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# Extended contraction mappings

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**Abstract.** We extend the Banach contraction principle and define a condition that applies to contraction mappings as well as nonexpansive mappings. The fixed point sets and domains of the mappings satisfying our theorems display interesting algebraic, geometric and dynamical features. Various examples substantiate our results.

## 1. Introduction

Banach [1] proved that if a selfmapping $T$ of a complete metric space $(X, \rho)$ satisfies:

$$\rho(Tx, Ty) \leq \lambda\rho(x, y), \ 0 \leq \lambda < 1, \tag{1}$$

then $T$ has a unique fixed point. Various useful applications and generalizations of this theorem have been obtained e. g. Boyd and Wong [4], Chatterjea [6], Ciric [7, 8], Kannan [12, 13], Meir-Keeler [14], Suzuki [24], Wardowski [25, 26]. In 2017, Pant and Pant [18] proved that the contractive type $(\epsilon, \delta)$ condition:

given $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that

$$\epsilon < \max\{d(x, fx), d(y, fy)\} < \epsilon + \delta \implies d(fx, fy) \leq \epsilon \tag{2}$$

applies to nonexpansive type mappings as well (see Theorem 2.9 [18]) and named such mappings as $(\epsilon - \delta)$ nonexpansive mappings. Condition (2) or its variants have been employed by researchers to find new solutions of Rhoades' problem [21] on continuity of contractive mappings at the fixed point, e. g., Bisht and Pant [2], Bisht and Rakocevic [3], Celik and Ozgur [5], Pant [16, 17], Pant et al [19, 20], Tas and Ozgur [23], Zheng and Wang [27].

In this paper we modify the Banach contraction condition (1) to make it applicable to contraction mappings as well as nonexpansive mappings. First, we give some relevant definitions.

**Definition 1.1 ([9, 10]).** *If $T$ is a self-mapping of a set $X$ then a point $x$ in $X$ is called an eventually fixed point of $T$ if there exists a natural number $N$ such that $T^{n+1}(x) = T^n(x)$ for $n \geq N$. If $Tx = x$ then $x$ is called a fixed point of $T$. A point $x$ in $X$ is called a periodic point of period $n$ if $T^n x = x$. The least positive integer $n$ for which $T^n x = x$ is called the prime period of $x$.*

**Definition 1.2.** *The set $\{x \in X : Tx = x\}$ is called the fixed point set of the mapping $T : X \to X$.*

**Definition 1.3.** *The function $T : (-\infty, \infty) \to (-\infty, \infty)$ such that $T(x)$ is the least integer not less than $x$ is called the least integer function or the ceiling function and is denoted by $T(x) = \lceil x \rceil$.*

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## 2. Main Results

**Theorem 2.1.** *Let $(X, \rho)$ be a complete metric space and $T : X \to X$ be such that for each $x, y$ in $X$ with $x \neq Tx$ or $y \neq Ty$ we have*

*(i)* $\rho(Tx, Ty) \leq \lambda\rho(x, y), 0 \leq \lambda < 1.$

*Then $T$ has a fixed point. $T$ has a unique fixed point $\iff$ (i) is satisfied for each $x \neq y$ in $X$.*

*Proof.* From (i) we infer that $T$ is continuous since $\rho(Tx, Ty) = \rho(x, y)$ when $x = Tx$ and $y = Ty$. Let $y_0$ be any point in $X$ and $\{y_n\}$ be the sequence defined by $y_n = Ty_{n-1}$, that is, $y_n = T^n y_0$. If $y_n = y_{n+1}$ for some $n$, then $y_n$ is a fixed point of $T$ and the theorem holds. Therefore, assume that $y_n \neq y_{n+1}$ for each $n \geq 0$. Then using (i), for each $n \geq 1$ and $p \geq 1$ we have

$$\begin{aligned}\rho(y_n, y_{n+p}) &= \rho(Ty_{n-1}, Ty_{n+p-1}) \\ &\leq \lambda\rho(y_{n-1}, y_{n+p-1}) \leq \lambda^2\rho(y_{n-2}, y_{n+p-2}) \leq \ldots \leq \lambda^n\rho(y_0, y_p).\end{aligned}$$

This implies that $\lim_{n\to\infty} \rho(y_n, y_{n+p}) = 0$, that is, $\{y_n\}$ is a Cauchy sequence. Since $X$ is complete, there exists $z$ in $X$ such that $\lim_{n\to\infty} y_n = z$ and $\lim_{n\to\infty} Ty_n = z$. Continuity of $T$ implies $\lim_{n\to\infty} Ty_n = Tz$, that is, $z = Tz$ and $z$ is a fixed point of $T$. Further, let $u$ be any point in X. Then, since $T^n y_0 = y_n$ is not a fixed point, using (i) we get

$$\rho(T^n u, T^n y_0) \leq \lambda\rho(T^{n-1}u, T^{n-1}y_0) \leq \lambda^2\rho(T^{n-2}u, T^{n-2}y_0) \leq \ldots \leq \lambda^n\rho(u, y_0).$$

This implies $\lim_{n\to\infty} \rho(T^n u, T^n y_0) = 0$, that is, $\lim_{n\to\infty} T^n u = z$. Thus, if there exists a point $y_0$ such that $T^{n+1}y_0 \neq T^n y_0$ for each $n$, then for each $u$ in $X$ the sequence of iterates $\{T^n u\}$ converges to $z$ and $z$ is the unique fixed point. Therefore,$T^{n+1}y_0 \neq T^n y_0, n \geq 0$, for some $y_0$ implies uniqueness of the fixed point.

Now, assume that condition (i) is satisfied for all $x, y$ in $X$. Then $T$ can have only one fixed point. Conversely, suppose that $T$ has a unique fixed point. Then for distinct $x, y$ we have $x \neq Tx$ or $y \neq Ty$ which implies that condition (i) holds for each $x \neq y$. This proves the theorem. □

**Example 2.2.** *Let $X = [1, \infty)$ and $\rho$ be the Euclidean metric. Let $T : X \to X$ be the signum function $Tx = sgn\, x$ defined as*

$$Tx = -1 \text{ if } x < 0, \quad T0 = 0, \quad Tx = 1 \text{ if } x > 0.$$

*Then $Tx = 1$ for each $x$ and $T$ is a contraction mapping that has a unique fixed point $x = 1$. If $x \neq 1$ then $Tx = T^2x$ and $x$ is an eventually fixed point.*

**Example 2.3.** *Let $F = \{re^{i\theta} : 0 \leq \theta \leq 2\pi, r = 1, 3, 3^2, \ldots\}$ be the self-similar family of concentric circles, each lying within larger circles having radii in a geometric progression, in the xy-plane. Let $X$ be the set of points of intersection of $F$ with the $N$ rays beginning at the origin and respectively making angles $0, \frac{2\pi}{N}, 2(\frac{2\pi}{N}), 3(\frac{2\pi}{N}), \ldots, (N-1)(\frac{2\pi}{N})$ measured counter clockwise with the positive x-axis and let $d$ be the usual metric on $X$. Define $T : X \to X$ by*

$$T(re^{i\theta}) = \lceil \frac{r}{3} \rceil e^{i\theta}$$

*where $\lceil x \rceil$ denotes the least integer not less than $x$. Then $T$ satisfies condition (i) with $\lambda = \frac{1}{2}$ and has $N$ fixed points $e^{i0}, e^{i(\frac{2\pi}{N})}, e^{i2(\frac{2\pi}{N})}, e^{i3(\frac{2\pi}{N})}, \ldots, e^{i(N-1)(\frac{2\pi}{N})}$. If $N = 1$ then $T$ is a Banach contraction mapping and has a unique fixed point $e^{i0} = 1$.*

**Example 2.4.** *Let $X = \{z = re^{i\theta} : 0 \leq \theta \leq 2\pi, r = 1, 3, 3^2, \ldots\}$ be the self-similar family of concentric circles, each lying within larger circles having radii in a geometric progression, in the xy-plane and let $d$ be the usual metric on $X$. Define $T : X \to X$ by $T(z) = \frac{z}{|z|} = \frac{z}{r}$.*
*Then $T$ satisfies (i) with $\lambda = \frac{1}{2}$ and each point on the unit circle $z = e^{i\theta}$ is a fixed point while every other point is an eventually fixed point. In this example, the unit circle is a fixed circle. Fixed circles are presently an active area of study (see [11, 15, 22]).*

**Example 2.5.** *Let $(X, \rho)$ be a metric space and $T$ be the identity mapping on $X$. Then each point is a fixed point and conditions (i) holds since there is no pair of points $(x, y)$ in $X$ that violates it.*

**Remark 2.6.** *The $N$ fixed points $e^{i0}, e^{i(\frac{2\pi}{N})}, e^{i2(\frac{2\pi}{N})}, e^{i3(\frac{2\pi}{N})}, \ldots, e^{i(N-1)(\frac{2\pi}{N})}$ in Example 2.3 are:*

A. *the $N^{th}$ roots of unity and these lie on the unit circle and form a cyclic group under multiplication,*
B. *vertices of a regular polygon of $N$ sides.*

*If $N = 2^n - 1$ then the fixed point set is identical with the set of periodic points of period $n$ for the doubling map which is important in dynamics of complex functions (see [9, 10]).*
*Also, the domain of the mapping in Example 2.4 is a self-similar family of circles. We thus see that the domain and the fixed point set of the mappings satisfying Theorem 2.1 may posses interesting algebraic, geometric and dynamical features. In place of the self-similar family of circles if we consider a self-similar family of spheres then the domain will be more intricate and visually attractive.*

## 3. Applications

We now give an application of condition (i) in determining the cardinality of the fixed point set of mappings for which Theorem 2.1 holds.

Suppose $(X, \rho)$ is a complete metric space and Theorem 2.1 holds for $T : X \to X$. Then $T$ has one or more fixed points. If condition (i) is satisfied for each $x \neq y$ in $X$ then $T$ has a unique fixed point. If $u, v$ are distinct fixed points of $T$ then $\rho(Tu, Tv) = \rho(u, v)$.

Suppose each set of $n + 1$ points $y_1, y_2, \ldots, y_{n+1}$ in $X$ satisfies

$$\begin{aligned}&\rho(Ty_1, Ty_2) + \rho(Ty_2, Ty_3) + \ldots + \rho(Ty_n, Ty_{n+1}) + \rho(Ty_{n+1}, Ty_1)\\&< \rho(y_1, y_2) + \rho(y_2, y_3) + \ldots + \rho(y_n, y_{n+1}) + \rho(y_{n+1}, y_1).\end{aligned}$$

Then, the number of fixed points of $T$ cannot exceed $n$. For, if $T$ has $n + 1$ fixed points, say $z_1, z_2, \ldots, z_{n+1}$, then we get

$$\begin{aligned}&\rho(Tz_1, Tz_2) + \rho(Tz_2, Tz_3) + \ldots + \rho(Tz_n, Tz_{n+1}) + \rho(Tz_{n+1}, Tz_1)\\&= \rho(z_1, z_2) + \rho(z_2, z_3) + \ldots + \rho(z_n, z_{n+1}) + \rho(z_{n+1}, z_1),\end{aligned}$$

which contradicts our assumption.

Next, suppose there exists a set of $n$ points $x_1, x_2, \ldots, x_n$ in $X$ such that $T$ does not satisfy

$$\begin{aligned}&\rho(Tx_1, Tx_2) + \rho(Tx_2, Tx_3) + \ldots + \rho(Tx_{n-1}, Tx_n) + \rho(Tx_n, Tx_1)\\&< \rho(x_1, x_2) + \rho(x_2, x_3) + \ldots + \rho(x_{n-1}, x_n) + \rho(x_n, x_1).\end{aligned} \tag{3}$$

By virtue of (i) this implies that each of $x_1, x_2, \ldots, x_n$ is a fixed point of $T$, otherwise $T$ will satisfy (3). This can be summarised as:

**Theorem 3.1.** *The cardinality of the set of fixed point of a selfmapping $T$ satisfying the conditions of Theorem 2.1 equals $n$ if and only if for each set of $n + 1$ points $y_1, y_2, \ldots, y_{n+1}$ we have*

$$\begin{aligned}&\rho(Ty_1, Ty_2) + \rho(Ty_2, Ty_3) + \ldots + \rho(Ty_n, Ty_{n+1}) + \rho(Ty_{n+1}, Ty_1)\\&< \rho(y_1, y_2) + \rho(y_2, y_3) + \ldots + \rho(y_n, y_{n+1}) + \rho(y_{n+1}, y_1),\end{aligned} \tag{4}$$

*while there exists a set of $n$ points $x_1, x_2, \ldots, x_n$ in $X$ that does not satisfy*

$$\begin{aligned}&\rho(Tx_1, Tx_2) + \rho(Tx_2, Tx_3) + \ldots + \rho(Tx_{n-1}, Tx_n) + \rho(Tx_n, Tx_1)\\&< \rho(x_1, x_2) + \rho(x_2, x_3) + \ldots + \rho(x_{n-1}, x_n) + \rho(x_n, x_1).\end{aligned} \tag{5}$$

**Remark 3.2.** *The proof of Theorem 2.1 shows that if for some $x$ in $X$ we have $T^n x \neq T^{n+1} x$ for each $n \geq 0$ then $T$ has a unique fixed point. This implies that if $T$ has more than one fixed point then the orbit $\{T^n x : n = 0, 1, \ldots\}$ of each $x$ in $X$ is a finite set, that is, starting the iteration with any initial point we reach a fixed point in a finite number of steps. This simplifies the search for fixed points. If $T$ has a finite number of fixed points, then the cardinality of the fixed point set can be determined by using inequalities (4) and (5).*

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