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A result on mixed skew and η-Jordan triple derivations on prime $*$-algebras

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Abstract. Let $\mathcal{A}$ be a prime $*$-algebra. In this paper, we establish under some mild conditions that every non-additive mixed $\eta-$Jordan triple $*$-derivation i.e., a non-additive map $\psi : \mathcal{A} \to \mathcal{A}$ satisfying

$$\psi(\mathscr{A} \circ_\eta \mathscr{B} \bullet \mathscr{C}) = \psi(\mathscr{A}) \circ_\eta \mathscr{B} \bullet \mathscr{C} + \mathscr{A} \circ_\eta \psi(\mathscr{B}) \bullet \mathscr{C} + \mathscr{A} \circ_\eta \mathscr{B} \bullet \psi(\mathscr{C})$$

for any $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathcal{A}$, is an additive $*$-derivation and $\psi(\eta\mathscr{A}) = \eta\psi(A)$ for all $\mathscr{A} \in \mathcal{A}$.

1. Introduction

Throughout, $\mathcal{A}$ represents a unital prime $*$-algebra with center $\mathscr{Z}(\mathcal{A})$. An additive map $\psi : \mathcal{A} \to \mathcal{A}$ is called $*$-derivation if $\psi(\mathscr{A}\mathscr{B}) = \psi(\mathscr{A})\mathscr{B} + \mathscr{A}\psi(\mathscr{B})$ and $\psi(\mathscr{A}^*) = \psi(\mathscr{A})^*$ for all $\mathscr{A}, \mathscr{B} \in \mathcal{A}$. We say that ψ preserves the mixed skew and η-Jordan triple derivation if

$$\psi(\mathscr{A} \circ_\eta \mathscr{B} \bullet \mathscr{C}) = \psi(\mathscr{A}) \circ_\eta \mathscr{B} \bullet \mathscr{C} + \mathscr{A} \circ_\eta \psi(\mathscr{B}) \bullet \mathscr{C} + \mathscr{A} \circ_\eta \mathscr{B} \bullet \psi(\mathscr{C})$$

for all $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathcal{A}$, where $\mathscr{A} \bullet \mathscr{B} = \mathscr{A}\mathscr{B} + \mathscr{B}\mathscr{A}^*$ and $\mathscr{A} \circ_\eta \mathscr{B} = \mathscr{A}\mathscr{B} + \eta\mathscr{B}\mathscr{A}$. The idea of mixed η-Jordan triple derivations originated from the definition of η-Jordan triple derivation which is defined by

$$\psi(\mathscr{A} \bullet_\eta \mathscr{B} \bullet_\eta \mathscr{C}) = \psi(\mathscr{A}) \bullet_\eta \mathscr{B} \bullet_\eta \mathscr{C} + \mathscr{A} \bullet_\eta \psi(\mathscr{B}) \bullet_\eta \mathscr{C} + \mathscr{A} \bullet_\eta \mathscr{B} \bullet_\eta \psi(\mathscr{C})$$

for all $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathcal{A}$. Such kind of maps have received a fair amount of attention, references witness a growing interest in litrature (cf. [3–5, 10, 11, 14, 16] and references therein).

For $\mathscr{A}, \mathscr{B} \in \mathcal{A}$, we define the new product, namely Jordan η-$*$-product of $\mathscr{A}$ and $\mathscr{B}$ by $\mathscr{A} \bullet_\eta \mathscr{B} = \mathscr{A}\mathscr{B} + \eta\mathscr{B}\mathscr{A}^*$, which is customarily called the Jordan 1-$*$-product and Jordan (-1)-$*$-product, respectively. The latter product also knows in litrature as skew Lie product. We say that the map ψ (not necessarily

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additive) with the property $\psi(\mathscr{A} \bullet_\eta \mathscr{B}) = \psi(\mathscr{A}) \bullet_\eta \mathscr{B} + \mathscr{A} \bullet_\eta \psi(\mathscr{B})$ is a η-Jordan $*$-derivable map. It is clear that for $\eta = -1$ and $\eta = 1$, the η-Jordan $*$-derivation is a $*$-Lie derivation and a $*$-Jordan derivation, respectively. In [6], Huo et al. considered the Jordan triple $\eta - *-$product of three elements $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ in a $*$-algebra $\mathcal{A}$ and defined $\mathscr{A} \bullet_\eta \mathscr{B} \bullet_\eta \mathscr{C} = (\mathscr{A} \bullet_\eta \mathscr{B}) \bullet_\eta \mathscr{C}$ (we should be aware that $\bullet_\eta$ is not necessarily associative). Given the consideration of $\eta - *-$Jordan derivations and $\eta - *-$Jordan triple derivations, Lin [8, 9] further developed them in more general way. Suppose that $n \geq 2$ is a fixed positive integer and η is a nonzero scalar. Let us see a sequence of polynomials with involution

$$\begin{aligned}
p_1(X_1) &= X_1, \\
p_2(X_1, X_2) &= p_1(X_1) \bullet_\eta X_2 = X_1 \bullet_\eta X_2, \\
p_3(X_1, X_2, X_3) &= p_2(X_1, X_2) \bullet_\eta X_3 = (X_1 \bullet_\eta X_2) \bullet_\eta X_3 = X_1 \bullet_\eta X_2 \bullet_\eta X_3, \\
p_4(X_1, X_2, X_3, X_4) &= p_2(X_1, X_2, X_3) \bullet_\eta X_4 = ((X_1 \bullet_\eta X_2) \bullet_\eta X_3) \bullet_\eta X_4 = X_1 \bullet_\eta X_2 \bullet_\eta X_3 \bullet_\eta X_4, \\
&\vdots \\
p_n(X_1, X_2, ..., X_n) &= p_{n-1}(X_1, X_2, ..., X_{n-1}) \bullet_\eta X_n \\
&= (...((X_1 \bullet_\eta X_2) \bullet_\eta X_3) \bullet_\eta ... \bullet_\eta X_{n-1}) \bullet_\eta X_n \\
&= X_1 \bullet_\eta X_2 \bullet_\eta ... \bullet_\eta X_{n-1} \bullet_\eta X_n.
\end{aligned}$$

A nonlinear $\eta - *-$Jordan n-derivation is a mapping $\psi : \mathcal{A} \to \mathcal{A}$ satisfying the condition

$$\psi(p_n(\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n)) = \sum_{k=1}^{n} p_n(\mathscr{A}_1, ..., \mathscr{A}_{k-1}, \psi(\mathscr{A}_k), \mathscr{A}_{k+1}, ..., \mathscr{A}_n), \tag{1}$$

for all $\mathscr{A}_1, \mathscr{A}_2, ..., \mathscr{A}_n \in \mathcal{A}$. By the definition, it is clear that every $\eta - *-$Jordan derivation is a $\eta - *-$Jordan 2-derivation and every $\eta - *-$Jordan triple derivation is a $\eta - *-$Jordan 3-derivation.

In [2], Daif initially proved that each non-additive derivation is additive on a 2- torsion free prime ring containing a nontrivial idempotent. In [12], Li et. al showed that if $\mathcal{A} \subseteq \mathcal{A}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert space is a von Neumann algebra without central abelian projections, then $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a nonlinear $\eta - *-$Jordan derivation if and only if ψ is an additive $*-$derivation and $\psi(\eta\mathscr{A}) = \eta\psi(\mathscr{A})$ for all $\mathscr{A} \in \mathcal{A}$. This result has generalized to the case of nonlinear 1-$*$-Jordan triple derivations by Zhao and Li in [16]. For more intresting results related to Jordan and Lie $*$-derivations, we refer the readers to [1, 8, 12, 13, 15].

In recent years, the study of mixed Lie and Jordan triple products and derivations has received a fair amount of attention (see [7, 17–19]). Recently, Taghavi et al. [14] demonstrated that if $\psi(I)$ is self-adjoint, then any non-linear map preserving η-Jordan triple derivation on a prime $*$-algebra $\mathcal{A}$ is an additive $*$-derivation. In this article, we investigate the above result on more general context for a mixed η-Jordan triple derivation on a prime $*$-algebra $\mathcal{A}$. In particular, we establish the following:

Main Theorem. Let $\mathcal{A}$ be a unital prime $*$-algebra with a nontrivial projection and $\eta \neq -1, \eta \in \mathbb{R}$. Then, the map $\psi : \mathcal{A} \to \mathcal{A}$ satisfying

$$\psi(\mathscr{A} \circ_\eta \mathscr{B} \bullet \mathscr{C}) = \psi(\mathscr{A}) \circ_\eta \mathscr{B} \bullet \mathscr{C} + \mathscr{A} \circ_\eta \psi(\mathscr{B}) \bullet \mathscr{C} + \mathscr{A} \circ_\eta \mathscr{B} \bullet \psi(\mathscr{C}) \tag{2}$$

for any $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathcal{A}$, is additive. Moreover, if $\psi(I)$ is skew self-adjoint i.e., $\psi(I)^* = -\psi(I)$, then ψ is an additive $*$-derivation and $\psi(\eta\mathscr{A}) = \eta\psi(\mathscr{A})$ for all $\mathscr{A} \in \mathcal{A}$.

We establish the proof of Main Theorem in two parts. In Section 2, we prove that the map ψ is additive. Then in Section 3, we demonstrate that ψ is an additive $*$-derivation.

2. ψ is additive

Theorem 2.1. *Let $\mathcal{A}$ be a unital prime $*$-algebra with a nontrivial projection and $\eta \neq -1, \eta \in \mathbb{R}$. Then, the map $\psi : \mathcal{A} \to \mathcal{A}$ satisfying*

$$\psi(\mathscr{A} \circ_\eta \mathscr{B} \bullet \mathscr{C}) = \psi(\mathscr{A}) \circ_\eta \mathscr{B} \bullet \mathscr{C} + \mathscr{A} \circ_\eta \psi(\mathscr{B}) \bullet \mathscr{C} + \mathscr{A} \circ_\eta \mathscr{B} \bullet \psi(\mathscr{C}) \tag{3}$$

for any $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathcal{A}$, is additive.

Take a projection $\mathcal{P}_1 \in \mathcal{A}$ and let $\mathcal{P}_2 = I - \mathcal{P}_1$. We write $\mathcal{A}_{jk} = \mathcal{P}_j\mathcal{A}\mathcal{P}_k$ for $j,k = 1,2$. Then by Peire's decomposition of $\mathcal{A}$, we have $\mathcal{A} = \mathcal{A}_{11} \oplus \mathcal{A}_{12} \oplus \mathcal{A}_{21} \oplus \mathcal{A}_{22}$. Note that any operator $\mathscr{A} \in \mathcal{A}$ can be written as $\mathscr{A} = \mathscr{A}_{11} + \mathscr{A}_{12} + \mathscr{A}_{21} + \mathscr{A}_{22}$. In view of the about facts, the proof of the theorem is given in the series of the following steps:

Step 1. $\psi(0) = 0$.

For $\mathscr{A} = \mathscr{B} = \mathscr{C} = 0$ in (3), we get

$$\psi(0) = \psi(0 \circ_\eta 0 \bullet 0) = \psi(0) \circ_\eta 0 \bullet 0 + 0 \circ_\eta \psi(0) \bullet 0 + 0 \circ_\eta \psi(0) \bullet 0 = 0.$$

Step 2. *We show that $\psi(\mathscr{A}_{12} + \mathscr{A}_{22}) = \psi(\mathscr{A}_{12}) + \psi(\mathscr{A}_{22})$ for every $\mathscr{A}_{12} \in \mathcal{A}_{12}$ and $\mathscr{A}_{22} \in \mathcal{A}_{22}$.*

We show that

$$\mathcal{T} = \psi(\mathscr{A}_{12} + \mathscr{A}_{22}) - \psi(\mathscr{A}_{12}) - \psi(\mathscr{A}_{22}) = 0.$$

For this reason, we write

$$\begin{aligned}
\psi(\mathcal{P}_1 \circ_\eta \mathcal{P}_1 \bullet (\mathscr{A}_{12} + \mathscr{A}_{22})) &= \psi(\mathcal{P}_1 \circ_\eta \mathcal{P}_1 \bullet \mathscr{A}_{12}) + \psi(\mathcal{P}_1 \circ_\eta \mathcal{P}_1 \bullet \mathscr{A}_{22}) \\
&= \psi(\mathcal{P}_1) \circ_\eta \mathcal{P}_1 \bullet \mathscr{A}_{12} + \mathcal{P}_1 \circ_\eta \psi(\mathcal{P}_1) \bullet \mathscr{A}_{12} + \mathcal{P}_1 \circ_\eta \mathcal{P}_1 \bullet \psi(\mathscr{A}_{12}) \\
&+ \psi(\mathcal{P}_1) \circ_\eta \mathcal{P}_1 \bullet \mathscr{A}_{22} + \mathcal{P}_1 \circ_\eta \psi(P_1) \bullet \mathscr{A}_{22} + \mathcal{P}_1 \circ_\eta \mathcal{P}_1 \bullet \psi(\mathscr{A}_{22}) \\
&= \psi(\mathcal{P}_1) \circ_\eta \mathcal{P}_1 \bullet (\mathscr{A}_{12} + \mathscr{A}_{22}) + \mathcal{P}_1 \circ_\eta \psi(\mathcal{P}_1) \bullet (\mathscr{A}_{12} + \mathscr{A}_{22}) \\
&+ \mathcal{P}_1 \circ_\eta \mathcal{P}_1 \bullet (\psi(\mathscr{A}_{12}) + \psi(\mathscr{A}_{22})).
\end{aligned}$$

So, $\mathcal{P}_1 \circ_\eta \mathcal{P}_1 \bullet \mathcal{T} = 0$, consequently $\mathcal{T}_{11} = \mathcal{T}_{12} = \mathcal{T}_{21} = 0$.

On the other hand, we have

$$\begin{aligned}
\psi(I \circ_\eta (\mathcal{P}_1 - \mathcal{P}_2) \bullet (\mathscr{A}_{12} + \mathscr{A}_{22})) &= \psi(I \circ_\eta (\mathcal{P}_1 - \mathcal{P}_2) \bullet \mathscr{A}_{12}) + \psi(I \circ_\eta (\mathcal{P}_1 - \mathcal{P}_2) \bullet \mathscr{A}_{22}) \\
&= \psi(I) \circ_\eta (\mathcal{P}_1 - \mathcal{P}_2) \bullet \mathscr{A}_{12} + I \circ_\eta \psi(\mathcal{P}_1 - \mathcal{P}_2) \bullet \mathscr{A}_{12} \\
&+ I \circ_\eta (\mathcal{P}_1 - \mathcal{P}_2) \bullet \psi(\mathscr{A}_{12}) + \psi(I) \circ_\eta (\mathcal{P}_1 - \mathcal{P}_2) \bullet \mathscr{A}_{12} \\
&+ I \circ_\eta \psi(\mathcal{P}_1 - \mathcal{P}_2) \bullet \mathscr{A}_{12} + I \circ_\eta (\mathcal{P}_1 - \mathcal{P}_2) \bullet \psi(\mathscr{A}_{12}) \\
&= (I \circ_\eta (\mathcal{P}_1 - \mathcal{P}_2) \bullet (\psi(\mathscr{A}_{12}) + \psi(\mathscr{A}_{22})) \\
&+ \psi(I) \circ_\eta (\mathcal{P}_1 - \mathcal{P}_2) \bullet (\mathscr{A}_{12} + \mathscr{A}_{22}) + I \circ_\eta \psi(\mathcal{P}_1 - \mathcal{P}_2) \bullet (\mathscr{A}_{12} + \mathscr{A}_{22}).
\end{aligned}$$

Therefore, $I \circ_\eta (\mathcal{P}_1 - \mathcal{P}_2) \bullet \mathcal{T} = 0$. Thus, $\mathcal{T}_{22} = 0$.

Step 3. *We can show that $\psi(\mathscr{A}_{11} + \mathscr{A}_{21}) = \psi(\mathscr{A}_{11}) + \psi(\mathscr{A}_{21})$ for every $\mathscr{A}_{11} \in \mathcal{A}_{11}$ and $\mathscr{A}_{21} \in \mathcal{A}_{21}$.*

The proof of this step is similar to Claim 1.

Step 4. *We show that $\psi(\mathscr{A}_{12} + \mathscr{A}_{21}) = \psi(\mathscr{A}_{12}) + \psi(\mathscr{A}_{21})$ for any $\mathscr{A}_{12} \in \mathcal{A}_{12}$ and $\mathscr{A}_{21} \in \mathcal{A}_{21}$*

We need to show that

$$\mathcal{T} = \psi(\mathscr{A}_{12} + \mathscr{A}_{21}) - \psi(\mathscr{A}_{12}) - \psi(\mathscr{A}_{21}) = 0.$$

For this purpose, we write

$$\begin{aligned}\psi(\mathcal{P}_1 \circ_\eta (\mathscr{A}_{12} + \mathscr{A}_{21}) \bullet \mathcal{P}_1) &= \psi(\mathcal{P}_1 \circ_\eta \mathscr{A}_{12} \bullet \mathcal{P}_1) + \psi(\mathcal{P}_1 \circ_\eta \mathscr{A}_{21} \bullet \mathcal{P}_1)\\ &= \psi(\mathcal{P}_1) \circ_\eta \mathscr{A}_{12} \bullet \mathcal{P}_1 + \mathcal{P}_1 \circ_\eta \psi(\mathscr{A}_{12}) \bullet \mathcal{P}_1 + \mathcal{P}_1 \circ_\eta \mathscr{A}_{12} \bullet \psi(\mathcal{P}_1)\\ &+ \psi(\mathcal{P}_1) \circ_\eta \mathscr{A}_{12} \bullet \mathcal{P}_1 + \mathcal{P}_1 \circ_\eta \psi(\mathscr{A}_{21}) \bullet \mathcal{P}_1 + \mathcal{P}_1 \circ_\eta \mathscr{A}_{21} \bullet \psi(\mathcal{P}_1)\\ &= \psi(\mathcal{P}_1) \circ_\eta (\mathscr{A}_{12} + \mathscr{A}_{21}) \bullet \mathcal{P}_1 + \mathcal{P}_1 \circ_\eta (\psi(\mathscr{A}_{12}) + \psi(\mathscr{A}_{21})) \bullet \mathcal{P}_1\\ &+ \mathcal{P}_1 \circ_\eta (\mathscr{A}_{12} + \mathscr{A}_{21}) \bullet \psi(\mathcal{P}_1). \qquad (4)\end{aligned}$$

Therefore, $\mathcal{T}_{21} = 0$ and

$$\mathcal{T}_{11} + \mathcal{T}_{11}^* = 0. \qquad (5)$$

By applying the same trick for $i\mathcal{P}_1$ instead of $\mathcal{P}_1$ in above (4), we obtain

$$\mathcal{T}_{11} - \mathcal{T}_{11}^* = 0. \qquad (6)$$

From (5) and (6) we have $\mathcal{T}_{11} = 0$.

Similarly, by applying $\mathcal{P}_2$ instead of $\mathcal{P}_1$ in above (4), we have $\mathcal{T}_{12} = 0$ and

$$\mathcal{T}_{22} + \mathcal{T}_{22}^* = 0. \qquad (7)$$

Moreover, by applying $i\mathcal{P}_2$ instead of $\mathcal{P}_1$ in (4), we obtain

$$\mathcal{T}_{22} - \mathcal{T}_{22}^* = 0. \qquad (8)$$

From (7) and (8) we have $\mathcal{T}_{22} = 0$.

Step 5. *For every $\mathscr{A}_{11} \in \mathcal{A}_{11}$, $\mathscr{A}_{12} \in \mathcal{A}_{12}$, $\mathscr{A}_{21} \in \mathcal{A}_{21}$, $\mathscr{A}_{22} \in \mathcal{A}_{22}$ we have*

1. $\psi(\mathscr{A}_{12} + \mathscr{A}_{21} + \mathscr{A}_{22}) = \psi(\mathscr{A}_{12}) + \psi(\mathscr{A}_{21}) + \psi(\mathscr{A}_{22})$.
2. $\psi(\mathscr{A}_{11} + \mathscr{A}_{12} + \mathscr{A}_{21}) = \psi(\mathscr{A}_{11}) + \psi(\mathscr{A}_{12}) + \psi(\mathscr{A}_{21})$.

We only prove the first part. The second one can be proved similarly.

We prove that $\mathcal{T} = \psi(\mathscr{A}_{12} + \mathscr{A}_{21} + \mathscr{A}_{22}) - \psi(\mathscr{A}_{12}) - \psi(\mathscr{A}_{21}) - \psi(\mathscr{A}_{22}) = 0$.

For this reason, we can write

$$\begin{aligned}\psi(\mathcal{P}_1 \circ_\eta \mathcal{P}_1 \bullet (\mathscr{A}_{12} + \mathscr{A}_{21} + \mathscr{A}_{22})) &= \psi(\mathcal{P}_1) \circ_\eta \mathcal{P}_1 \bullet \mathscr{A}_{12} + \mathcal{P}_1 \circ_\eta \psi(\mathcal{P}_1) \bullet \mathscr{A}_{12} + \mathcal{P}_1 \circ_\eta \mathcal{P}_1 \bullet \psi(\mathscr{A}_{12})\\ &+ \psi(\mathcal{P}_1) \circ_\eta \bullet \mathcal{P}_1 \bullet \mathscr{A}_{21} + \mathcal{P}_1 \circ_\eta \psi(\mathcal{P}_1) \bullet \mathscr{A}_{21}\\ &+ \mathcal{P}_1 \circ_\eta \mathcal{P}_1 \bullet \psi(\mathscr{A}_{21}) + \psi(\mathcal{P}_1) \circ_\eta \mathcal{P}_1 \bullet \mathscr{A}_{22} + \mathcal{P}_1 \circ_\eta \psi(\mathcal{P}_1) \bullet \mathscr{A}_{22}\\ &+ \mathcal{P}_1 \circ_\eta \mathcal{P}_1 \bullet \psi(\mathscr{A}_{22})\\ &= \mathcal{P}_1 \circ_\eta \psi(\mathcal{P}_1) \bullet (\mathscr{A}_{12} + \mathscr{A}_{21} + \mathscr{A}_{22}) + \psi(\mathcal{P}_1) \circ_\eta \mathcal{P}_1 \bullet (\mathscr{A}_{12} + \mathscr{A}_{21} + \mathscr{A}_{22})\\ &+ \mathcal{P}_1 \circ_\eta \mathcal{P}_1 \bullet (\psi(\mathscr{A}_{12}) + \psi(\mathscr{A}_{21}) + \psi(\mathscr{A}_{22})).\end{aligned}$$

Hence, $\mathcal{P}_1 \circ_\eta \mathcal{P}_1 \bullet \mathcal{T} = 0$. So, $\mathcal{T}_{11} = \mathcal{T}_{21} = \mathcal{T}_{12} = 0$.

On the other hand, we can verify that

$$\begin{aligned}\psi(\mathcal{P}_2 \circ_\eta (\mathscr{A}_{12} + \mathscr{A}_{21} + \mathscr{A}_{22}) \bullet \mathcal{P}_2) &= \psi(\mathcal{P}_2 \circ_\eta \mathscr{A}_{12} \bullet \mathcal{P}_2) + \psi(\mathcal{P}_2 \circ_\eta \mathscr{A}_{21} \bullet \mathcal{P}_2) + \psi(\mathcal{P}_2 \circ_\eta \mathscr{A}_{22} \bullet \mathcal{P}_2)\\ &= \psi(\mathcal{P}_2) \circ_\eta \mathscr{A}_{12} \bullet \mathcal{P}_2 + \mathcal{P}_2 \circ_\eta \psi(\mathscr{A}_{12}) \bullet \mathcal{P}_2 + \mathcal{P}_2 \circ_\eta \mathscr{A}_{12} \bullet \psi(\mathcal{P}_2)\\ &+ \psi(\mathcal{P}_2) \circ_\eta \mathscr{A}_{21} \bullet \mathcal{P}_2 + \mathcal{P}_2 \circ_\eta \psi(\mathscr{A}_{21}) \bullet \mathcal{P}_2 + \mathcal{P}_2 \circ_\eta \mathscr{A}_{21} \bullet \psi(\mathcal{P}_2)\\ &+ \psi(\mathcal{P}_2) \circ_\eta \mathscr{A}_{22} \bullet \mathcal{P}_2 + \mathcal{P}_2 \circ_\eta \psi(\mathscr{A}_{22}) \bullet \mathcal{P}_2 + \mathcal{P}_2 \circ_\eta \mathscr{A}_{22} \bullet \psi(\mathcal{P}_2)\\ &= \psi(\mathcal{P}_2) \circ_\eta (\mathscr{A}_{12} + \mathscr{A}_{21} + \mathscr{A}_{22}) \bullet \mathcal{P}_2\\ &+ \mathcal{P}_2 \circ_\eta (\psi(\mathscr{A}_{12}) + \psi(\mathscr{A}_{21}) + \psi(\mathscr{A}_{22})) \bullet \mathcal{P}_2\\ &+ \mathcal{P}_2 \circ_\eta (\mathscr{A}_{12} + \mathscr{A}_{21} + \mathscr{A}_{22}) \bullet \psi(\mathcal{P}_2).\end{aligned}$$

Hence, we obtain $\mathcal{P}_2 \circ_\eta \mathcal{T} \bullet \mathcal{P}_2 = 0$. Then, $\mathcal{T}_{22} + \mathcal{T}_{22}^* = 0$.

Likewise, by applying $i\mathcal{P}_2$ instead of $\mathcal{P}_2$ in above we have $\mathcal{T}_{22} - \mathcal{T}_{22}^* = 0$. So, $\mathcal{T}_{22} = 0$.

Step 6. *For every* $\mathscr{A}_{ij} \in \mathcal{A}_{ij}$, *we show that* $\psi\Big(\sum_{i,j=1}^{2} \mathscr{A}_{ij}\Big) = \sum_{i,j=1}^{2} \psi(\mathscr{A}_{ij})$.

We prove that $\mathcal{T} = \psi\Big(\sum_{i,j=1}^{2} \mathscr{A}_{ij}\Big) - \sum_{i,j=1}^{2} \psi(\mathscr{A}_{ij}) = 0$.

It is easy to verify that

$$\begin{aligned}
\psi\Big(\mathcal{P}_2 \circ_\eta \mathcal{P}_2 \bullet \sum_{i,j=1}^{2} \mathscr{A}_{ij}\Big) &= \psi(\mathcal{P}_2 \circ_\eta \mathcal{P}_2 \bullet \mathscr{A}_{11}) + \psi(\mathcal{P}_2 \circ_\eta \mathcal{P}_2 \bullet \mathscr{A}_{12}) + \psi(\mathcal{P}_2 \circ_\eta \mathcal{P}_2 \bullet \mathscr{A}_{21}) + \psi(\mathcal{P}_2 \circ_\eta \mathcal{P}_2 \bullet \mathscr{A}_{22}) \\
&= \psi(\mathcal{P}_2) \circ_\eta \mathcal{P}_2 \bullet \mathscr{A}_{11} + \mathcal{P}_2 \circ_\eta \psi(\mathcal{P}_2) \bullet \mathscr{A}_{11} + \mathcal{P}_2 \circ_\eta \mathcal{P}_2 \bullet \psi(\mathscr{A}_{11}) + \psi(\mathcal{P}_2) \circ_\eta \mathcal{P}_2 \bullet \mathscr{A}_{12} \\
&+ \mathcal{P}_2 \circ_\eta \psi(\mathcal{P}_2)\mathscr{A}_{12} + \mathcal{P}_2 \circ_\eta \mathcal{P}_2 \bullet \psi(\mathscr{A}_{12}) + \mathcal{P}_2 \circ_\eta \psi(\mathcal{P}_2) \bullet \mathscr{A}_{21} \\
&+ \psi(\mathcal{P}_2) \circ_\eta \mathcal{P}_2 \bullet \mathscr{A}_{21} + \mathcal{P}_2 \circ_\eta \mathcal{P}_2 \bullet \psi(\mathscr{A}_{21}) + \psi(\mathcal{P}_2) \circ_\eta \mathcal{P}_2 \bullet \mathscr{A}_{22} \\
&+ \mathcal{P}_2 \circ_\eta \psi(\mathcal{P}_2) \bullet \mathscr{A}_{22} + \mathcal{P}_2 \circ_\eta \mathcal{P}_2 \bullet \psi(\mathscr{A}_{22}) \\
&= \psi(\mathcal{P}_2) \circ_\eta \mathcal{P}_2 \bullet (\mathscr{A}_{11} + \mathscr{A}_{12} + \mathscr{A}_{21} + \mathscr{A}_{22}) + \mathcal{P}_2 \circ_\eta \psi(\mathcal{P}_2) \bullet (\mathscr{A}_{11} + \mathscr{A}_{12} + \mathscr{A}_{21} + \mathscr{A}_{22}) \\
&+ \mathcal{P}_2 \circ_\eta \mathcal{P}_2 \bullet (\psi(\mathscr{A}_{11}) + \psi(\mathscr{A}_{12}) + \psi(\mathscr{A}_{21}) + \psi(\mathscr{A}_{22})).
\end{aligned}$$

Thus, $\mathcal{P}_2 \circ_\eta \mathcal{P}_2 \bullet \mathcal{T} = 0$. So, $\mathcal{T}_{22} = \mathcal{T}_{12} = 0$.

Similarly, by applying $\mathcal{P}_1$ instead of $\mathcal{P}_2$ in above, we have $\mathcal{T}_{11} = \mathcal{T}_{21} = 0$.

Step 7. *For any* $\mathscr{A}_{ij}, \mathscr{B}_{ij} \in \mathcal{A}_{ij}$ *with* $i \neq j$, $\psi(\mathscr{A}_{ij} + \mathscr{B}_{ij}) = \psi(\mathscr{A}_{ij}) + \psi(\mathscr{B}_{ij})$.

It follows from Step 2 that

$$\begin{aligned}
\psi(\mathscr{A}_{ij} + \mathscr{B}_{ij}) &= \psi\Big(\frac{1}{1+\eta} I \circ_\eta \mathcal{P}_i \bullet (2\mathcal{P}_j + \mathscr{A}_{ij} + \mathscr{B}_{ij})\Big) \\
&= \psi\Big(\frac{1}{1+\eta} I\Big) \circ_\eta \mathcal{P}_i \bullet (2\mathcal{P}_j + \mathscr{A}_{ij} + \mathscr{B}_{ij}) \\
&+ \frac{1}{1+\eta} I \circ_\eta \psi(\mathcal{P}_i) \bullet (2\mathcal{P}_j + \mathscr{A}_{ij} + \mathscr{B}_{ij}) \\
&+ \frac{1}{1+\eta} I \circ_\eta \mathcal{P}_i \bullet \psi(2\mathcal{P}_j + \mathscr{A}_{ij} + \mathscr{B}_{ij}) \\
&= \psi\Big(\frac{1}{1+\eta} I\Big) \circ_\eta \mathcal{P}_i \bullet (2\mathcal{P}_j + \mathscr{A}_{ij} + \mathscr{B}_{ij}) \\
&+ \frac{1}{1+\eta} I \circ_\eta \psi(\mathcal{P}_i) \bullet (2\mathcal{P}_j + \mathscr{A}_{ij} + \mathscr{B}_{ij}) \\
&+ \frac{1}{1+\eta} I \circ_\eta \mathcal{P}_i \bullet (\psi(\mathcal{P}_j) + \psi(\mathscr{A}_{ij}) + \psi(\mathcal{P}_j) + \psi(\mathscr{B}_{ij})) \\
&= 2\psi\Big(\frac{1}{1+\eta} I \circ_\eta \mathcal{P}_i \bullet \mathcal{P}_j\Big) + \psi\Big(\frac{1}{1+\eta} I \circ_\eta \mathcal{P}_i \bullet \mathscr{A}_{ij}\Big) \\
&+ \psi\Big(\frac{1}{1+\eta} I \circ_\eta \mathcal{P}_i \bullet \mathscr{B}_{ij}\Big) \\
&= \psi(\mathscr{A}_{ij}) + \psi(\mathscr{B}_{ij}).
\end{aligned}$$

Step 8. *For any* $\mathscr{A}_{ii}, \mathscr{B}_{ii} \in \mathcal{A}_{ii}$ *such that* $i \in \{1, 2\}$, $\psi(\mathscr{A}_{ii} + \mathscr{B}_{ii}) = \psi(\mathscr{A}_{ii}) + \psi(\mathscr{B}_{ii})$.

Suppose That $\mathcal{T} = \psi(\mathscr{A}_{ii} + \mathscr{B}_{ii}) - (\psi(\mathscr{A}_{ii}) + \psi(\mathscr{B}_{ii}))$. It is easy to find

$$\begin{aligned}
&\psi(\mathcal{P}_i) \circ_\eta (\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \mathcal{P}_j + \mathcal{P}_i \circ_\eta \psi(\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \mathcal{P}_j + \mathcal{P}_i \circ_\eta (\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \psi(\mathcal{P}_j)\\
&= \psi(\mathcal{P}_i \circ_\eta (\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \mathcal{P}_j)\\
&= \psi(\mathcal{P}_i \circ_\eta \mathscr{A}_{ii} \bullet \mathcal{P}_j) + \psi(\mathcal{P}_i \circ_\eta \mathscr{B}_{ii} \bullet \mathcal{P}_j)\\
&= \psi(\mathcal{P}_i) \circ_\eta \mathscr{A}_{ii} \bullet \mathcal{P}_j + \mathcal{P}_i \circ_\eta \psi(\mathscr{A}_{ii}) \bullet \mathcal{P}_j + \mathcal{P}_i \circ_\eta \mathscr{A}_{ii} \bullet \psi(\mathcal{P}_j)\\
&+ \psi(\mathcal{P}_i) \circ_\eta \mathscr{B}_{ii} \bullet \mathcal{P}_j + \mathcal{P}_i \circ_\eta \psi(\mathscr{B}_{ii}) \bullet \mathcal{P}_j + \mathcal{P}_i \circ_\eta \mathscr{B}_{ii} \bullet \psi(\mathcal{P}_j)\\
&+ \psi(\mathcal{P}_i) \circ_\eta (\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \mathcal{P}_j + \mathcal{P}_i \circ_\eta (\psi(\mathscr{A}_{ii}) + \psi(\mathscr{B}_{ii})) \bullet \mathcal{P}_j\\
&+ \mathcal{P}_i \circ_\eta (\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \psi(\mathcal{P}_j).
\end{aligned}$$

Also

$$\begin{aligned}
&\psi(\mathcal{P}_j) \circ_\eta (\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \mathcal{P}_i + \mathcal{P}_j \circ_\eta \psi(\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \mathcal{P}_i + \mathcal{P}_j \circ_\eta (\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \psi(\mathcal{P}_i)\\
&= \psi(\mathcal{P}_j \circ_\eta (\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \mathcal{P}_i)\\
&= \psi(\mathcal{P}_j \circ_\eta \mathscr{A}_{ii} \bullet \mathcal{P}_i) + \psi(\mathcal{P}_j \circ_\eta \mathscr{B}_{ii} \bullet \mathcal{P}_i)\\
&= \psi(\mathcal{P}_j) \circ_\eta \mathscr{A}_{ii} \bullet \mathcal{P}_i + \mathcal{P}_j \circ_\eta \psi(\mathscr{A}_{ii}) \bullet \mathcal{P}_i + \mathcal{P}_j \circ_\eta \mathscr{A}_{ii} \bullet \psi(\mathcal{P}_i)\\
&+ \psi(\mathcal{P}_j) \circ_\eta \mathscr{B}_{ii} \bullet \mathcal{P}_i + \mathcal{P}_j \circ_\eta \psi(\mathscr{B}_{ii}) \bullet \mathcal{P}_i + \mathcal{P}_j \circ_\eta \mathscr{B}_{ii} \bullet \psi(\mathcal{P}_i)\\
&= \psi(\mathcal{P}_j) \circ_\eta (\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \mathcal{P}_i + \mathcal{P}_j \circ_\eta (\psi(\mathscr{A}_{ii}) + \psi(\mathscr{B}_{ii})) \bullet \mathcal{P}_i\\
&+ \mathcal{P}_j \circ_\eta (\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \psi(\mathcal{P}_i).
\end{aligned}$$

From above, we have $\mathcal{P}_i \circ_\eta \mathcal{T} \bullet \mathcal{P}_j = \mathcal{P}_j \circ_\eta \mathcal{T} \bullet \mathcal{P}_i = 0$. This yields $\mathcal{T}_{12} = \mathcal{T}_{21} = 0$. Next, according to Step 2 and Step 7, for any $\mathscr{C}_{ji} \in \mathscr{M}_{ji}$ with $i \neq j$, we have

$$\begin{aligned}
&\psi(\mathscr{C}_{ji}) \circ_\eta (\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \mathcal{P}_i + \mathscr{C}_{ji} \circ_\eta \psi(\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \mathcal{P}_i + \mathscr{C}_{ji} \circ_\eta (\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \psi(\mathcal{P}_i)\\
&= \psi(\mathscr{C}_{ji} \circ_\eta (\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \mathcal{P}_i)\\
&= \psi(\mathscr{C}_{ji}\mathscr{A}_{ii} + \mathscr{A}_{ii}^*\mathscr{C}_{ji}^* + \mathscr{C}_{ji}\mathscr{B}_{ii} + \mathscr{B}_{ii}^*\mathscr{C}_{ji}^*)\\
&= \psi(\mathscr{C}_{ji}\mathscr{A}_{ii} + \mathscr{C}_{ji}\mathscr{B}_{ii}) + \psi(\mathscr{A}_{ii}^*\mathscr{C}_{ji}^* + \mathscr{B}_{ii}^*\mathscr{C}_{ji}^*)\\
&= \psi(\mathscr{C}_{ji}\mathscr{A}_{ii} + \mathscr{A}_{ii}^*\mathscr{C}_{ji}^*) + \psi(\mathscr{C}_{ji}\mathscr{B}_{ii} + \mathscr{B}_{ii}^*\mathscr{C}_{ji}^*)\\
&= \psi(\mathscr{C} ji \circ_\eta \mathscr{A}_{ii} \bullet \mathcal{P}_i) + \psi(\mathscr{C} ji \circ_\eta \mathscr{B}_{ii} \bullet \mathcal{P}_i)\\
&= \psi(\mathscr{C}_{ji}) \circ_\eta (\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \mathcal{P}_i + \mathscr{C}_{ji} \circ_\eta (\psi(\mathscr{A}_{ii}) + \psi(\mathscr{B}_{ii})) \bullet \mathcal{P}_i\\
&+ \mathscr{C}_{ji} \circ_\eta (\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \psi(\mathcal{P}_i).
\end{aligned}$$

Similarly, for any $\mathscr{C}_{ij} \in \mathcal{A}_{ij}$ with $i \neq j$, we get

$$\begin{aligned}
&\psi(\mathscr{C}_{ij}) \circ_\eta (\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \mathcal{P}_j + \mathscr{C}_{ij} \circ_\eta \psi(\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \mathcal{P}_j + \mathscr{C}_{ij} \circ_\eta (\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \psi(\mathcal{P}_j)\\
&= \psi(\mathscr{C}_{ij}) \circ_\eta (\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \mathcal{P}_j + \mathscr{C}_{ij} \circ_\eta (\psi(\mathscr{A}_{ii}) + \psi(\mathscr{B}_{ii})) \bullet \mathcal{P}_j + \mathscr{C}_{ij} \circ_\eta (\mathscr{A}_{ii} + \mathscr{B}_{ii}) \bullet \psi(\mathcal{P}_j).
\end{aligned}$$

Hence $\mathscr{C}_{ji} \circ_\eta \mathcal{T} \bullet \mathcal{P}_i = \mathscr{C}_{ij} \circ_\eta \mathcal{T} \bullet \mathcal{P}_j = 0$. This gives $\mathcal{T}_{11} = \mathcal{T}_{22} = 0$. Thus $\mathcal{T} = 0$ i.e.,

$$\psi(\mathscr{A}_{ii} + \mathscr{B}_{ii}) = \psi(\mathscr{A}_{ii}) + \psi(\mathscr{B}_{ii}).$$

Proof of Theorem 2.1. Additivity of ψ is follow from Steps 1−8. □

3. ψ is a $*$-derivation

Theorem 3.1. *Let $\mathcal{A}$ be a unital prime $*$-algebra with a nontrivial projection and $\eta \neq -1, \eta \in \mathbb{R}$. Then, the map $\psi : \mathcal{A} \to \mathcal{A}$ satisfying*

$$\psi(\mathscr{A} \circ_\eta \mathscr{B} \bullet \mathscr{C}) = \psi(\mathscr{A}) \circ_\eta \mathscr{B} \bullet \mathscr{C} + \mathscr{A} \circ_\eta \psi(\mathscr{B}) \bullet \mathscr{C} + \mathscr{A} \circ_\eta \mathscr{B} \bullet \psi(\mathscr{C}) \tag{9}$$

for any $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathcal{A}$ and skew self-adjoint $\psi(I)$ is an additive $*$-derivation and $\psi(\eta\mathscr{A}) = \eta\psi(\mathscr{A})$ for all $\mathscr{A} \in \mathcal{A}$.

We will prove our theorem with help of following claims:

Claim 1. *$\psi(i\mathscr{A}) = i\psi(\mathscr{A})$ for all $\mathscr{A} \in \mathcal{A}$.*

For any $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathcal{A}$, it is straight forward to verify that

$$i\mathscr{A} \circ_\eta \mathscr{B} \bullet I = \mathscr{A} \circ_\eta i\mathscr{B} \bullet I.$$

Thus,

$$\psi(i\mathscr{A} \circ_\eta \mathscr{B} \bullet I) = \psi(i\mathscr{A}) \circ_\eta \mathscr{B} \bullet I + i\mathscr{A} \circ_\eta \psi(\mathscr{B}) \bullet I + i\mathscr{A} \circ_\eta \mathscr{B} \bullet \psi(I) \tag{10}$$

for all $\mathscr{A}, \mathscr{B} \in \mathcal{A}$. On the other hand,

$$\psi(\mathscr{A} \circ_\eta i\mathscr{B} \bullet I) = \psi(\mathscr{A}) \circ_\eta i\mathscr{B} \bullet I + \mathscr{A} \circ_\eta \psi(i\mathscr{B}) \bullet I + \mathscr{A} \circ_\eta i\mathscr{B} \bullet \psi(I) \tag{11}$$

for all $\mathscr{A}, \mathscr{B} \in \mathcal{A}$. Observe from (10) and (11) that

$$\psi(i\mathscr{A}) \circ_\eta \mathscr{B} \bullet I + i\mathscr{A} \circ_\eta \psi(\mathscr{B}) \bullet I = \psi(\mathscr{A}) \circ_\eta i\mathscr{B} \bullet I + \mathscr{A} \circ_\eta \psi(i\mathscr{B}) \bullet I \tag{12}$$

for all $\mathscr{A}, \mathscr{B} \in \mathcal{A}$. This implies

$$(\psi(i\mathscr{A}) - i\psi(\mathscr{A})) \circ_\eta \mathscr{B} \bullet I = \mathscr{A} \circ_\eta (\psi(i\mathscr{B}) - i\psi(\mathscr{B})) \bullet I \tag{13}$$

for all $\mathscr{A}, \mathscr{B} \in \mathcal{A}$. Replace $\mathscr{A}$ by $i\mathscr{A}$ in (13) and using the fact that ψ is additivie, we get

$$(-\psi(\mathscr{A}) - i\psi(i\mathscr{A})) \circ_\eta \mathscr{B} \bullet I = i\mathscr{A} \circ_\eta (\psi(i\mathscr{B}) - i\psi(\mathscr{B})) \bullet I \tag{14}$$

for all $\mathscr{A}, \mathscr{B} \in \mathcal{A}$. Multiply equation (13) by i and combined the so obtained relation with (14), we obtain

$$(\psi(\mathscr{A}) + i\psi(i\mathscr{A})) \circ_\eta \mathscr{B} \bullet I = 0$$

for all $\mathscr{A}, \mathscr{B} \in \mathcal{A}$. This can be written as $(\psi(i\mathscr{A}) - i\psi(\mathscr{A})) \circ_\eta \mathscr{B} \bullet I = 0$ for all $\mathscr{A}, \mathscr{B} \in \mathcal{A}$. In particular, for $\mathscr{B} = I$, we have

$$\psi(i\mathscr{A}) - i\psi(\mathscr{A}) + \psi(i\mathscr{A})^* + i\psi(\mathscr{A})^* = 0 \tag{15}$$

for all $\mathscr{A} \in \mathcal{A}$. Setting $\mathscr{A}$ as $i\mathscr{A}$ in last expression, we obtain

$$\psi(\mathscr{A}) + i\psi(i\mathscr{A}) + \psi(\mathscr{A})^* - i\psi(i\mathscr{A})^* = 0 \tag{16}$$

for all $\mathscr{A} \in \mathcal{A}$. Also, multiply (15) by i, we get

$$i\psi(i\mathscr{A}) + \psi(\mathscr{A}) + i\psi(i\mathscr{A})^* - \psi(\mathscr{A})^* = 0 \tag{17}$$

for all $\mathscr{A} \in \mathcal{A}$. Combining (16) and (17) yields $i\psi(i\mathscr{A}) + \psi(\mathscr{A}) = 0$ for all $\mathscr{A} \in \mathcal{A}$ and hence, $\psi(i\mathscr{A}) = i\psi(\mathscr{A})$. for all $\mathscr{A} \in \mathcal{A}$.

Claim 2. *We show that $\psi(I) = \psi(iI) = 0$ for any $\eta \neq -1, \eta \in \mathbb{R}$.*

By Claim 1 and the fact that $\psi(I)$ is skew self-adjoint, we have

$$\psi(iI)^* = -i\psi(I)^* = i\psi(I) = \psi(iI).$$

We have

$$\begin{aligned}\psi(iI \circ_\eta I \bullet I) &= \psi(iI) \circ_\eta I \bullet I + iI \circ_\eta \psi(I) \bullet I + iI \circ_\eta I \bullet \psi(I) \\ &= (1+\eta)\psi(iI) \bullet I + (1+\eta)i\psi(I) \bullet I + (1+\eta)iI \bullet \psi(I) \\ &= (1+\eta)\psi(iI) + (1+\eta)\psi(iI)^* + (1+\eta)i\psi(I) - (1+\eta)i\psi(I)^* + (1+\eta)i\psi(I) - (1+\eta)i\psi(I) \\ &= 4(1+\eta)i\psi(I).\end{aligned}$$

On the other hand,

$$iI \circ_\eta I \bullet I = 0.$$

Since $\eta \neq -1$, hence $\psi(I) = 0$ and consequently $\psi(iI) = 0$.

Claim 3. *For any $\mathscr{A} \in \mathcal{A}$, $\psi(\eta\mathscr{A}) = \eta\psi(\mathscr{A})$.*

It follows from Claim 2 that

$$\psi(I \circ_\eta I \bullet \mathscr{A}) = I \circ_\eta I \bullet \psi(\mathscr{A}).$$

This yields $\psi(\eta\mathscr{A}) = \eta\psi(\mathscr{A})$.

Claim 4. *$\psi(\mathscr{A}^*) = \psi(\mathscr{A})^*$ for all $\mathscr{A} \in \mathcal{A}$.*

Observe that

$$\psi(I \circ_\eta \mathscr{A} \bullet I) = I \circ_\eta \psi(\mathscr{A}) \bullet I.$$

This gives

$$(1+\eta)\psi(\mathscr{A} + \mathscr{A}^*) = (1+\eta)\psi(\mathscr{A}) + (1+\eta)\psi(\mathscr{A})^*.$$

Therefore, we have $\psi(\mathscr{A}^*) = \psi(\mathscr{A})^*$ for all $\mathscr{A} \in \mathcal{A}$.

Claim 5. *ψ is a derivation.*

For any $\mathscr{A}, \mathscr{B} \in \mathcal{A}$, we see that

$$\begin{aligned}(1+\eta)\psi(\mathscr{A}\mathscr{B} + \mathscr{B}\mathscr{A}^*) &= \psi(I \circ_\eta \mathscr{A} \bullet \mathscr{B}) \\ &= I \circ_\eta \psi(\mathscr{A}) \bullet \mathscr{B} + I \circ_\eta \mathscr{A} \bullet \psi(\mathscr{B}) \\ &= (1+\eta)(\psi(\mathscr{A})\mathscr{B} + \mathscr{B}\psi(\mathscr{A})^*) \\ &\quad + (1+\eta)(\mathscr{A}\psi(\mathscr{B}) + \psi(\mathscr{B})\mathscr{A}^*)\end{aligned} \tag{18}$$

Setting $\mathscr{A}$ (resp. $\mathscr{B}$) as $-i\mathscr{A}$ (resp. $i\mathscr{B}$) in above relation gives

$$\begin{aligned}(1+\eta)\psi(\mathscr{A}\mathscr{B} - \mathscr{B}\mathscr{A}^*) &= (1+\eta)(\psi(\mathscr{A})\mathscr{B} - \mathscr{B}\psi(\mathscr{A})^*) \\ &\quad + (1+\eta)(\mathscr{A}\psi(\mathscr{B}) - \psi(\mathscr{B})\mathscr{A}^*)\end{aligned} \tag{19}$$

for all $\mathscr{A}, \mathscr{B} \in \mathcal{A}$. Addition of (18) and (19) yields

$$\psi(\mathscr{A}\mathscr{B}) = \psi(\mathscr{A})\mathscr{B} + \mathscr{A}\psi(\mathscr{B})$$

for all $\mathscr{A}, \mathscr{B} \in \mathcal{A}$. Thereby the proof of the theorem is completed.

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