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On new trapezoid and midpoint type inequalities for generalized quantum integrals

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Abstract. In this article, by utilizing the functions with bounded second derivatives, we first prove some trapezoid and midpoint type inequalities for generalized quantum integrals which are introduced in the recent papers. Then we establish some new quantum integral inequalities for mappings whose second quantum derivatives are bounded. Moreover, we obtain some new weighted trapezoid and midpoint type inequalities for generalized quantum integrals by using the functions with bounded second derivatives. Finally, we investigate the connections between our results and those in earlier works.

1. Introduction

Quantum calculus, occasionally known as calculus without limits, is equivalent to the traditional infinitesimal calculus without the notion of limits. Many researchers have recently been studied extensively in the field of q-calculus. Euler started out on this subject because of the very excessive demand of mathematics that fashions quantum computing q-calculus seemed like a connection between physics and mathematics. It has programs in several areas of arithmetic, along with combinatorics, quantity principle, basic hypergeometric functions, and orthogonal polynomials, and in fields of other sciences, which include mechanics, the idea of relativity, and quantum idea [10–15, 18]. Seemingly, Euler became the founder of this branch of mathematics, through the usage of the parameter q in Newtons work on the infinite collection. Later, the q-calculus turned into first given through Jackson [16]. In 1908–1909, Jackson described the general q-integral and q-difference operator [15]. In 1969, Agarwal described the q-fractional derivative for the primary time [1]. In 1966–1967, Al-Salam delivered a q-analog of the Riemann−Liouville fractional integral operator and q-fractional vital operator [5]. In 2004, Rajkovic gave a definition of the Riemann-type q-fundamental which generalized to Jackson q-essential. In 2013, Tariboon delivered ${}_{\sigma}D_{q}$-difference operator [2].

Many integral inequalities have been presented, utilizing quantum integrals for numerous type of functions. The interested readers are suggested to see [2, 4, 6, 8, 17, 21, 24, 27–29, 31, 33, 34]. The authors used quantum integrals to prove Hermite-Hadamard type integral inequalities and their left-right estimates for convex, coordinate convex and various other classes of functions. Noor et al. estimated a generalized version of quantum integral inequalities in [24]. Nwaeze et al. proved certain parametrized

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quantum integral inequalities for generalized quasi-convex functions in [26]. Khan et al. exhibited quantum Hermite-Hadamard inequality using green function in [20], Vivas-Cortez et al. [9] and Ali et al. [32] proved new quantum Simpson's and quantum Newton's type inequalities for convex and coordinated convex functions.

In this article, motivated by these continuing proceedings, we exhibit a generalized form of quantum Midpoint and quantum Trapezoid type inequalities using the functions with bounded twice differentiable derivatives, these newly established inequalities are the generalizations of previously proved results.

2. Preliminaries of q-Calculus and Some Inequalities

In this section, we discuss some required definitions of quantum calculus and important quantum integral inequalities for Hermite-Hadamard on left and right sides bounds:

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1},\ \ q \in (0,1).$$

Jackson derived the q-Jackson integral in [15] from 0 to ρ for $0 < q < 1$ as follows:

$$\int_0^\rho \mathcal{F}(\kappa)\, d_q\kappa = (1-q)\,\rho \sum_{n=0}^{\infty} q^n \mathcal{F}(\rho q^n)$$

provided the sum converge absolutely.
The q-Jackson integral in a generic interval $[\sigma, \rho]$ was given by in [15] and defined as follows:

$$\int_\sigma^\rho \mathcal{F}(\kappa)\, d_q\kappa = \int_0^\rho \mathcal{F}(\kappa)\, d_q\kappa - \int_0^\sigma \mathcal{F}(\kappa)\, d_q\kappa.$$

Definition 2.1. *Let us suppose that a function $\mathcal{F} : [\sigma, \rho] \to \mathbb{R}$ is continuous, then q_σ-derivative [29] and q^ρ-derivative [7] of $\mathcal{F}$ at $\kappa \in [\sigma, \rho]$ are defined as follows*

$${}_\sigma D_q \mathcal{F}(\kappa) = \frac{\mathcal{F}(\kappa) - \mathcal{F}(q\kappa + (1-q)\sigma)}{(1-q)(\kappa - \sigma)},\ \kappa \neq \sigma$$

and

$${}^\rho D_q \mathcal{F}(\kappa) = \frac{\mathcal{F}(q\kappa + (1-q)\rho) - \mathcal{F}(\kappa)}{(1-q)(\rho - \kappa)},\ \kappa \neq \rho.$$

Definition 2.2. *We assume that a function $\mathcal{F} : [\sigma, \rho] \to \mathbb{R}$ is continuous, then the q_σ-definite integral [30] and q^ρ-definite integral [7] on $[\sigma, \rho]$ are defined as follows*

$$\int_\sigma^\rho \mathcal{F}(\kappa)\ {}_\sigma d_q\kappa = (1-q)(\rho-\sigma) \sum_{n=0}^{\infty} q^n \mathcal{F}(q^n\rho + (1-q^n)\sigma) = (\rho - \sigma)\int_0^1 \mathcal{F}((1-t)\sigma + t\rho)\, d_q t$$

and

$$\int_\sigma^\rho \mathcal{F}(\kappa)\ {}^\rho d_q\kappa = (1-q)(\rho-\sigma) \sum_{n=0}^{\infty} q^n\ \mathcal{F}(q^n\sigma + (1-q^n)\rho) = (\rho - \sigma)\int_0^1 \mathcal{F}(t\sigma + (1-t)\rho)\, d_q t,$$

respectively.

In [2] and [7], Alp et al. and Bermudo et al. established the q_σ-Hermite-Hadamard and q_ρ-Hermite-Hadamard inequalities for convexity, which are defined as follows, respectively.

Theorem 2.3. *Let $\mathcal{F} : [\sigma, \rho] \to \mathbb{R}$ be a convex differentiable function on $[\sigma, \rho]$ and $0 < q < 1$. Then q-Hermite-Hadamard inequalities are as follows:*

$$\mathcal{F}\left(\frac{q\sigma + \rho}{1+q}\right) \leq \frac{1}{\rho - \sigma} \int_{\sigma}^{\rho} \mathcal{F}(\kappa)\ {}_{\sigma}d_q\kappa \leq \frac{q\mathcal{F}(\sigma) + \mathcal{F}(\rho)}{1+q} \tag{1}$$

and

$$\mathcal{F}\left(\frac{\sigma + q\rho}{1+q}\right) \leq \frac{1}{\rho - \sigma} \int_{\sigma}^{\rho} \mathcal{F}(\kappa)\ {}^{\rho}d_q\kappa \leq \frac{\mathcal{F}(\sigma) + q\mathcal{F}(\rho)}{1+q}. \tag{2}$$

The authors of [23] and [2] have set certain boundaries for the left and right sides of the inequality (1). On the other hand, Budak has set certain boundaries for the left and right sides of the inequality (2). From inequality (1) and inequality (2), one can the following inequalities:

Corollary 2.4. *[7] For any convex function $\mathcal{F} : [\sigma, \rho] \to \mathbb{R}$ and $0 < q < 1$, we have*

$$\mathcal{F}\left(\frac{q\sigma + \rho}{1+q}\right) + \mathcal{F}\left(\frac{\sigma + q\rho}{1+q}\right) \leq \frac{1}{\rho - \sigma} \left\{ \int_{\sigma}^{\rho} \mathcal{F}(\kappa)\ {}_{\sigma}d_q\kappa + \int_{\sigma}^{\rho} \mathcal{F}(\kappa)\ {}^{\rho}d_q\kappa \right\} \leq \mathcal{F}(\sigma) + \mathcal{F}(\rho)$$

and

$$\mathcal{F}\left(\frac{\sigma + \rho}{2}\right) \leq \frac{1}{2(\rho - \sigma)} \left\{ \int_{\sigma}^{\rho} \mathcal{F}(\kappa)\ {}_{\sigma}d_q\kappa + \int_{\sigma}^{\rho} \mathcal{F}(\kappa)\ {}^{\rho}d_q\kappa \right\} \leq \frac{\mathcal{F}(\sigma) + \mathcal{F}(\rho)}{2}.$$

By using the area of trapezoids, Alp and Sarikaya introduced the following generalized quantum integral which we will called ${}_{\sigma}T_q$-integral.

Definition 2.5. *[3] Let $\mathcal{F} : [\sigma, \rho] \to \mathbb{R}$ is continuous function. For $\kappa \in [\sigma, \rho]$*

$$\int_{\sigma}^{\rho} \mathcal{F}(t)\ {}_{\sigma}d_q^T\, t = \frac{(1-q)(\rho-\sigma)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n \mathcal{F}(q^n\rho + (1-q^n)\sigma) - \mathcal{F}(\rho)\right], \tag{3}$$

where $0 < q < 1$.

Theorem 2.6 (${}_{a}T_q$-Hermite-Hadamard). *[3] Suppose that $\mathcal{F} : [\sigma, \rho] \to \mathbb{R}$ is a convex continuous function on $[\sigma, \rho]$ and $0 < q < 1$. Then we have*

$$\mathcal{F}\left(\frac{\sigma + \rho}{2}\right) \leq \frac{1}{\rho - \sigma} \int_{\sigma}^{\rho} \mathcal{F}(\kappa)\ {}_{\sigma}d_q^T\kappa \leq \frac{\mathcal{F}(\sigma) + \mathcal{F}(\rho)}{2}.$$

In [19], Kara et al. introduced the following generalized quantum integral which is called ${}^{\rho}T_q$-integral by using the area of trapezoids.

Definition 2.7. *[19] Assume that* $\mathcal{F}:[\sigma,\rho]\to\mathbb{R}$ *is continuous function. For* $\kappa\in[\sigma,\rho]$,

$$\int_{\sigma}^{\rho}\mathcal{F}(t)\ {}^{\rho}d_q^T\,t=\frac{(1-q)(\rho-\sigma)}{2q}\left[(1+q)\sum_{n=0}^{\infty}q^n\,\mathcal{F}(q^n\sigma+(1-q^n)\rho)-\mathcal{F}(\sigma)\right], \tag{4}$$

where $0<q<1$. *This integral is called* ${}^{\rho}T_q$*-integral.*

Theorem 2.8. *[19][3] Let* $\mathcal{F}:[\sigma,\rho]\to\mathbb{R}$ *be a function and* $0<q<1$. *Then we have*

$$\int_{0}^{1}\mathcal{F}(t\rho+(1-t)\sigma)\ {}_0d_q^T t=\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathcal{F}(t)\ {}_\sigma d_q^T t$$

and

$$\int_{0}^{1}\mathcal{F}(t\rho+(1-t)\sigma)\ {}^1d_q^T t=\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathcal{F}(t)\ {}^\rho d_q^T t.$$

Theorem 2.9 (${}^bT_q$-Hermite-Hadamard). *[19] If* $\mathcal{F}:[\sigma,\rho]\to\mathbb{R}$ *is a convex continuous function on* $[\sigma,\rho]$ *and* $0<q<1$, *then we have*

$$\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\leq\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathcal{F}(\kappa)\ {}^\rho d_q^T\kappa\leq\frac{\mathcal{F}(\sigma)+\mathcal{F}(\rho)}{2}.$$

Lemma 2.10. *[22] Let us note that* $\mathcal{F}:[\sigma,\rho]\to\mathbb{R}$ *is a twice differentiable mapping so that there exist real constants* m *and* M *such that* $m\leq\mathcal{F}''\leq M$. *Then for* $\gamma\in[0,1]$, *we have*

$$m\frac{\gamma(1-\gamma)}{2}(\rho-\sigma)^2\leq(1-\gamma)\mathcal{F}(\sigma)+\gamma\mathcal{F}(\rho)-\mathcal{F}(\gamma\rho+(1-\gamma)\sigma)\leq M\frac{\gamma(1-\gamma)}{2}(\rho-\sigma)^2. \tag{5}$$

Lemma 2.11. *[22] Let us consider that* $\mathcal{F}:[\sigma,\rho]\to\mathbb{R}$ *is a twice differentiable mapping so that there exist real constants* m *and* M *such that* $m\leq\mathcal{F}''\leq M$. *Then, the following inequalities*

$$m\frac{(1-2\gamma)^2}{8}(\rho-\sigma)^2\leq\frac{\mathcal{F}(\gamma\sigma+(1-\gamma)\rho)+\mathcal{F}((1-\gamma)\sigma+\gamma\rho)}{2}-\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\leq M\frac{(1-2\gamma)^2}{8}(\rho-\sigma)^2 \tag{6}$$

are valid for all $\gamma\in[0,1]$.

3. Quantum Midpoint and Trapezoid-type Inequalities

In this section, we prove some quantum trapezoid and midpoint type inequalities for functions whose second derivatives are bounded.

Theorem 3.1. *If* $\mathcal{F}:[\sigma,\rho]\to\mathbb{R}$ *is a twice differentiable mapping, then there exist real constants* m *and* M *such that* $m\leq\mathcal{F}''\leq M$. *Then, the following double inequality*

$$m\frac{(\rho-\sigma)^2q}{4[3]_q}\leq\frac{\mathcal{F}(\sigma)+\mathcal{F}(\rho)}{2}-\frac{1}{(\rho-\sigma)}\int_{\sigma}^{\rho}\mathcal{F}(\kappa)\ {}^\rho d_q^T\kappa\leq M\frac{(\rho-\sigma)^2q}{4[3]_q} \tag{7}$$

is valid for $0<q<1$.

Proof. By ${}^{\rho}T_q$ integrating of (5) with respect to the γ from 0 to 1, we have

$$m\frac{(\rho-\sigma)^2}{2}\int_0^1 \gamma(1-\gamma)\ {}^1d_q^T\gamma \le \mathcal{F}(\sigma)\int_0^1 (1-\gamma)\ {}^1d_q^T\gamma + \mathcal{F}(\rho)\int_0^1 \gamma\ {}^1d_q^T\gamma \tag{8}$$

$$-\int_0^1 \mathcal{F}(\gamma\rho+(1-\gamma)\sigma)\ {}^1d_q^T\gamma \le M\frac{(\rho-\sigma)^2}{2}\int_0^1 \gamma(1-\gamma)\ {}^1d_q^T\gamma.$$

With the help of the equality (4), we obtain the following equalities

$$\int_0^1 \gamma\ {}^1d_q^T\gamma = \frac{(1-q)}{2q}(1+q)\sum_{n=0}^{\infty} q^n(1-q^n) = \frac{1-q^2}{2q}\left[\frac{1}{1-q}-\frac{1}{1-q^2}\right] = \frac{1}{2}, \tag{9}$$

$$\begin{aligned}\int_0^1 (1-\gamma)\ {}^1d_q^T\gamma &= \frac{(1-q)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n(1-(1-q^n))-1\right] \\ &= \frac{1-q}{2q}\left[(1+q)\frac{1}{1-q^2}-1\right] = \frac{1}{2},\end{aligned} \tag{10}$$

$$\begin{aligned}\int_0^1 \gamma^2\ {}^1d_q^T\gamma &= \frac{(1-q)}{2q}(1+q)\sum_{n=0}^{\infty} q^n(1-q^n)^2 \\ &= \frac{\left(1-q^2\right)}{2q}\sum_{n=0}^{\infty}\left(q^n-2q^{2n}+q^{3n}\right) \\ &= \frac{\left(1-q^2\right)}{2q}\left(\frac{1}{1-q}-\frac{2}{1-q^2}+\frac{1}{1-q^3}\right) \\ &= \frac{1+q^2}{2[3]_q},\end{aligned} \tag{11}$$

$$\int_0^1 \gamma(1-\gamma)\ {}^1d_q^T\gamma = \int_0^1 \gamma\ {}^1d_q^T\gamma - \int_0^1 \gamma^2\ {}^1d_q^T\gamma = \frac{q}{2[3]_q}, \tag{12}$$

and

$$\begin{aligned}\int_0^1 \mathcal{F}(\gamma\rho+(1-\gamma)\sigma)\ {}^1d_q^T\gamma &= \frac{(1-q)}{2q}\left[(1+q)\sum_{n=0}^{\infty} q^n\mathcal{F}((1-q^n)\rho+q^n\sigma)-\mathcal{F}(\sigma)\right] \\ &= \frac{1}{(\rho-\sigma)}\int_\sigma^\rho \mathcal{F}(\kappa)\ {}^{\rho}d_q^T\kappa.\end{aligned} \tag{13}$$

If we substitute the equalities (9)-(13) in the double inequality (8), then we obtain the desired result. □

Theorem 3.2. *If $\mathcal{F} : [\sigma, \rho] \to \mathbb{R}$ is a twice differentiable mapping, then there exist real constants m and M so that $m \leq \mathcal{F}'' \leq M$. Then, the following double inequality holds:*

$$m\frac{(\rho-\sigma)^2 q}{4[3]_q} \leq \frac{\mathcal{F}(\sigma)+\mathcal{F}(\rho)}{2} - \frac{1}{(\rho-\sigma)}\int_\sigma^\rho \mathcal{F}(\kappa)\ {}_\sigma d_q^T\kappa \leq M\frac{(\rho-\sigma)^2 q}{4[3]_q} \tag{14}$$

for $0 < q < 1$.

Proof. Integrating double inequality (5) with respect to the γ over $[0,1]$, we get

$$m\frac{(\rho-\sigma)^2}{2}\int_0^1 \gamma(1-\gamma)\ {}_0d_q^T\gamma \leq \mathcal{F}(\sigma)\int_0^1 (1-\gamma)\ {}_0d_q^T\gamma + \mathcal{F}(\rho)\int_0^1 \gamma\ {}_0d_q^T\gamma - \int_0^1 \mathcal{F}(\gamma\rho+(1-\gamma)\sigma)\ {}_0d_q^T\gamma \tag{15}$$
$$\leq M\frac{(\rho-\sigma)^2}{2}\int_0^1 \gamma(1-\gamma)\ {}_0d_q^T\gamma.$$

By using the equality (3), we have the following equalities

$$\int_0^1 \gamma\ {}_0d_q^T\gamma = \frac{(1-q)}{2q}(1+q)\sum_{n=0}^\infty q^n(1-q^n) = \frac{1-q^2}{2q}\left[\frac{1}{1-q} - \frac{1}{1-q^2}\right] = \frac{1}{2}, \tag{16}$$

$$\int_0^1 (1-\gamma)\ {}_0d_q^T\gamma = \frac{(1-q)}{2q}\left[(1+q)\sum_{n=0}^\infty q^n(1-(1-q^n)) - 1\right] = \frac{1}{2}, \tag{17}$$

$$\int_0^1 \gamma^2\ {}_0d_q^T\gamma = \frac{(1-q)}{2q}(1+q)\sum_{n=0}^\infty q^n(1-q^n)^2 \tag{18}$$
$$= \frac{\left(1-q^2\right)}{2q}\sum_{n=0}^\infty \left(q^n - 2q^{2n} + q^{3n}\right) = \frac{1+q^2}{2[3]_q},$$

$$\int_0^1 \gamma(1-\gamma)\ {}_0d_q^T\gamma = \int_0^1 \gamma\ {}_0d_q^T\gamma - \int_0^1 \gamma^2\ {}_0d_q^T\gamma = \frac{q}{2[3]_q} \tag{19}$$

and

$$\int_0^1 \mathcal{F}(\gamma\rho+(1-\gamma)\sigma)\ {}_0d_q^T\gamma = \frac{(1-q)}{2q}\left[(1+q)\sum_{n=0}^\infty q^n\mathcal{F}(q^n\rho+(1-q^n)\sigma) - \mathcal{F}(\rho)\right] \tag{20}$$
$$= \frac{1}{(\rho-\sigma)}\int_\sigma^\rho \mathcal{F}(\kappa)\ {}_\sigma d_q^T\kappa.$$

By substituting the inequalities (16)-(20) in the double inequality (15), we establish required result. □

Theorem 3.3. *Assume that $\mathcal{F}:[\sigma,\rho]\to\mathbb{R}$ is a twice differentiable mapping. Then there exist real constants m and M such that $m\leq\mathcal{F}''\leq M$. Then, the following double inequality*

$$m\frac{(\rho-\sigma)^2\left(1-q+q^2\right)}{8[3]_q}\leq\frac{1}{2(\rho-\sigma)}\left[\int_{\sigma}^{\rho}\mathcal{F}(\kappa)\ {}_{\sigma}d_q^T\kappa+\int_{\sigma}^{\rho}\mathcal{F}(\kappa)\ {}^{\rho}d_q^T\kappa\right]-\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\leq M\frac{(\rho-\sigma)^2\left(1-q+q^2\right)}{8[3]_q} \tag{21}$$

is valid for $0<q<1$.

Proof. Let us ${}_{\sigma}T_q$ integrate double inequality (6) with respect to the γ over from 0 to 1, we obtain

$$\begin{aligned}m\frac{(\rho-\sigma)^2}{8}\int_0^1(1-2\gamma)^2\ {}_0d_q^T\gamma&\leq\frac{1}{2}\left[\int_0^1\mathcal{F}(\gamma\sigma+(1-\gamma)\rho)\ {}_0d_q^T\gamma\right.\\&\left.\quad+\int_0^1\mathcal{F}((1-\gamma)\sigma+\gamma\rho)\ {}_0d_q^T\gamma-\int_0^1\mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\ {}_0d_q^T\gamma\right]\\&\leq M\frac{(\rho-\sigma)^2}{8}\int_0^1(1-2\gamma)^2\ {}_0d_q^T\gamma.\end{aligned}$$

With the help of the equality (3), we get the following equalities

$$\begin{aligned}\int_0^1(1-2\gamma)^2\ {}_0d_q^T\gamma&=\frac{(1-q)}{2q}\left[(1+q)\sum_{n=0}^{\infty}q^n(1-2q^n)^2-1\right]\\&=\frac{(1-q)}{2q}\left[(1+q)\left(\frac{1}{1-q}-\frac{4}{1-q^2}+\frac{4}{1-q^3}\right)-1\right]\\&=\frac{1-q+q^2}{[3]_q},\end{aligned} \tag{22}$$

$$\begin{aligned}\int_0^1\mathcal{F}(\gamma\sigma+(1-\gamma)\rho)\ {}_0d_q^T\gamma&=\frac{(1-q)}{2q}\left[(1+q)\sum_{n=0}^{\infty}q^n\mathcal{F}(q^n\sigma+(1-q^n)\rho)-\mathcal{F}(\sigma)\right]\\&=\frac{1}{(\rho-\sigma)}\int_{\sigma}^{\rho}\mathcal{F}(\kappa)\ {}^{\rho}d_q^T\kappa,\end{aligned} \tag{23}$$

and

$$\begin{aligned}\int_0^1\mathcal{F}((1-\gamma)\sigma+\gamma\rho)\ {}_0d_q^T\gamma&=\frac{(1-q)}{2q}\left[(1+q)\sum_{n=0}^{\infty}q^n\mathcal{F}((1-q^n)\sigma+q^n\rho)-\mathcal{F}(\rho)\right]\\&=\frac{1}{(\rho-\sigma)}\int_{\sigma}^{\rho}\mathcal{F}(\kappa)\ {}_{\sigma}d_q^T\kappa.\end{aligned} \tag{24}$$

If we substitute the inequalities (22)-(24) in (21), then we establish desired result. □

4. Quantum Integral Inequalities for Functions with Bounded Quantum Derivatives

In this section, we present some quantum trapezoid type inequalities for function whose second quantum derivatives are bounded. Now, we first prove the following Lemma.

Lemma 4.1. *Let $\mathcal{F}:[\sigma,\rho]\to\mathbb{R}$ be twice q-differentiable. If ${}^{\rho}D_q^2\mathcal{F}$ is integrable on $[\sigma,\rho]$, then we have*

$$\frac{q^3}{[2]_q(\rho-\sigma)}\int_{\sigma}^{\rho}(\kappa-\sigma)(\rho-\kappa)\ {}^{\rho}D_q^2\mathcal{F}(\kappa)\ {}^{\rho}d_q^T\kappa$$
$$=\frac{(1-q)\mathcal{F}(\sigma)+q\mathcal{F}(\rho)+\mathcal{F}(q\sigma+(1-q)\rho)}{2}-\frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathcal{F}(\kappa)\ {}^{\rho}d_q^T\kappa$$

for $0<q<1$.

Proof. By using Definition 2.1, ${}^{\rho}D_q^2\mathcal{F}(\kappa)$ is obtained as

$${}^{\rho}D_q^2\mathcal{F}(\kappa)=\frac{q\mathcal{F}(\kappa)-(1+q)\mathcal{F}\left(q\kappa+(1-q)\rho+\mathcal{F}\left(q^2\kappa+\left(1-q^2\right)\rho\right)\right)}{q(1-q)^2(\rho-\kappa)^2}. \tag{25}$$

By equality (25), we get

$$\int_{\sigma}^{\rho}(\kappa-\sigma)(\rho-\kappa)\ {}^{\rho}D_q^2\mathcal{F}(\kappa)\ {}^{\rho}d_q^T\kappa$$
$$=\frac{1}{q(1-q)^2}\left[q\int_{\sigma}^{\rho}\frac{\kappa-\sigma}{\rho-\kappa}\mathcal{F}(\kappa)\ {}^{\rho}d_q^T\kappa-(1+q)\int_{\sigma}^{\rho}\frac{\kappa-\sigma}{\rho-\kappa}\mathcal{F}(q\kappa+(1-q)\rho)\ {}^{\rho}d_q^T\kappa\right.$$
$$\left.+\int_{\sigma}^{\rho}\frac{\kappa-\sigma}{\rho-\kappa}\mathcal{F}\left(q^2\kappa+\left(1-q^2\right)\rho\right)\ {}^{\rho}d_q^T\kappa\right].$$

Using Definition 2.7, we obtain

$$\int_{\sigma}^{\rho}(\kappa-\sigma)(\rho-\kappa)\ {}^{\rho}D_q^2\mathcal{F}(\kappa)\ {}^{\rho}d_q^T\kappa$$
$$=\frac{(\rho-\sigma)(1+q)}{2q^2(1-q)}\left[q\sum_{n=0}^{\infty}(1-q^n)\mathcal{F}(q^n\sigma+(1-q^n)\rho)\right.$$
$$-(1+q)\sum_{n=0}^{\infty}(1-q^n)\mathcal{F}(q^n\sigma+(1-q^n)\rho)$$
$$\left.+\sum_{n=0}^{\infty}(1-q^n)\mathcal{F}\left(q^{n+2}\sigma+\left(1-q^{n+2}\right)\rho\right)\right].$$

By using properties of series, we can write

$$\int_{\sigma}^{\rho}(\kappa-\sigma)(\rho-\kappa)\ {}^{\rho}D_q^2\mathcal{F}(\kappa)\ {}^{\rho}d_q^T\kappa$$

$$
\begin{aligned}
&= \frac{(\rho-\sigma)(1+q)}{2q^2(1-q)} \\
&\times \left[q\sum_{n=0}^{\infty} \mathcal{F}(q^n\sigma + (1-q^n)\rho) - (1+q)\sum_{n=0}^{\infty} \mathcal{F}\left(q^{n+1}\sigma + \left(1-q^{n+1}\right)\rho\right) \right. \\
&+ \sum_{n=0}^{\infty} \mathcal{F}\left(q^{n+2}\sigma + \left(1-q^{n+2}\right)\rho\right) - q\sum_{n=0}^{\infty} q^n\mathcal{F}(q^n\sigma + (1-q^n)\rho) \\
&\left. + (1+q)\sum_{n=0}^{\infty} q^n\mathcal{F}\left(q^{n+1}\sigma + \left(1-q^{n+1}\right)\rho\right) - \sum_{n=0}^{\infty} q^n\mathcal{F}\left(q^{n+2}\sigma + \left(1-q^{n+2}\right)\rho\right) \right] \\
&= \frac{(\rho-\sigma)(1+q)}{2q^2(1-q)} \left[q\sum_{n=0}^{\infty} \left[\mathcal{F}(q^n\sigma + (1-q^n)\rho) - \mathcal{F}\left(q^{n+1}\sigma + \left(1-q^{n+1}\right)\rho\right)\right] \right. \\
&+ \sum_{n=0}^{\infty} \left[\mathcal{F}\left(q^{n+2}\sigma + \left(1-q^{n+2}\right)\rho\right) - \mathcal{F}\left(q^{n+1}\sigma + \left(1-q^{n+1}\right)\rho\right)\right] \\
&- q\sum_{n=0}^{\infty} q^n\mathcal{F}(q^n\sigma + (1-q^n)\rho) + \frac{1+q}{q}\left(\sum_{n=0}^{\infty} q^n\mathcal{F}(q^n\sigma + (1-q^n)\rho) - \mathcal{F}(\sigma)\right) \\
&\left. - \frac{1}{q^2}\left(\sum_{n=0}^{\infty} q^n\mathcal{F}(q^n\sigma + (1-q^n)\rho) - \mathcal{F}(\sigma) - q\mathcal{F}(q\sigma + (1-q)\rho)\right) \right] \\
&= \frac{(\rho-\sigma)(1+q)}{2q^2(1-q)} \left[q\left[\mathcal{F}(\sigma) - \mathcal{F}(\rho)\right] + \mathcal{F}(\rho) - \mathcal{F}(q\sigma + (1-q)\rho) \right. \\
&+ \left(-q + \frac{1+q}{q} - \frac{1}{q^2}\right)\sum_{n=0}^{\infty} q^n\mathcal{F}(q^n\sigma + (1-q^n)\rho) \\
&\left. + \left(-\frac{1+q}{q} + \frac{1}{q^2}\right)\mathcal{F}(\sigma) + \frac{1}{q}\mathcal{F}(q\sigma + (1-q)\rho) \right] \\
&= \frac{(\rho-\sigma)(1+q)(1-q)}{2q^3}\mathcal{F}(\sigma) + \frac{(\rho-\sigma)(1+q)}{2q^2}\mathcal{F}(\rho) \\
&+ \frac{(\rho-\sigma)(1+q)}{2q^3}\mathcal{F}(q\sigma + (1-q)\rho) - \frac{1+q}{q^3}\int_{\sigma}^{\rho} \mathcal{F}(\kappa)\ {}^{\rho}d_q^T\kappa.
\end{aligned}
$$

This ends the proof of Lemma 4.1. □

Theorem 4.2. *Let $\mathcal{F} : [\sigma, \rho] \to \mathbb{R}$ be twice q-differentiable. If ${}^{\rho}D_q^2\mathcal{F}$ is integrable on $[\sigma, \rho]$ and $m \leq {}^{\rho}D_q^2\mathcal{F}(\kappa) \leq M$, then we have*

$$
\begin{aligned}
&\frac{mq^4(\rho-\sigma)^2}{2[2]_q[3]_q} \\
&\leq \frac{(1-q)\mathcal{F}(\sigma) + q\mathcal{F}(\rho) + \mathcal{F}(q\sigma + (1-q)\rho)}{2} - \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho} \mathcal{F}(\kappa)\ {}^{\rho}d_q^T\kappa \\
&\leq \frac{Mq^4(\rho-\sigma)^2}{2[2]_q[3]_q}
\end{aligned}
\tag{26}
$$

for $0 < q < 1$.

Proof. Since $m \leq {}^{\rho}D_q^2\mathcal{F}(\kappa) \leq M$, we get

$$m(\kappa-\sigma)(\rho-\kappa) \leq (\kappa-\sigma)(\rho-\kappa)\ {}^{\rho}D_q^2\mathcal{F}(\kappa) \leq M(\kappa-\sigma)(\rho-\kappa) \tag{27}$$

for $\forall\kappa \in [\sigma,\rho]$. Integrating (27) on $[\sigma,\rho]$ in the sense of ${}^{\rho}T_q$-integral, we have

$$\begin{aligned}
&\frac{mq^3}{[2]_q(\rho-\sigma)}\int_{\sigma}^{\rho}(\kappa-\sigma)(\rho-\kappa)\ {}^{\rho}d_q^T\kappa \\
&\leq \frac{q^3}{[2]_q(\rho-\sigma)}\int_{\sigma}^{\rho}(\kappa-\sigma)(\rho-\kappa)\ {}^{\rho}D_q^2\mathcal{F}(\kappa)\ {}^{\rho}d_q^T\kappa \\
&\leq \frac{Mq^3}{[2]_q(\rho-\sigma)}\int_{\sigma}^{\rho}(\kappa-\sigma)(\rho-\kappa)\ {}^{\rho}d_q^T\kappa
\end{aligned}$$

and

$$\begin{aligned}
\int_{\sigma}^{\rho}(\kappa-\sigma)(\rho-\kappa)\ {}^{\rho}d_q^T\kappa &= \frac{(1-q)(\rho-\sigma)^3(1+q)}{2q}\sum_{n=0}^{\infty}q^{2n}(1-q^n) \\
&= \frac{(\rho-\sigma)^3 q}{2[3]_q}.
\end{aligned} \tag{28}$$

By using the equality (28) and using the Lemma 4.1, the inequality (26) is obtained. The proof is completed. □

Lemma 4.3. *Let $\mathcal{F} : [\sigma,\rho] \to \mathbb{R}$ be twice q-differentiable and let ${}_{\sigma}D_q^2\mathcal{F}$ be integrable on $[\sigma,\rho]$, then we have*

$$\begin{aligned}
&\frac{q^3}{[2]_q(\rho-\sigma)}\int_{\sigma}^{\rho}(\kappa-\sigma)(\rho-\kappa)\ {}_{\sigma}D_q^2\mathcal{F}(\kappa)\ {}_{\sigma}d_q^T\kappa \\
&= \frac{(1-q)\mathcal{F}(\rho)+q\mathcal{F}(\sigma)+\mathcal{F}(q\sigma+(1-q)\rho)}{2} - \frac{1}{\rho-\sigma}\int_{\sigma}^{\rho}\mathcal{F}(\kappa)\ {}_{\sigma}d_q^T\kappa
\end{aligned}$$

for $0 < q < 1$.

Proof. By using Definition 2.1, ${}_{\sigma}D_q^2\mathcal{F}(\kappa)$ is obtained as

$${}_{\sigma}D_q^2\mathcal{F}(\kappa) = \frac{q\mathcal{F}(\kappa)-(1+q)\mathcal{F}q\kappa+(1-q)\sigma)+\mathcal{F}\left(q^2\kappa+\left(1-q^2\right)\sigma\right)}{q(1-q)^2(\kappa-\sigma)^2}. \tag{29}$$

By equality (29), we get

$$\begin{aligned}
&\int_{\sigma}^{\rho}(\kappa-\sigma)(\rho-\kappa)\ {}_{\sigma}D_q^2\mathcal{F}(\kappa)\ {}_{\sigma}d_q^T\kappa \\
&= \frac{1}{q(1-q)^2}\left[q\int_{\sigma}^{\rho}\frac{\rho-\kappa}{\kappa-\sigma}\mathcal{F}(\kappa)\ {}_{\sigma}d_q^T\kappa-(1+q)\int_{\sigma}^{\rho}\frac{\rho-\kappa}{\kappa-\sigma}\mathcal{F}(q\kappa+(1-q)\sigma)\ {}_{\sigma}d_q^T\kappa\ {}_{\sigma}d_q^T\kappa\right.
\end{aligned}$$

$$+\int_{\sigma}^{\rho}\frac{\rho-\kappa}{\kappa-\sigma}\mathcal{F}\left(q^{2}\kappa+\left(1-q^{2}\right)\sigma\right)\ {}_{\sigma}d_{q}^{T}\kappa\Bigg].$$

Using Definition 2.5, we obtain

$$\begin{aligned}
&\int_{\sigma}^{\rho}(\kappa-\sigma)(\rho-\kappa)\ {}_{\sigma}D_{q}^{2}\mathcal{F}(\kappa)\ {}_{\sigma}d_{q}^{T}\kappa\\
&=\frac{(\rho-\sigma)(1+q)}{2q^{2}(1-q)}\left[q\sum_{n=0}^{\infty}(1-q^{n})\mathcal{F}(q^{n}\rho+(1-q^{n})\sigma)\right.\\
&\quad-(1+q)\sum_{n=0}^{\infty}(1-q^{n})\mathcal{F}(q^{n}\rho+(1-q^{n})\sigma)\\
&\quad\left.+\sum_{n=0}^{\infty}(1-q^{n})\mathcal{F}\left(q^{n+2}\rho+\left(1-q^{n+2}\right)\sigma\right)\right].
\end{aligned}$$

By using properties of series, we can write

$$\begin{aligned}
&\int_{\sigma}^{\rho}(\kappa-\sigma)(\rho-\kappa)\ {}_{\sigma}D_{q}^{2}\mathcal{F}(\kappa)\ {}_{\sigma}d_{q}^{T}\kappa\\
&=\frac{(\rho-\sigma)(1+q)}{2q^{2}(1-q)}\\
&\quad\times\left[q\sum_{n=0}^{\infty}\mathcal{F}(q^{n}\rho+(1-q^{n})\sigma)-(1+q)\sum_{n=0}^{\infty}\mathcal{F}\left(q^{n+1}\rho+\left(1-q^{n+1}\right)\sigma\right)\right.\\
&\quad+\sum_{n=0}^{\infty}\mathcal{F}\left(q^{n+2}\rho+\left(1-q^{n+2}\right)\sigma\right)-q\sum_{n=0}^{\infty}q^{n}\mathcal{F}(q^{n}\rho+(1-q^{n})\sigma)\\
&\quad\left.+(1+q)\sum_{n=0}^{\infty}q^{n}\mathcal{F}\left(q^{n+1}\rho+\left(1-q^{n+1}\right)\sigma\right)-\sum_{n=0}^{\infty}q^{n}\mathcal{F}\left(q^{n+2}\rho+\left(1-q^{n+2}\right)\sigma\right)\right]\\
&=\frac{(\rho-\sigma)(1+q)}{2q^{2}(1-q)}\left[q\sum_{n=0}^{\infty}\left[\mathcal{F}(q^{n}\rho+(1-q^{n})\sigma)-\mathcal{F}\left(q^{n+1}\rho+\left(1-q^{n+1}\right)\sigma\right)\right]\right.\\
&\quad+\sum_{n=0}^{\infty}\left[\mathcal{F}\left(q^{n+2}\rho+\left(1-q^{n+2}\right)\sigma\right)-\mathcal{F}\left(q^{n+1}\rho+\left(1-q^{n+1}\right)\sigma\right)\right]\\
&\quad-q\sum_{n=0}^{\infty}q^{n}\mathcal{F}(q^{n}\rho+(1-q^{n})\sigma)+\frac{1+q}{q}\left(\sum_{n=0}^{\infty}q^{n}\mathcal{F}(q^{n}\rho+(1-q^{n})\sigma)-\mathcal{F}(\rho)\right)\\
&\quad\left.-\frac{1}{q^{2}}\left(\sum_{n=0}^{\infty}q^{n}\mathcal{F}(q^{n}\rho+(1-q^{n})\sigma)-\mathcal{F}(\rho)-q\mathcal{F}(q\rho+(1-q)\sigma)\right)\right]\\
&=\frac{(\rho-\sigma)(1+q)}{2q^{2}(1-q)}\left[q\left[\mathcal{F}(\rho)-\mathcal{F}(\sigma)\right]+\mathcal{F}(\sigma)-\mathcal{F}(q\rho+(1-q)\sigma)\right.\\
&\quad+\left(-q+\frac{1+q}{q}-\frac{1}{q^{2}}\right)\sum_{n=0}^{\infty}q^{n}\mathcal{F}(q^{n}\rho+(1-q^{n})\sigma)\\
&\quad\left.+\left(-\frac{1+q}{q}+\frac{1}{q^{2}}\right)\mathcal{F}(\sigma)+\frac{1}{q}\mathcal{F}(q\rho+(1-q)\sigma)\right]
\end{aligned}$$

$$= \frac{(\rho-\sigma)(1+q)(1-q)}{2q^3}\mathcal{F}(\rho) + \frac{(\rho-\sigma)(1+q)}{2q^2}\mathcal{F}(\sigma)$$
$$+ \frac{(\rho-\sigma)(1+q)}{2q^3}\mathcal{F}(q\rho+(1-q)\sigma) - \frac{1+q}{q^3}\int_\sigma^\rho \mathcal{F}(\kappa)\ {}_\sigma d_q^T\kappa.$$

□

Theorem 4.4. *Let $\mathcal{F}:[\sigma,\rho]\to\mathbb{R}$ be twice q-differentiable. If ${}^\rho D_q^2\mathcal{F}$ is integrable on $[\sigma,\rho]$ and $m \le {}_\sigma D_q^2\mathcal{F}(\kappa) \le M$, then*

$$\frac{mq^4(\rho-\sigma)^2}{2[2]_q[3]_q} \tag{30}$$
$$\le \frac{(1-q)\mathcal{F}(\rho)+q\mathcal{F}(\sigma)+\mathcal{F}(q\rho+(1-q)\sigma)}{2} - \frac{1}{\rho-\sigma}\int_\sigma^\rho \mathcal{F}(\kappa)\ {}_\sigma d_q^T\kappa$$
$$\le \frac{Mq^4(\rho-\sigma)^2}{2[2]_q[3]_q}.$$

Proof. Since $m \le {}_\sigma D_q^2\mathcal{F}(\kappa) \le M$, we get

$$(\kappa-\sigma)(\rho-\kappa)m \le (\kappa-\sigma)(\rho-\kappa)\ {}_\sigma D_q^2\mathcal{F}(\kappa) \le (\kappa-\sigma)(\rho-\kappa)M \tag{31}$$

for $\forall\kappa\in[\sigma,\rho]$. Integrating (31) on $[\sigma,\rho]$ in the sense of ${}_\sigma T_q$-integral, we have

$$\frac{mq^3}{[2]_q(\rho-\sigma)}\int_\sigma^\rho (\kappa-\sigma)(\rho-\kappa)\ {}_\sigma d_q^T\kappa$$
$$\le \frac{q^3}{[2]_q(\rho-\sigma)}\int_\sigma^\rho (\kappa-\sigma)(\rho-\kappa)\ {}_\sigma D_q^2\mathcal{F}(\kappa)\ {}_\sigma d_q^T\kappa$$
$$\le \frac{Mq^3}{[2]_q(\rho-\sigma)}\int_\sigma^\rho (\kappa-\sigma)(\rho-\kappa)\ {}_\sigma d_q^T\kappa$$

and

$$\int_\sigma^\rho (\kappa-\sigma)(\rho-\kappa)\ {}_\sigma d_q^T\kappa = \frac{(1-q)(\rho-\sigma)^3(1+q)}{2q}\sum_{n=0}^{\infty} q^{2n}(1-q^n) \tag{32}$$
$$= \frac{(\rho-\sigma)^3 q}{2[3]_q}.$$

By using the equality (32) and using the Lemma 4.3, the inequality (30) is obtained. This is the end of proof of Theorem 4.4. □

5. Quantum Fejer-type Inequalities

In this section, we establish some weighted trapezoid and midpoint type inequalities for generalized quantum integrals by using the functions whose second derivatives are bounded.

Theorem 5.1. *Let $\mathcal{F} : [\sigma, \rho] \to \mathbb{R}$ be a twice differentiable function such that there exist real constants m and M so that $m \leq \mathcal{F}'' \leq M$ and also $\mathcal{G} : [\sigma, \rho] \to \mathbb{R}$ is nonnegative, ${}^{\rho}T_q$-integrable function. Then we have the following inequality:*

$$\frac{m}{2}\int_{\sigma}^{\rho}(\kappa-\sigma)(\rho-\kappa)\mathcal{G}(\kappa)\,{}^{\rho}d_q^T\kappa$$

$$\leq \frac{\mathcal{F}(\sigma)}{(\rho-\sigma)}\int_{\sigma}^{\rho}(\rho-\kappa)\mathcal{G}(\kappa)\,{}^{\rho}d_q^T\kappa + \frac{\mathcal{F}(\rho)}{(\rho-\sigma)}\int_{\sigma}^{\rho}(\kappa-\sigma)\mathcal{G}(\kappa)\,{}^{\rho}d_q^T\kappa - \int_{\sigma}^{\rho}\mathcal{F}(\kappa)\mathcal{G}(\kappa)\,{}^{\rho}d_q^T\kappa$$

$$\leq \frac{M}{2}\int_{\sigma}^{\rho}(\kappa-\sigma)(\rho-\kappa)\mathcal{G}(\kappa)\,{}^{\rho}d_q^T\kappa$$

for $0 < q < 1$.

Proof. Multiplying both sides of the inequality (5) by $\mathcal{G}(\gamma\rho + (1-\gamma)\sigma)$ and then integrating the inequality with respect to γ over $[0, 1]$ as ${}^{\rho}T_q$-integral. We obtain

$$\frac{m(\rho-\sigma)^2}{2}\int_0^1 \gamma(1-\gamma)\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\,{}^{1}d_q^T\gamma \tag{33}$$

$$\leq \mathcal{F}(\sigma)\int_0^1(1-\gamma)\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\,{}^{1}d_q^T\gamma$$
$$+\mathcal{F}(\rho)\int_0^1\gamma\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\,{}^{1}d_q^T\gamma$$
$$-\int_0^1\mathcal{F}(\gamma\rho+(1-\gamma)\sigma)\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\,{}^{1}d_q^T\gamma$$

$$\leq \frac{M(\rho-\sigma)^2}{2}\int_0^1\gamma(1-\gamma)\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\,{}^{1}d_q^T\gamma.$$

Calculating the integrals in the inequality (33), we have

$$\int_0^1\gamma(1-\gamma)\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\,{}^{1}d_q^T\gamma = \frac{1}{(\rho-\sigma)^3}\int_{\sigma}^{\rho}(\kappa-\sigma)(\rho-\kappa)\mathcal{G}(\kappa)\,{}^{\rho}d_q^T\kappa, \tag{34}$$

$$\int_0^1(1-\gamma)\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\,{}^{1}d_q^T\gamma = \frac{1}{(\rho-\sigma)^2}\int_{\sigma}^{\rho}(\rho-\kappa)\mathcal{G}(\kappa)\,{}^{\rho}d_q^T\kappa, \tag{35}$$

$$\int_0^1 \gamma \mathcal{G}(\gamma\rho + (1-\gamma)\sigma)\ {}^1d_q^T\gamma = \frac{1}{(\rho-\sigma)^2}\int_\sigma^\rho (\kappa-\sigma)\mathcal{G}(\kappa)\ {}^\rho d_q^T\kappa \tag{36}$$

and

$$\int_0^1 \mathcal{F}(\gamma\rho + (1-\gamma)\sigma)\mathcal{G}(\gamma\rho + (1-\gamma)\sigma)^1 d_q^T\gamma = \frac{1}{(\rho-\sigma)}\int_\sigma^\rho \mathcal{F}(\kappa)\mathcal{G}(\kappa)\ {}^\rho d_q^T\kappa. \tag{37}$$

Substituting from (34) to (37) into the inequality (33) and then multiplying both sides of the resulting inequality by $(\rho-\sigma)$, we obtain desired inequality. □

Theorem 5.2. *Let $\mathcal{F} : [\sigma,\rho] \to \mathbb{R}$ be a twice differentiable function such that there exist real constants m and M so that $m \leq \mathcal{F}'' \leq M$ and also $\mathcal{G} : [\sigma,\rho] \to \mathbb{R}$ is non-negative, ${}_\sigma T_q$-integrable function. Then we have the following inequalities:*

$$\frac{m}{2}\int_\sigma^\rho (\kappa-\sigma)(\rho-\kappa)\mathcal{G}(\kappa)\ {}_\sigma d_q^T\kappa$$

$$\leq \frac{\mathcal{F}(\sigma)}{(\rho-\sigma)}\int_\sigma^\rho (\rho-\kappa)\mathcal{G}(\kappa)\ {}_\sigma d_q^T\kappa + \frac{\mathcal{F}(\rho)}{(\rho-\sigma)}\int_\sigma^\rho (\kappa-\sigma)\mathcal{G}(\kappa)\ {}_\sigma d_q^T\kappa - \int_\sigma^\rho \mathcal{F}(\kappa)\mathcal{G}(\kappa)\ {}_\sigma d_q^T\kappa$$

$$\leq \frac{M}{2}\int_\sigma^\rho (\kappa-\sigma)(\rho-\kappa)\mathcal{G}(\kappa)\ {}_\sigma d_q^T\kappa$$

for $0 < q < 1$.

Proof. Multiplying both sides of the inequality (5) by $\mathcal{G}(\gamma\rho + (1-\gamma)\sigma)$ and then integrating the inequality we obtain with respect to γ over $[0,1]$ as ${}_\sigma T_q$-integral, we have

$$\frac{m(\rho-\sigma)^2}{2}\int_0^1 \gamma(1-\gamma)\mathcal{G}(\gamma\rho + (1-\gamma)\sigma)\ {}_0d_q^T\gamma \tag{38}$$

$$\begin{aligned}&\leq \mathcal{F}(\sigma)\int_0^1 (1-\gamma)\mathcal{G}(\gamma\rho + (1-\gamma)\sigma)\ {}_0d_q^T\gamma \\ &\quad + \mathcal{F}(\rho)\int_0^1 \gamma\mathcal{G}(\gamma\rho + (1-\gamma)\sigma)\ {}_0d_q^T\gamma \\ &\quad - \int_0^1 \mathcal{F}(\gamma\rho + (1-\gamma)\sigma)\mathcal{G}(\gamma\rho + (1-\gamma)\sigma)\ {}_0d_q^T\gamma\end{aligned}$$

$$\leq \frac{M(\rho-\sigma)^2}{2}\int_0^1 \gamma(1-\gamma)\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\ {}_0d_q^T\gamma.$$

By the definition of ${}_\sigma T_q$-integral, there are following equalities:

$$\int_0^1 \gamma(1-\gamma)\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\ {}_0d_q^T\gamma = \frac{1}{(\rho-\sigma)^3}\int_\sigma^\rho (\kappa-\sigma)(\rho-\kappa)\mathcal{G}(\kappa)\ {}_\sigma d_q^T\kappa, \tag{39}$$

$$\int_0^1 (1-\gamma)\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\ {}_0d_q^T\gamma = \frac{1}{(\rho-\sigma)^2}\int_\sigma^\rho (\rho-\kappa)\mathcal{G}(\kappa)\ {}_\sigma d_q^T\kappa, \tag{40}$$

$$\int_0^1 \gamma\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\ {}_0d_q^T\gamma = \frac{1}{(\rho-\sigma)^2}\int_\sigma^\rho (\kappa-\sigma)\mathcal{G}(\kappa)\ {}_\sigma d_q^T\kappa \tag{41}$$

and

$$\int_0^1 \mathcal{F}(\gamma\rho+(1-\gamma)\sigma)\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\ {}_0d_q^T\gamma = \frac{1}{(\rho-\sigma)}\int_\sigma^\rho \mathcal{F}(\kappa)\mathcal{G}(\kappa)\ {}_\sigma d_q^T\kappa. \tag{42}$$

When we put the statements (39)-(42) into the inequality (38), then the hypothesis of the theorem is obtained. □

Theorem 5.3. *Let $\mathcal{F}:[\sigma,\rho]\to\mathbb{R}$ be a twice differentiable function such that there exist real constants m and M so that $m\leq\mathcal{F}''\leq M$ and also $\mathcal{G}:[\sigma,\rho]\to\mathbb{R}$ is nonnegative, ${}^\rho T_q$-integrable and symmetric about $\kappa=\frac{\sigma+\rho}{2}$ (i.e. $\mathcal{G}(\kappa)=\mathcal{G}(\sigma+\rho-\kappa)$). Then, we have the following inequalities:*

$$\frac{m}{8(\rho-\sigma)}\int_\sigma^\rho (\sigma+\rho-2\kappa)^2\mathcal{G}(\kappa)\ {}^\rho d_q^T\kappa$$

$$\leq \frac{1}{(\rho-\sigma)}\left[\int_\sigma^\rho \mathcal{F}(\kappa)\mathcal{G}(\kappa)\ {}_\sigma d_q^T\kappa + \int_\sigma^\rho \mathcal{F}(\kappa)\mathcal{G}(\kappa)\ {}^\rho d_q^T\kappa\right] - \mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\int_\sigma^\rho \mathcal{G}(\kappa)\ {}^\rho d_q^T\kappa$$

$$\leq \frac{M}{8(\rho-\sigma)}\int_\sigma^\rho (\sigma+\rho-2\kappa)^2\mathcal{G}(\kappa)\ {}^\rho d_q^T\kappa$$

for $0<q<1$.

Proof. Multiplying both sides of the inequality (6) by $\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)$ and then integrating the inequality with respect to γ over $[0,1]$ as ${}^\rho T_q$-integral. We obtain

$$m\frac{(\rho-\sigma)^2}{8}\int_0^1 (1-2\gamma)^2\ \mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\ {}^1d_q^T\gamma \tag{43}$$

$$\leq \frac{1}{2}\left[\int_0^1 \mathcal{F}(\gamma\sigma+(1-\gamma)\rho)\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\ {}^1d_q^T\gamma + \int_0^1 \mathcal{F}(\gamma\rho+(1-\gamma)\sigma)\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\ {}^1d_q^T\gamma\right] - \mathcal{F}(\frac{\sigma+\rho}{2})\int_0^1 \mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\ {}^1d_q^T\gamma$$

$$\leq M\frac{(\rho-\sigma)^2}{8}\int_0^1 (1-2\gamma)^2\ \mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\ {}^1d_q^T\gamma.$$

When we calculate the integrals in the inequality (43) using the definition of ${}^\rho T_q$-integral, we see the following equalities:

$$\int_0^1 (1-2\gamma)^2\ \mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\ {}^1d_q^T\gamma = \frac{1}{(\rho-\sigma)^3}\int_\sigma^\rho (\sigma+\rho-2\kappa)^2\mathcal{G}(\kappa)\ {}^\rho d_q^T\kappa, \tag{44}$$

$$\int_0^1 \mathcal{F}(\gamma\sigma+(1-\gamma)\rho)\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\ {}^1d_q^T\gamma = \frac{1}{\rho-\sigma}\int_\sigma^\rho \mathcal{F}(\kappa)\mathcal{G}(\kappa)\ {}_\sigma d_q^T\kappa \tag{45}$$

and

$$\int_0^1 \mathcal{F}(\gamma\rho+(1-\gamma)\sigma)\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\ {}^1d_q^T\gamma = \frac{1}{\rho-\sigma}\int_\sigma^\rho \mathcal{F}(\kappa)\mathcal{G}(\kappa)\ {}^\rho d_q^T\kappa. \tag{46}$$

Writing (44)-(46) into the inequality (43), we obtain desired inequality. □

Theorem 5.4. *Let $\mathcal{F} : [\sigma,\rho] \to \mathbb{R}$ be a twice differentiable function such that there exist real constants m and M so that $m \leq \mathcal{F}'' \leq M$ and also $\mathcal{G} : [\sigma,\rho] \to \mathbb{R}$ is non-negative, ${}_\sigma T_q$-integrable and symmetric about $\kappa = \frac{\sigma+\rho}{2}$ (i.e. $\mathcal{G}(\kappa) = \mathcal{G}(\sigma+\rho-\kappa)$). Then we have the following inequality:*

$$\frac{m}{8(\rho-\sigma)}\int_\sigma^\rho (\sigma+\rho-2\kappa)^2\mathcal{G}(\kappa)\ {}_\sigma d_q^T\kappa$$

$$\leq \frac{1}{(\rho-\sigma)}\left[\int_\sigma^\rho \mathcal{F}(\kappa)\mathcal{G}(\kappa)\ {}_\sigma d_q^T\kappa + \int_\sigma^\rho \mathcal{F}(\kappa)\mathcal{G}(\kappa)\ {}^\rho d_q^T\kappa\right] - \mathcal{F}\left(\frac{\sigma+\rho}{2}\right)\int_\sigma^\rho \mathcal{G}(\kappa)\ {}_\sigma d_q^T\kappa$$

$$\leq \frac{M}{8(\rho-\sigma)}\int_\sigma^\rho (\sigma+\rho-2\kappa)^2\mathcal{G}(\kappa)\ {}_\sigma d_q^T\kappa$$

for $0 < q < 1$.

Proof. We multiply the inequality (6) by $\mathcal{G}(\gamma\rho + (1-\gamma)\sigma)$ and then integrate the inequality with respect to γ over $[0,1]$ we obtain ${}_{\sigma}T_q$-integral. We have

$$m\frac{(\rho-\sigma)^2}{8}\int_0^1 (1-2\gamma)^2\,\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\;{}_0d_q^T\gamma \tag{47}$$

$$\le \frac{1}{2}\left[\int_0^1 \mathcal{F}(\gamma\sigma+(1-\gamma)\rho)\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\;{}_0d_q^T\gamma + \int_0^1 \mathcal{F}(\gamma\rho+(1-\gamma)\sigma)\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\;{}_0d_q^T\gamma\right] - \mathcal{F}(\frac{\sigma+\rho}{2})\int_0^1 \mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\;{}_0d_q^T\gamma$$

$$\le M\frac{(\rho-\sigma)^2}{8}\int_0^1 (1-2\gamma)^2\,\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\;{}_0d_q^T\gamma.$$

By calculating the integrals in the inequality (47) using the definition of ${}_{\sigma}T_q$-integral, we have

$$\int_0^1 (1-2\gamma)^2\,\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\;{}_0d_q^T\gamma = \frac{1}{(\rho-\sigma)^3}\int_\sigma^\rho (\sigma+\rho-2\kappa)^2\mathcal{G}(\kappa)\;{}_{\sigma}d_q^T\kappa, \tag{48}$$

$$\int_0^1 \mathcal{F}(\gamma\sigma+(1-\gamma)\rho)\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\;{}_0d_q^T\gamma = \frac{1}{\rho-\sigma}\int_\sigma^\rho \mathcal{F}(\kappa)\mathcal{G}(\kappa)\;{}^{\rho}d_q^T\kappa \tag{49}$$

and

$$\int_0^1 \mathcal{F}(\gamma\rho+(1-\gamma)\sigma)\mathcal{G}(\gamma\rho+(1-\gamma)\sigma)\;{}_0d_q^T\gamma = \frac{1}{\rho-\sigma}\int_\sigma^\rho \mathcal{F}(\kappa)\mathcal{G}(\kappa)\;{}_{\sigma}d_q^T\kappa. \tag{50}$$

Putting the statements (48)-(50) into the inequality (47), the proof is completed. □

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All participants have been informed about the purposes of the study and declare that there is no unethical behaviour. It has not been used elsewhere and has not been posted. Compliance with the publishing ethics rules of the universities has been approved.

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