. *Fixed-Point Theorems for* L_γ *Contractions in Branciari Distance Spaces.* Axioms **2022**, 11, 479.
[29] N. S. Hayel, M. Imdad, I. A. Khan and M. D. Hasanuzzaman, *Fuzzy* Θ_f*-contractive mappings and their fixed points with applications,* J. Intell. Fuzzy Syst. 39 (**2020**), no. 5, 7097-7106.