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Numerical radius peak multilinear mappings on ℓ_1

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Abstract. For $n \geq 2$ and a Banach space E, $\mathcal{L}(^nE : E)$ denotes the space of all continuous n-linear mappings from E to itself. We let

$$\Pi(E) = \{[x^*, x_1, \ldots, x_n] : x^*(x_j) = \|x^*\| = \|x_j\| = 1 \text{ for } j = 1, \ldots, n\}.$$

An element $[x^*, x_1, \ldots, x_n] \in \Pi(E)$ is called a *numerical radius point* of $T \in \mathcal{L}(^nE : E)$ if

$$|x^*(T(x_1, \ldots, x_n))| = v(T),$$

where the numerical radius $v(T) = \sup_{[y^*, y_1, \ldots, y_n] \in \Pi(E)} \left| y^*\Big(T(y_1, \ldots, y_n)\Big)\right|$. For $T \in \mathcal{L}(^nE : E)$, we define

$$\text{Nradius}(T) = \{[x^*, x_1, \ldots, x_n] \in \Pi(E) : [x^*, x_1, \ldots, x_n] \text{ is a numerical radius point of } T\}.$$

Nradius(T) is called *the set of all numerical radius points* for T. T is called *numerical radius peak n-linear mapping* if

$$\text{Nradius}(T) = \{\pm[x^*, x_1, \ldots, x_n]\}.$$

In this paper we investigate Nradius(T) for every $T \in \mathcal{L}(^n\ell_1 : \ell_1)$ and characterize all numerical radius peak multilinear mappings in $\mathcal{L}(^n\ell_1 : \ell_1)$, where ℓ_1 is a real or complex space.

1. Introduction

Let us sketch a brief history of norm or numerical radius attaining multilinear forms and polynomials on Banach spaces. In 1961 Bishop and Phelps [2] initiated and showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, especially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [1], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [3] showed that the Radon-Nikodym property is also sufficient for the denseness of norm attaining polynomials. Jiménez-Sevilla and Payá [5] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz

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sequence spaces. Recently, Kim [10] investigated the polynomial numerical index $n^{(k)}(\ell_p)$, the symmetric multilinear numerical index $n_s^{(k)}(\ell_p)$, and the multilinear numerical index $n_m^{(k)}(\ell_p)$ of ℓ_p spaces for $1 \leq p \leq \infty$. He proved that $n_s^{(k)}(\ell_1) = n_m^{(k)}(\ell_1) = 1$ for every $k \geq 2$. He also showed that for $1 < p < \infty$, $n_I^{(k)}(\ell_p^{j+1}) \leq n_I^{(k)}(\ell_p^j)$ for every $j \in \mathbb{N}$ and $n_I^{(k)}(\ell_p) = \lim_{j\to\infty} n_I^{(k)}(\ell_p^j)$ for every $I = s, m$, where $\ell_p^j = (\mathbb{C}^j, \|\cdot\|_p)$ or $(\mathbb{R}^j, \|\cdot\|_p)$.

Let $n \in \mathbb{N}, n \geq 2$. We write S_E for the unit sphere of a Banach space E. We denote by $\mathcal{L}(^nE : E)$ the Banach space of all continuous n-linear mappings from E into itself endowed with the norm $\|T\| = \sup_{(x_1,\cdots,x_n)\in S_E\times\cdots\times S_E} \|T(x_1, \cdots, x_n)\|$. $\mathcal{L}_s(^nE : E)$ denotes the closed subspace of all continuous symmetric n-linear mappings on E. We let

$$\Pi(E) = \Big\{[x^*, x_1, \ldots, x_n] : x^*(x_j) = \|x^*\| = \|x_j\| = 1 \text{ for } j = 1, \ldots, n\Big\}.$$

An element $[x^*, x_1, \ldots, x_n] \in \Pi(E)$ is called a *numerical radius point* of $T \in \mathcal{L}(^nE : E)$ if $|x^*(T(x_1, \ldots, x_n))| = v(T)$, where the numerical radius

$$v(T) = \sup_{[y^*,y_1,\ldots,y_n]\in\Pi(E)} \Big|y^*\Big(T(y_1, \ldots, y_n)\Big)\Big|.$$

For $T \in \mathcal{L}(^nE : E)$, we define in [13]

$$\begin{aligned}\text{Nradius}(T) \;=\; & \Big\{[x^*, x_1, \ldots, x_n] \in \Pi(E) : [x^*, x_1, \ldots, x_n] \\ & \text{is a numerical radius point of } T\Big\}.\end{aligned}$$

Nradius(T) is called the *numerical radius points set* of T. Notice that $[x^*, x_1, \ldots, x_n] \in$ Nradius(T) if and only if $[-x^*, -x_1, \ldots, -x_n] \in$ Nradius(T). T is called *numerical radius peak n-linear mapping* if

$$\text{Nradius}(T) = \{\pm[x^*, x_1, \ldots, x_n]\}.$$

An element $(x_1, \ldots, x_n) \in E^n$ is called a *norming point* of $T \in \mathcal{L}(^nE : E)$ if $\|x_1\| = \cdots = \|x_n\| = 1$ and $\|T(x_1, \ldots, x_n)\| = \|L\|$. We define in [6]

$$\text{Norm}(T) = \Big\{(x_1, \ldots, x_n) \in E^n : (x_1, \ldots, x_n) \text{ is a norming point of } T\Big\}.$$

Norm(T) is called the *norming set* of T.

A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form L on the product $E \times \cdots \times E$ such that $P(x) = L(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. An element $[x^*, x] \in \pi(E)$ is called a *numerical radius point* of $P \in \mathcal{P}(^nE : E)$ if $|x^*(P(x))| = v(P)$, where the numerical radius

$$v(P) = \sup_{[y^*,y]\in\Pi(E)} \Big|y^*(P(y))\Big|.$$

We define in [13]

$$\text{Nradius}(P) = \Big\{[x^*, x] \in \Pi(E) : [x^*, x] \text{ is a numerical radius point of } P\Big\}.$$

Nradius(P) is called the *numerical radius points set* of P. Notice that $[x^*, x] \in$ Nradius(P) if and only if $[-x^*, -x] \in$ Nradius(P).

An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^nE)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. For $P \in \mathcal{P}(^nE)$, we define in [7]

$$\text{Norm}(P) = \Big\{x \in E : x \text{ is a norming point of } P\Big\}.$$

Norm(P) is called the *norming set* of P.

For $m \in \mathbb{N}$, let $\ell_1^m := \mathbb{R}^m$ with the the ℓ_1-norm and $\ell_\infty^2 = \mathbb{R}^2$ with the supremum norm. Notice that if $E = \ell_1^m$ or ℓ_∞^2 and $T \in \mathcal{L}(^nE)$, Norm$(T) \neq \emptyset$ since S_E is compact. Kim [7] classified Norm(P) for every $P \in \mathcal{P}(^2\ell_\infty^2)$. Kim [6, 8, 9, 12] classified Norm(T) for every $T \in \mathcal{L}_s(^2\ell_\infty^2), \mathcal{L}(^2\ell_\infty^2), \mathcal{L}(^2\ell_1^2), \mathcal{L}_s(^2\ell_1^3)$ or $\mathcal{L}_s(^3\ell_1^2)$. Kim [11] classified Nradius(T) for every $T \in \mathcal{L}(^2\ell_\infty^2 : \ell_\infty^2)$ in connection with the set of the norm attaining points of T. He also characterized all numerical radius peak mappings in $\mathcal{L}(^m\ell_\infty^n : \ell_\infty^n)$ for $n, m \geq 2$, where $\ell_\infty^n = \mathbb{R}^n$ with the supremum norm. Kim [13] presented explicit formulaes for the numerical radius of T for every $T \in \mathcal{L}(^nE : E)$ for $E = c_0$ or ℓ_∞. Using these formulaes he showed that there are no numerical radius peak mappings of $\mathcal{L}(^nc_0 : c_0)$. Recently, Kim [14] also classified Norm(T) for every $T \in \mathcal{L}(^2\mathbb{R}^2_{h(w)})$, where $\mathbb{R}^2_{h(w)}$ denotes the plane with the hexagonal norm with weight $0 < w < 1$ $\|(x, y)\|_{h(w)} = \max\left\{|y|, |x| + (1 - w)|y|\right\}$.

If $T \in \mathcal{L}(^nE)$ or $\mathcal{L}(^nE : E)$ and Norm$(T) \neq \emptyset$, T is called a *norm attaining* and if $T \in \mathcal{L}(^nE : E)$ and Nradius$(T) \neq \emptyset$, T is called a *numerical radius attaining*. Similarly, if $P \in \mathcal{P}(^nE)$ or $\mathcal{P}(^nE : E)$ and Norm$(P) \neq \emptyset$, P is called a *norm attaining* and if $P \in \mathcal{P}(^nE)$ or $\mathcal{P}(^nE : E)$ and Nradius$(P) \neq \emptyset$, P is called a *numerical radius attaining*. (See [3])

For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [4].

It seems to be natural and interesting to study about Nradius(T) for $T \in \mathcal{L}(^nE : E)$ and numerical radius peak multilinear mappings on E. Kim [13] showed that there are no numerical radius peak n-linear mappings on c_0.

In this paper, we investigate Nradius(T) for every $T \in \mathcal{L}(^n\ell_1 : \ell_1)$ and characterize all numerical radius peak multilinear mappings in $\mathcal{L}(^n\ell_1 : \ell_1)$, where ℓ_1 is a real or complex space.

2. Results

Let $\{e_n\}_{n\in\mathbb{N}}$ be the canonical basis of real or complex space ℓ_1 and $\{e_n^*\}_{n\in\mathbb{N}}$ the biorthogonal functionals associated to $\{e_n\}_{n\in\mathbb{N}}$. In [10], some explicit formulae for the numerical radius and the norm of T for every $T \in \mathcal{L}(^n\ell_1 : \ell_1)$ was given.

Theorem 2.1 [10]. *Let $n \geq 2$. Let $T = \sum_{j\in\mathbb{N}} T_je_j \in \mathcal{L}(^n\ell_1 : \ell_1)$ be such that*

$$T_j\Big(\sum_{i\in\mathbb{N}} x_i^{(1)}e_i, \cdots, \sum_{i\in\mathbb{N}} x_i^{(n)}e_i\Big) = \sum_{i_1,\ldots,i_n\in\mathbb{N}} a^{(j)}_{i_1\cdots i_n}\, x^{(1)}_{i_1}\cdots x^{(n)}_{i_n} \in \mathcal{L}(^n\ell_1)$$

for some $a^{(j)}_{i_1\cdots i_n} \in \mathbb{R}$. Then

$$\sup\Big\{\sum_{j\in\mathbb{N}}\left|a^{(j)}_{i_1\cdots i_n}\right| : (i_1, \ldots, i_n) \in \mathbb{N}^n\Big\} = v(T) = \|T\|.$$

In [11] the author classified Nradius(T) for every $T \in \mathcal{L}(^2\ell_\infty^2 : \ell_\infty^2)$ in connection with the set of the norm attaining points of T. We can now describe Nradius(T) for a certain $T \in \mathcal{L}(^n\ell_1 : \ell_1)$.

Theorem 2.2. *Let $n \geq 2$. Let $T = \sum_{j\in\mathbb{N}} T_je_j \in \mathcal{L}(^n\ell_1 : \ell_1)$ be such that $T \neq 0$ and*

$$T_j\Big(\sum_{i\in\mathbb{N}} x_i^{(1)}e_i, \cdots, \sum_{i\in\mathbb{N}} x_i^{(n)}e_i\Big) = \sum_{i_1,\ldots,i_n\in\mathbb{N}} a^{(j)}_{i_1\cdots i_n}\, x^{(1)}_{i_1}\cdots x^{(n)}_{i_n} \in \mathcal{L}(^n\ell_1)$$

for some $a^{(j)}_{i_1\cdots i_n} \in \mathbb{R}$. If there is $(i_1', \ldots, i_n') \in \mathbb{N}^n$ such that

$$\sum_{j\in\mathbb{N}}\left|a^{(j)}_{i_1'\cdots i_n'}\right| = v(T) > \sum_{j\in\mathbb{N}}\left|a^{(j)}_{i_1\cdots i_n}\right|$$

for every $(i_1, \ldots, i_n) \in \mathbb{N}^n \backslash \{(i_1', \ldots, i_n')\}$*, then*

$$\begin{aligned} &\text{Nradius}(T) \\ = \quad &\Big\{ \pm \Big[\Big(\sum_{l\in A} sign\Big(a^{(l)}_{i_1'\cdots i_n'}\Big)e_l^* + \sum_{j\in B} \lambda_j e_j^* + \sum_{j\in\mathbb{N}\backslash(A\cup B)} sign\Big(a^{(j)}_{i_1'\cdots i_n'}\Big)e_j^* \Big), \\ &\overline{sign\Big(a^{(i_1')}_{i_1'\cdots i_n'}\Big)}e_{i_1}, \ \ldots, \ \overline{sign\Big(a^{(i_n')}_{i_1'\cdots i_n'}\Big)}e_{i_n} \Big] \Big\}, \end{aligned}$$

where $A = \{i_1', \ldots, i_n'\}$ *and* $B = \Big\{ j \in \mathbb{N}\backslash A : a^{(j)}_{i_1'\cdots i_n'} = 0 \Big\}$.

Proof. The inclusion ($\supseteq$) is obvious.

($\subseteq$). Let $[z^*, x_1, \ldots, x_n] \in \text{Nradius}(T)$. Write $z^* = \sum_{j\in\mathbb{N}} z_j e_j^*$ and $x_l = \sum_{j\in\mathbb{N}} x_j^{(l)} e_j$ for $l = 1, \ldots, n$.

Claim. $|x^{(1)}_{i_1}| \cdots |x^{(n)}_{i_n}| = 0$ for every $(i_1, \ldots, i_n) \in \mathbb{N}^n \backslash \{(i_1', \ldots, i_n')\}$.

Assume that $|x^{(1)}_{k_1}| \cdots |x^{(n)}_{k_n}| \neq 0$ for some $(k_1, \ldots, k_n) \in \mathbb{N}^n \backslash \{(i_1', \ldots, i_n')\}$.
It follows that

$$\begin{aligned} v(T) &= \Big| z^*(T(x_1, \ldots, x_n)) \Big| = \Big| \sum_{j\in\mathbb{N}} z_j T_j(x_1, \ldots, x_n) \Big| \\ &\leq \sum_{j\in\mathbb{N}} |z_j| \, |T_j(x_1, \ldots, x_n)| \\ &= \Big(\sum_{j\in\mathbb{N}} |z_j| \Big| a^{(j)}_{i_1'\cdots i_n'} \Big| \Big) |x^{(1)}_{i_1'}| \cdots |x^{(n)}_{i_n'}| + \Big(\sum_{j\in\mathbb{N}} |z_j| \Big| a^{(j)}_{k_1\cdots k_n} \Big| \Big) \ |x^{(1)}_{k_1}| \cdots |x^{(n)}_{k_n}| \\ &+ \sum_{(i_1,\ldots,i_n)\in\mathbb{N}^n\backslash\{(i_1',\ldots,i_n'),(k_1,\ldots,k_n)\}} \Big(\sum_{j\in\mathbb{N}} |z_j| \Big| a^{(j)}_{i_1\cdots i_n} \Big| \Big) |x^{(1)}_{i_1}| \cdots |x^{(n)}_{i_n}| \\ &< \Big(\sum_{j\in\mathbb{N}} \Big| a^{(j)}_{i_1'\cdots i_n'} \Big| \Big) |x^{(1)}_{i_1'}| \cdots |x^{(n)}_{i_n'}| + v(T) \, |x^{(1)}_{k_1}| \cdots |x^{(n)}_{k_n}| \\ &+ \sum_{(i_1,\ldots,i_n)\in\mathbb{N}^n\backslash\{(i_1',\ldots,i_n'),(k_1,\ldots,k_n)\}} \Big(\sum_{j\in\mathbb{N}} \Big| a^{(j)}_{i_1\cdots i_n} \Big| \Big) |x^{(1)}_{i_1}| \cdots |x^{(n)}_{i_n}| \\ &\leq v(T) \Big(\sum_{j\in\mathbb{N}} |x^{(1)}_j| \Big) \cdots \Big(\sum_{j\in\mathbb{N}} |x^{(n)}_j| \Big) = v(T), \end{aligned}$$

which is impossible. Hence, the claim holds. Therefore,

$$v(T) = \Big(\sum_{j\in\mathbb{N}} |z_j| \Big| a^{(j)}_{i_1'\cdots i_n'} \Big| \Big) |x^{(1)}_{i_1'}| \cdots |x^{(n)}_{i_n'}| = \Big(\sum_{j\in\mathbb{N}} \Big| a^{(j)}_{i_1'\cdots i_n'} \Big| \Big) |x^{(1)}_{i_1'}| \cdots |x^{(n)}_{i_n'}|.$$

Since $a^{(j)}_{i_1'\cdots i_n'} \neq 0$ for all $j \in \mathbb{N}\backslash(A\cup B)$, $|z_j| = 1$ for all $j \in \mathbb{N}\backslash(A\cup B)$ and $|x^{(l)}_{i_l'}| = 1$ for $l = 1, \ldots, n$. Hence, $x_l = \lambda_l e_{i_l'}$ and $z_{i_l'} = \overline{\lambda_{i_l'}}$ for $l = 1, \ldots, n$ and some $\lambda_l \in \mathbb{C}$ with $|\lambda_l| = 1$.

It follows that

$$\begin{aligned} v(T) &= \Big| z^*\Big(T(\lambda_1 e_{i_1'}, \ldots, \lambda_n e_{i_n'}) \Big) \Big| = \Big| z^*\Big(T(e_{i_1'}, \ldots, e_{i_n'}) \Big) \Big| \\ &= \Big| \sum_{j\in\mathbb{N}} z_j \ T_j(e_{i_1'}, \ldots, e_{i_n'}) \Big| \\ &\leq \sum_{j\in\mathbb{N}} |z_j| \Big| T_j(e_{i_1'}, \ldots, e_{i_n'}) \Big| = \sum_{j\in\mathbb{N}\backslash B} |z_j| \Big| a^{(j)}_{i_1'\cdots i_n'} \Big| \\ &= \sum_{j\in\mathbb{N}\backslash B} \Big| a^{(j)}_{i_1'\cdots i_n'} \Big| = v(T), \end{aligned}$$

which shows that

$$\sum_{j\in\mathbb{N}\backslash B}\left|a^{(j)}_{i'_1\cdots i'_n}\right| = v(T) = \Big|\sum_{j\in\mathbb{N}\backslash B} z_j\, T_j(e_{i'_1},\ldots,e_{i'_n})\Big| = \Big|\sum_{j\in\mathbb{N}\backslash B} z_j\, a^{(j)}_{i'_1\cdots i'_n}\Big|.$$

Hence, $z_j = \text{sign}(a^{(j)}_{i'_1\cdots i'_n})$ for every $j\in\mathbb{N}\backslash B$. Therefore,

$$[z^*,x_1,\ldots,x_n] = \pm\Big[\Big(\sum_{l\in A}\text{sign}\Big(a^{(l)}_{i'_1\cdots i'_n}\Big)e^*_l + \sum_{j\in B}\lambda_j e^*_j + \sum_{j\in\mathbb{N}\backslash(A\cup B)}\text{sign}\Big(a^{(j)}_{i'_1\cdots i'_n}\Big)e^*_j\Big),$$
$$\overline{\text{sign}\Big(a^{(i'_1)}_{i'_1\cdots i'_n}\Big)}e_{i_1},\ \ldots,\ \overline{\text{sign}\Big(a^{(i'_n)}_{i'_1\cdots i'_n}\Big)}e_{i_n}\Big]$$

for some $\lambda_j\in\mathbb{C}, |\lambda_j|\le 1$ $(j\in B)$. Therefore, the inclusion $(\subseteq)$ holds. □

The author [11] characterized all numerical radius peak mappings in $\mathcal{L}(^m\ell^n_\infty : \ell^n_\infty)$ for $n,m\ge 2$, where $\ell^n_\infty = \mathbb{R}^n$ with the supremum norm. The author [13] showed that there are no numerical radius peak mappings of $\mathcal{L}(^n c_0 : c_0)$. Using Theorem 2.2, we characterize all numerical radius peak n-linear mappings in $\mathcal{L}(^n\ell_1 : \ell_1)$.

Theorem 2.3. *Let $n\ge 2$. Let $T=\sum_{j\in\mathbb{N}} T_j e_j \in \mathcal{L}(^n\ell_1 : \ell_1)$ be such that $T\neq 0$ and*

$$T_j\Big(\sum_{i\in\mathbb{N}} x^{(1)}_i e_i,\cdots,\sum_{i\in\mathbb{N}} x^{(n)}_i e_i\Big) = \sum_{i_1,\ldots,i_n\in\mathbb{N}} a^{(j)}_{i_1\cdots i_n}\, x^{(1)}_{i_1}\cdots x^{(n)}_{i_n} \in \mathcal{L}(^n\ell_1)$$

for some $a^{(j)}_{i_1\cdots i_n}\in\mathbb{R}$. Then T is a numerical radius peak mapping if and only if there is $(i'_1,\ldots,i'_n)\in\mathbb{N}^n$ such that

$$v(T)=\sum_{j\in\mathbb{N}}\left|a^{(j)}_{i'_1\cdots i'_n}\right|,\ a^{(j)}_{i'_1\cdots i'_n}\neq 0 \text{ for all } j\in\mathbb{N}\backslash\{i'_1,\ldots,i'_n\},\ v(T)>\sum_{j\in\mathbb{N}}\left|a^{(j)}_{i_1\cdots i_n}\right|$$

for every $(i_1,\ldots,i_n)\in\mathbb{N}^n\backslash\{(i'_1,\ldots,i'_n)\}$.

Proof. $(\Rightarrow)$. Assume the contrary. We consider two cases.

Case 1. There are $i'_1,\ldots,i'_n,k'_1,\ldots,k'_n\in\mathbb{N}$ such that $(i'_1,\ldots,i'_n)\neq(k'_1,\ldots,k'_n)$ and

$$v(T)=\sum_{j\in\mathbb{N}}\left|a^{(j)}_{i'_1\cdots i'_n}\right| = \sum_{j\in\mathbb{N}}\left|a^{(j)}_{k'_1\cdots k'_n}\right|.$$

Let $A_1=\{i'_1,\ldots,i'_n\}$ and $A_2=\{k'_1,\ldots,k'_n\}$.

Notice that

$$\pm\Big[\Big(\sum_{l\in A_1}\text{sign}(a^{(l)}_{i'_1\cdots i'_n})e^*_l + \sum_{j\in\mathbb{N}\backslash A_1}\text{sign}(a^{(j)}_{i'_1\cdots i'_n})e^*_j\Big),\ \overline{\text{sign}(a^{(i'_1)}_{i'_1\cdots i'_n})}e_{i'_1},\ldots,\overline{\text{sign}(a^{(i'_n)}_{i'_1\cdots i'_n})}e_{i'_n}\Big],$$
$$\pm\Big[\Big(\sum_{l\in A_2}\text{sign}(a^{(l)}_{k'_1\cdots k'_n})e^*_l + \sum_{j\in\mathbb{N}\backslash A_2}\text{sign}(a^{(j)}_{k'_1\cdots k'_n})e^*_j\Big),\ \overline{\text{sign}(a^{(k'_1)}_{k'_1\cdots k'_n})}e_{k'_1},\ldots,\overline{\text{sign}(a^{(k'_n)}_{k'_1\cdots k'_n})}e_{k'_n}\Big]$$
$$\in\ \text{Nradius}(T).$$

Hence, T is not a numerical radius peak mapping. This is a contradiction.

Case 2. There is $j_0\in\mathbb{N}\backslash\{i'_1,\ldots,i'_n\}$ such that $v(T)=\sum_{j\in\mathbb{N}}\left|a^{(j)}_{i'_1\cdots i'_n}\right|$ and $a^{(j_0)}_{i'_1\cdots i'_n}=0$.

Notice that for every $\lambda \in \mathbb{C}, |\lambda| \leq 1$,

$$\pm\Big[\Big(\sum_{l\in A_1}\mathrm{sign}(a^{(l)}_{i'_1\cdots i'_n})e^*_l + \lambda e^*_{j_0} + \sum_{j\in\mathbb{N}\backslash(A_1\cup\{j_0\})}\mathrm{sign}(a^{(j)}_{i'_1\cdots i'_n})e^*_j\Big),$$
$$\overline{\mathrm{sign}(a^{(i'_1)}_{i'_1\cdots i'_n})}e_{i'_1},\ \ldots,\ \overline{\mathrm{sign}(a^{(i'_n)}_{i'_1\cdots i'_n})}e_{i'_n}\Big] \in \mathrm{Nradius}(T).$$

Hence, T is not a numerical radius peak mapping. This is a contradiction.

(⇐). Let $A = \{i'_1, \ldots, i'_n\}$. Then $B := \{j \in \mathbb{N}\backslash A : a^{(j)}_{i'_1\cdots i'_n} = 0\} = \emptyset$. By Theorem 2.2,

$$\begin{aligned} &\mathrm{Nradius}(T)\\ =\ &\Big\{\pm\Big[\Big(\sum_{l\in A}\mathrm{sign}\Big(a^{(l)}_{i'_1\cdots i'_n}\Big)e^*_l + \sum_{j\in\mathbb{N}\backslash(A\cup B)}\mathrm{sign}\Big(a^{(j)}_{i'_1\cdots i'_n}\Big)e^*_j\Big),\\ &\overline{\mathrm{sign}\Big(a^{(i'_1)}_{i'_1\cdots i'_n}\Big)}e_{i_1},\ \ldots,\ \overline{\mathrm{sign}\Big(a^{(i'_n)}_{i'_1\cdots i'_n}\Big)}e_{i_n}\Big]\Big\}.\end{aligned}$$

Hence, T is a numerical radius peak mapping. □

Corollary 2.4. *Let $n \geq 2$. Let $T = \sum_{j\in\mathbb{N}} T_j e_j \in \mathcal{L}_s(^n\ell_1 : \ell_1)$ be the same as in Theorem 2.3. Then T is a numerical radius peak mapping if and only if there is $i_0 \in \mathbb{N}$ such that*

$$v(T) = \sum_{j\in\mathbb{N}}\Big|a^{(j)}_{i_0\cdots i_0}\Big|,\ a^{(j)}_{i_0\cdots i_0} \neq 0 \textit{ for all } j \in \mathbb{N}\backslash\{i_0\},\ v(T) > \sum_{j\in\mathbb{N}}\Big|a^{(j)}_{i_1\cdots i_n}\Big|$$

for every $(i_1, \ldots, i_n) \in \mathbb{N}^n\backslash\{(i_0, \ldots, i_0)\}$.

Theorem 2.5. *Let $n \geq 2$. Let $T = \sum_{j\in\mathbb{N}} T_j e_j \in \mathcal{L}(^n\ell_1 : \ell_1)$ be the same as in Theorem 2.3. Then* Nradius$(T) \neq \emptyset$ *if and only if there is $(i'_1, \ldots, i'_n) \in \mathbb{N}^n$ such that*

$$v(T) = \sum_{j\in\mathbb{N}}\Big|a^{(j)}_{i'_1\cdots i'_n}\Big|.$$

Proof. (⇒). Assume the contrary. By Theorem 2.1,

$$v(T) > \sum_{j\in\mathbb{N}}\Big|a^{(j)}_{i_1\cdots i_n}\Big|$$

for every $(i_1, \ldots, i_n) \in \mathbb{N}^n$. Let $[z^*, x_1, \ldots, x_n] \in \mathrm{Nradius}(T)$. Write $z^* = \sum_{j\in\mathbb{N}} z_j e^*_j$ and $x_l = \sum_{j\in\mathbb{N}} x^{(l)}_j e_j$ for $l = 1, \ldots, n$.

Claim. $x^{(1)}_{\tilde{i}_1} \cdots x^{(n)}_{\tilde{i}_n} = 0$ for every $(\tilde{i}_1, \ldots, \tilde{i}_n) \in \mathbb{N}^n$.

Assume that $|x^{(1)}_{\tilde{i}_1}|\cdots|x^{(n)}_{\tilde{i}_n}| > 0$. It follows that

$$\begin{aligned}
v(T) &= \Big|z^*(T(x_1,\ldots,x_n))\Big| = \Big|\sum_{j\in\mathbb{N}} z_j T_j(x_1,\ldots,x_n)\Big| \\
&\le \sum_{j\in\mathbb{N}} |z_j|\,|T_j(x_1,\ldots,x_n)| \\
&= \Big(\sum_{j\in\mathbb{N}} |z_j|\,\Big|a^{(j)}_{\tilde{i}_1,\cdots,\tilde{i}_n}\Big|\Big)\,|x^{(1)}_{\tilde{i}_1}|\cdots|x^{(n)}_{\tilde{i}_n}| \\
&+ \sum_{(i_1,\ldots,i_n)\in\mathbb{N}^n\setminus\{(\tilde{i}_1,\ldots,\tilde{i}_n)\}} \Big(\sum_{j\in\mathbb{N}} |z_j|\,\Big|a^{(j)}_{i_1,\ldots,i_n}\Big|\Big)\,|x^{(1)}_{i_1}|\cdots|x^{(n)}_{i_n}| \\
&\le \Big(\sum_{j\in\mathbb{N}} \Big|a^{(j)}_{\tilde{i}_1,\cdots,\tilde{i}_n}\Big|\Big)\,|x^{(1)}_{\tilde{i}_1}|\cdots|x^{(n)}_{\tilde{i}_n}| \\
&+ \sum_{(i_1,\ldots,i_n)\in\mathbb{N}^n\setminus\{(\tilde{i}_1,\ldots,\tilde{i}_n)\}} \Big(\sum_{j\in\mathbb{N}} |z_j|\,\Big|a^{(j)}_{i_1,\ldots,i_n}\Big|\Big)\,|x^{(1)}_{i_1}|\cdots|x^{(n)}_{i_n}| \\
&< v(T)\,|x^{(1)}_{\tilde{i}_1}|\cdots|x^{(n)}_{\tilde{i}_n}| \\
&+ \sum_{(i_1,\ldots,i_n)\in\mathbb{N}^n\setminus\{(\tilde{i}_1,\ldots,\tilde{i}_n)\}} \Big(\sum_{j\in\mathbb{N}} |z_j|\,\Big|a^{(j)}_{i_1,\ldots,i_n}\Big|\Big)\,|x^{(1)}_{i_1}|\cdots|x^{(n)}_{i_n}| \\
&\le v(T)\Big(\sum_{j\in\mathbb{N}} |x^{(1)}_j|\Big)\cdots\Big(\sum_{j\in\mathbb{N}} |x^{(n)}_j|\Big) = v(T),
\end{aligned}$$

which is impossible. Hence, the claim holds. Hence, $v(T) = 0$. This is a contradiction because

$$0 = v(T) > \sum_{j\in\mathbb{N}} \Big|a^{(j)}_{i_1\cdots i_n}\Big|$$

for every $(i_1,\ldots,i_n)\in\mathbb{N}^n$.

($\Leftarrow$). Let $A = \{i'_1,\ldots,i'_n\}$, $B = \Big\{\mathbb{N}\setminus A : a^{(j)}_{i'_1,\cdots,i'_n} = 0\Big\}$. Notice that

$$\Big[\Big(\sum_{l\in A}\text{sign}\Big(a^{(l)}_{i'_1\cdots i'_n}\Big)e^*_l + \sum_{j\in\mathbb{N}\setminus(A\cup B)}\text{sign}\Big(a^{(j)}_{i'_1\cdots i'_n}\Big)e^*_j\Big),$$
$$\overline{\text{sign}\Big(a^{(i'_1)}_{i'_1\cdots i'_n}\Big)}e_{i_1},\ \ldots,\ \overline{\text{sign}\Big(a^{(i'_n)}_{i'_1\cdots i'_n}\Big)}e_{i_n}\Big] \in \text{Nradius}(T) \neq \emptyset,$$

which is a contradiction. □

Proposition 2.6. *Let $n \ge 2$. Let $T = \sum_{j\in\mathbb{N}} T_j e_j \in \mathcal{L}(^n\ell_1 : \ell_1)$ be the same as in Theorem 2.3. If $T \neq 0$ is norm attaining, then*

$$\text{Norm}(T) \supseteq \Big\{\Big(t_1e_{i_1},\ldots,t_ne_{i_n}\Big) : t_1,\ldots,t_n\in\mathbb{C}, |t_k| = 1, k = 1,\ldots,n,$$
$$v(T) = \sum_{j\in\mathbb{N}}\Big|a^{(j)}_{i_1\cdots i_n}\Big| \textit{ for some } (i_1,\ldots,i_n)\in\mathbb{N}^n\Big\}.$$

Proof. Suppose that $v(T) = \sum_{j\in\mathbb{N}}\Big|a^{(j)}_{i_1\cdots i_n}\Big|$ for some $(i_1,\ldots,i_n)\in\mathbb{N}^n$.

Claim. $(t_1e_{i_1},\ldots,t_ne_{i_n}) \in \text{Norm}(T)$ for every $t_1,\ldots,t_n\in\mathbb{C}, |t_k| = 1, k = 1,\ldots,n$.

Let

$$z^* = \Big(\sum_{l\in A} \mathrm{sign}(a^{(l)}_{i_1\cdots i_n})e^*_l + \sum_{j\in B} \lambda_j e^*_j + \sum_{j\in\mathbb{N}\setminus(A\cup B)} \mathrm{sign}(a^{(j)}_{i_1\cdots i_n})e^*_j\Big).$$

Then

$$\Big[z^*, \overline{\mathrm{sign}(a^{(i_1)}_{i_1\cdots i_n})}e_{i_1}, \ldots, \overline{\mathrm{sign}(a^{(i_n)}_{i_1\cdots i_n})}e_{i_n}\Big] \in \Pi((\ell_1)^n).$$

It follows that

$$\begin{aligned} \|T\| &= v(T) \\ &= \Big|z^*\Big(T\Big(\overline{\mathrm{sign}(a^{(i_1)}_{i_1\cdots i_n})}e_{i_1}, \ldots, \overline{\mathrm{sign}(a^{(i_n)}_{i_1\cdots i_n})}e_{i_n}\Big)\Big)\Big| \\ &\le \|T(e_{i_1}, \ldots, e_{i_n})\|_1 \le \|T\|, \end{aligned}$$

Hence,

$$\|T(t_1e_{i_1}, \ldots, t_ne_{i_n})\|_1 = \|T(e_{i_1}, \ldots, e_{i_n})\|_1 = \|T\|.$$

Therefore, the claim holds. □

Proposition 2.7. *Let $n \ge 2$. Let $T = \sum_{j\in\mathbb{N}} T_j e_j \in \mathcal{L}_s(^n\ell_1 : \ell_1)$ be the same as in Theorem 2.3. Suppose that T is not a numerical peak mapping in $\mathcal{L}(^n\ell_1 : \ell_1)$. If there is $(i'_1, \ldots, i'_n) \in \mathbb{N}^n$ such that*

$$\sum_{j\in\mathbb{N}} \Big|a^{(j)}_{i'_1\cdots i'_n}\Big| = v(T) > \sum_{j\in\mathbb{N}} \Big|a^{(j)}_{i_1\cdots i_n}\Big|$$

for every $(i_1, \ldots, i_n) \in \mathbb{N}^n\setminus\{(i'_1, \ldots, i'_n)\}$, then Nradius$(T)$ *is an infinite set.*

Proof. Let $A = \{i'_1, \ldots, i'_n\}, B = \{\mathbb{N}\setminus A : a^{(j)}_{i'_1,\cdots,i'_n} = 0\}$. Since T is not a numerical radius peak mapping, by Theorem 2.3, $B \neq \emptyset$. Let $\lambda_j \in \mathbb{C}, |\lambda_j| \le 1$ $(j \in B)$. By Theorem 2.2,

$$\Big[\Big(\sum_{l\in A} \mathrm{sign}(a^{(l)}_{i'_1,\cdots,i'_n})e^*_l + \sum_{j\in B} \lambda_j e^*_j + \sum_{j\in\mathbb{N}\setminus(A\cup B)} \mathrm{sign}(a^{(j)}_{i'_1,\cdots,i'_n})e^*_j\Big),$$
$$\overline{\mathrm{sign}(a^{(i'_1)}_{i'_1,\cdots,i'_n})}e_{i'_1}, \ldots, \overline{\mathrm{sign}(a^{(i'_n)}_{i'_1,\cdots,i'_n})}e_{i'_n}\Big] \in \mathrm{Nradius}(T),$$

which shows that Nradius(T) is infinite. □

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