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On new versions of Hermite-Hadamard-type inequalities based on tempered fractional integrals

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Abstract. This research is on the new versions of Hermite–Hadamard type inequalities. These inequalities established by means of convex mappings include tempered fractional integral operators. Obtaining these inequalities, well-known Hölder inequality and power mean inequality are also utilized. The resulting Hermite–Hadamard type inequalities are a generalization of some of the studies on this subject, including Riemann-Liouville fractional integrals. What's more, new results are obtained through special choices.

1. Introduction & Preliminaries

Convex theory is a subject that has been used in many fields of optimization theory, energy systems, engineering applications, and physics. Convexity theory has an important place in these branches of mathematics, especially in inequalities. Hermite-Hadamard, midpoint-type and trapezoid-type inequalities are the most well-known of these inequalities. Hermite–Hadamard-type inequalities which have been first investigated by C. Hermite and J. Hadamard for the case of convex functions. If $f : I \to \mathbb{R}$ is a convex function on the interval I of real numbers and $a, b \in I$ with $a < b$, then the following double inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Midpoint-type inequalities which are the left hand side of (1) and trapezoid-type inequalities which are the right hand side of (1). If f is concave, then both inequalities in are valid in the reverse direction.

On the other hand, the fractional calculus has a long history in the literature. The origin of fractional calculus can be traced back to the letter between Leibniz and L'Hopital. The improvement of the theories of fractional calculus is well contributed by several mathematicians and physicists in the past three centuries. Hence, the books covering fractional calculus began to emerge from the last century, such as Oldham and Spanier (1974), Samko, Kilbas and Marichev (1993), Podlubny (1999), and so on. In recent years, more theories and experiments show that a broad range of non-classical phenomena appeared in the applied

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sciences and engineering can be defined by fractional calculus [14, 17, 18]. Owing to its good mathematical features, nowadays fractional calculus has become a powerful tool in depicting the anomalous kinetics which arises in physics, biology, chemistry, and other complex dynamics [14]. In practical applications, some different kinds of fractional derivatives, such as Riemann-Liouville fractional derivative, Caputo fractional derivative [17, 18], Riesz fractional derivative [18], and Hilfer fractional derivative [6, 22] are introduced.

Now, we will present several necessary definitions in order to create our main results.

Definition 1.1. *The gamma function, incomplete gamma function, λ-incomplete gamma function are defined by*

$$\Gamma(\alpha) := \int_0^\infty t^{\alpha-1}e^{-t}dt,$$

$$\curlyvee(\alpha, x) := \int_0^x t^{\alpha-1}e^{-t}dt,$$

and

$$\curlyvee_\lambda(\alpha, x) := \int_0^x t^{\alpha-1}e^{-\lambda t}dt,$$

respectively. Here, $0 < \alpha < \infty$ *and* $\lambda \geq 0$.

There are some properties λ-incomplete gamma function as follows:

Remark 1.2. *[15] For the real numbers* $\alpha > 0$*;* $x, \lambda \geq 0$ *and* $a < b$*, we have*

1. $\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha, 1) = \int_0^1 t^{\alpha-1}e^{-\lambda\left(\frac{b-a}{2}\right)t}dt = \left(\frac{2}{b-a}\right)^\alpha \curlyvee_\lambda(\alpha, b-a),$
2. $\int_0^1 \curlyvee_{\lambda(b-a)}(\alpha, x)\,dx = \frac{\curlyvee_\lambda(\alpha, b-a)}{(b-a)^\alpha} - \frac{\curlyvee_\lambda(\alpha+1, b-a)}{(b-a)^{\alpha+1}}.$

Riemann-Liouville integral operators are defined by as follows:

Definition 1.3. *[9] For* $f \in L_1[a, b]$*, the Riemann-Liouville integrals of order* $\alpha > 0$ *are given by*

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a \tag{2}$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b. \tag{3}$$

Obviously, the Riemann-Liouville integrals will be equal to classical integrals for the condition $\alpha = 1$.

Dragomir and Agarwal first established trapezoid-type inequalities for the case of convex functions in [4], while Kirmacı first proved midpoint-type inequalities for the case of convex functions in [10]. Sarikaya et al. and Iqbal et al. presented some fractional midpoint-type inequalities and trapezoid-type inequalities for the case of convex functions in papers [7] and [19], respectively. For more information about fractional integral inequalities, see [2, 16] and the references cited therein.

Now, we recall the fundamental definitions and new notations of tempered fractional operators.

Definition 1.4. *[11, 13] The fractional tempered integral operators $\mathcal{J}_{a+}^{(\alpha,\lambda)}f$ and $\mathcal{J}_{b-}^{(\alpha,\lambda)}f$ of order $\alpha > 0$ and $\lambda \geq 0$ are presented by*

$$\mathcal{J}_{a+}^{(\alpha,\lambda)}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}e^{-\lambda(x-t)}f(t)dt, \quad x \in [a,b] \tag{4}$$

and

$$\mathcal{J}_{b-}^{(\alpha,\lambda)}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}e^{-\lambda(t-x)}f(t)dt, \quad x \in [a,b], \tag{5}$$

respectively for $f \in L_1[a,b]$.

Obviously, if we consider $\lambda = 0$, then the fractional integral in (4) reduces to the Riemann-Liouville fractional integral in (2). Furthermore, the fractional integral in (5) coincides with the Riemann-Liouville fractional integral in (3) when $\lambda = 0$.

Tempered fractional calculus can be recognized as the generalization of fractional calculus. To the best of our knowledge, the definitions of fractional integration with weak singular and exponential kernels were firstly reported in Buschman's earlier work [3]. For the other different definitions of the tempered fractional integration, see the books [12, 18, 21] and references therein. Mohammed et al. [15] are established some Hermite–Hadamard-type involved the tempered fractional integrals for the case of convex functions which cover the previously published result such as Riemann integrals, Riemann-Liouville fractional integrals. More precisely, the authors followed the Sarikaya et al. [19] and Sarikaya and Yildirim [20] technique to establish some Hermite–Hadamard-type inequalities (including both trapezoidal and midpoint type) which involved the tempered fractional integrals.

With the help of the continuing research and mentioned papers above, we will acquire several Hermite–Hadamard-type inequalities via differentiable convex mappings involving tempered fractional integral operators. The entire form of study takes the form of five sections including the introduction. Here, the fundamentals definitions of Riemann-Liouville integral operators and tempered fractional integrals are explained for building our main results. In addition, recalls will be made about gamma, incomplete gamma function, and λ-incomplete gamma function, which are well-known in the literature. In Section 2, we prove some new version of Hermite–Hadamard-type inequalities via convex mappings with the help of tempered fractional integrals. More precisely, Hölder and power-mean inequalities, which are well-known in the literature, will use in some of the proven inequalities. In Section 3, we present some new version of trapezoid-type inequalities via convex mappings involving tempered fractional integrals. In Section 4, we acquire some new version of midpoint-type inequalities for convex functions including tempered fractional integrals. Furthermore, we also present some corollaries and remarks. Finally, in Section 5, ideas that will guide the researchers will be given. Interested researchers will be informed that new versions of the inequalities we have acquired can be derived via different fractional integrals.

2. Hermite–Hadamard-type Inequality Involving Tempered Fractional Integrals

In this section, we use tempered fractional integrals to construct Hermite–Hadamard type inequalities via differentiable convex functions. First, let's set up the following identity to obtain Hermite–Hadamard type inequalities.

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$. Then, the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha)}{2\ \gamma_\lambda\left(\alpha, \frac{b-a}{2}\right)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right) + \mathcal{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right] \leq \frac{f(a)+f(b)}{2} \tag{6}$$

with $\lambda \geq 0, \alpha > 0$.

Proof. Since f is convex function on $[a,b]$, we can write

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \tag{7}$$

for all $x, y \in [a,b]$.

If we take $x = \frac{1+t}{2}a + \frac{1-t}{2}b$ and $y = \frac{1-t}{2}a + \frac{1+t}{2}b$ with $t \in [0,1]$ in (7) and then using convexity of f, we obtain

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) + f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right] \\ &\leq \frac{1}{2}\left[\frac{1+t}{2}f(a) + \frac{1-t}{2}f(b) + \frac{1-t}{2}f(a) + \frac{1+t}{2}f(b)\right] \\ &= \frac{f(a)+f(b)}{2}. \end{aligned}$$

That is,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left[f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) + f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right] \leq \frac{f(a)+f(b)}{2}. \tag{8}$$

Multiplying the double inequality (8) by $t^{\alpha}e^{-\lambda\frac{(b-a)}{2}t}$ and then integrating the resulting inequality respect to t over $[0,1]$, we have

$$\begin{aligned} & f\left(\frac{a+b}{2}\right)\int_0^1 t^{\alpha-1}e^{-\lambda\frac{(b-a)}{2}t}dt \\ & \leq \frac{1}{2}\left[\int_0^1 t^{\alpha-1}e^{-\lambda\frac{(b-a)}{2}t} f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)dt + \int_0^1 t^{\alpha-1}e^{-\lambda\frac{(b-a)}{2}t} f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)dt\right] \\ & \leq \frac{f(a)+f(b)}{2}\int_0^1 t^{\alpha-1}e^{-\lambda\frac{(b-a)}{2}t}dt. \end{aligned} \tag{9}$$

Now, we calculate the integrals in (9). It is clear that

$$\int_0^1 t^{\alpha-1}e^{-\lambda\frac{(b-a)}{2}t}dt = \left(\frac{2}{b-a}\right)^{\alpha} \curlyvee_{\lambda}\left(\alpha, \frac{b-a}{2}\right). \tag{10}$$

Using the change of variable $x = \frac{1+t}{2}a + \frac{1-t}{2}b$ for $t \in [0,1]$, we obtain

$$\int_0^1 t^{\alpha-1} e^{-\lambda \frac{(b-a)}{2} t} f\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right) dt \tag{11}$$

$$= \int_0^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right)^{\alpha-1} \left(\frac{2}{b-a}\right)^{\alpha} e^{-\lambda\left(\frac{a+b}{2}-x\right)} f(x)dx$$

$$= \left(\frac{2}{b-a}\right)^{\alpha} \Gamma(\alpha)\, J_{a+}^{(\alpha,\lambda)} f\left(\frac{a+b}{2}\right).$$

Similarly, we have

$$\int_0^1 t^{\alpha-1} e^{-\lambda \frac{(b-a)}{2} t} f\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right) dt \tag{12}$$

$$= \left(\frac{2}{b-a}\right)^{\alpha} \Gamma(\alpha)\, J_{b-}^{(\alpha,\lambda)} f\left(\frac{a+b}{2}\right).$$

Substituting the equalities (10)-(12) into (9) and then multiplying the obtained result by $\frac{\left(\frac{2}{b-a}\right)^{\alpha}}{\curlyvee_{\lambda}\left(\alpha, \frac{b-a}{2}\right)}$, we deduce that

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha)}{2\ \curlyvee_{\lambda}\left(\alpha, \frac{b-a}{2}\right)} \left[\mathcal{J}_{a+}^{(\alpha,\lambda)} f\left(\frac{a+b}{2}\right) + \mathcal{J}_{b-}^{(\alpha,\lambda)} f\left(\frac{a+b}{2}\right)\right] \leq \frac{f(a)+f(b)}{2}.$$

Consequently, we obtained the desired result. □

Remark 2.2. *If we choose $\lambda = 0$ in (6), it gives the double inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha)}{2\ \curlyvee_{\lambda}\left(\alpha, \frac{b-a}{2}\right)} \left[J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right)\right] \leq \frac{f(a)+f(b)}{2},$$

which is given by Ertugral et al. in [5].

3. Trapezoid-type Inequality Involving Tempered Fractional Integrals

In this section, we use tempered fractional integrals to construct trapezoid-type inequalities for differentiable convex mappings. First, let's set up the following identity to establish trapezoid-type inequalities.

Lemma 3.1. *Consider that $f : [a,b] \to \mathbb{R}$ is a differentiable mapping on (a,b) such that $f' \in L_1[a,b]$. Then, the* *following equality holds:*

$$\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha)}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathcal{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right] \tag{13}$$

$$=\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{1}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}$$

$$\times\int_0^1\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\left[f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt.$$

Proof. Employing the integration by parts, it follows

$$I_1=\int_0^1\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\left[f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right]dt \tag{14}$$

$$=\frac{2}{b-a}\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\,f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\Big|_0^1$$

$$-\frac{2}{b-a}\int_0^1 t^{\alpha-1}e^{-\lambda\left(\frac{b-a}{2}\right)t}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt$$

$$=\left(\frac{2}{b-a}\right)^{\alpha+1}\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)f(b)-\left(\frac{2}{b-a}\right)^{\alpha+1}\Gamma(\alpha)\,J_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right).$$

Similarly,

$$I_2=\int_0^1\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\left[f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt \tag{15}$$

$$=-\left(\frac{2}{b-a}\right)^{\alpha+1}\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)f(a)-\left(\frac{2}{b-a}\right)^{\alpha+1}\Gamma(\alpha)\,J_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right).$$

From (14) and (15), if we examine the following calculation

$$\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{[I_1-I_2]}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}$$

$$=\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha)}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[J_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+J_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right],$$

which completes the proof of Lemma 3.1. □

Theorem 3.2. *Suppose that $f:[a,b]\to\mathbb{R}$ is a differentiable mapping on (a,b) and $\left|f'\right|$ is convex on $[a,b]$. Under these conditions, the following inequality is derived:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha)}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathcal{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{\varphi_1(\alpha,\lambda)}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left\{\left|f'(b)\right|+\left|f'(a)\right|\right\},$$

where

$$\varphi_1(\alpha,\lambda)=\int_0^1\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\,dt=\left(\frac{2}{b-a}\right)^{\alpha}\left[\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)-\frac{2}{b-a}\curlyvee_{\lambda}\left(\alpha+1,\frac{b-a}{2}\right)\right].\tag{16}$$

Proof. If we take the absolute value of both sides of (13), then we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha)}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathcal{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right]\right|\tag{17}$$

$$\leq\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{1}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\int_0^1\left|\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt$$

$$+\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{1}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\int_0^1\left|\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt.$$

It is known that $\left|f'\right|$ is convex on $[a,b]$. It follows

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha)}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathcal{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{1}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}$$

$$\times\int_0^1\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\left(\frac{1-t}{2}\left|f'(b)\right|+\frac{1+t}{2}\left|f'(a)\right|+\frac{1-t}{2}\left|f'(a)\right|+\frac{1+t}{2}\left|f'(b)\right|\right)dt$$

$$=\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{1}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\int_0^1\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\left\{\left|f'(b)\right|+\left|f'(a)\right|\right\}dt.$$

Thus, the proof of Theorem 3.2 is finished. □

Remark 3.3. *If we set $\lambda = 0$ in Theorem 3.2, then we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{b-}^{\alpha} f\left(\frac{a+b}{2}\right) + J_{a+}^{\alpha} f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \frac{b-a}{4(\alpha+1)}\left[\left|f'(b)\right| + \left|f'(a)\right|\right],$$

which is given by Budak et al. in [1, Remark 7].

Theorem 3.4. *Suppose that $f : [a,b] \to \mathbb{R}$ is a differentiable mapping on (a,b), such that $f' \in L_1[a,b]$. In addition, suppose that $\left|f'\right|^q$ is convex on $[a,b]$ with $q > 1$. Then the following double inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha)}{2\curlyvee_{\lambda}\left(\alpha, \frac{b-a}{2}\right)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)} f\left(\frac{a+b}{2}\right) + \mathcal{J}_{b-}^{(\alpha,\lambda)} f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \left(\frac{b-a}{2}\right)^{\alpha+1}\frac{\left(\psi_1^p(\alpha,\lambda)\right)^{\frac{1}{p}}}{2\curlyvee_{\lambda}\left(\alpha, \frac{b-a}{2}\right)}\left\{\left(\frac{\left|f'(b)\right|^q + 3\left|f'(a)\right|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{\left|f'(a)\right|^q + 3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\}$$

$$\leq \left(\frac{b-a}{2}\right)^{\alpha+1}\frac{\left(4\psi_1^p(\alpha,\lambda)\right)^{\frac{1}{p}}}{2\curlyvee_{\lambda}\left(\alpha, \frac{b-a}{2}\right)}\left\{\left|f'(a)\right| + \left|f'(b)\right|\right\}.$$

Here, $\frac{1}{p} + \frac{1}{q} = 1$ and

$$\psi_1^p(\alpha,\lambda) = \int_0^1 \left(\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right)^p dt.$$

Proof. If Hölder's inequality is used in (17), then we acquire

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha)}{2\curlyvee_{\lambda}\left(\alpha, \frac{b-a}{2}\right)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)} f\left(\frac{a+b}{2}\right) + \mathcal{J}_{b-}^{(\alpha,\lambda)} f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \left(\frac{b-a}{2}\right)^{\alpha+1}\frac{1}{2\curlyvee_{\lambda}\left(\alpha, \frac{b-a}{2}\right)}\left(\int_0^1 \left|\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(\frac{1-t}{2}a + \frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+ \left(\frac{b-a}{2}\right)^{\alpha+1}\frac{1}{2\curlyvee_{\lambda}\left(\alpha, \frac{b-a}{2}\right)}\left(\int_0^1 \left|\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(\frac{1+t}{2}a + \frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}.$$

Applying the convexity of $\left|f'\right|^q$ on $[a,b]$, we have the following inequality

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha)}{2\curlyvee_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left[J_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+J_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{1}{2\curlyvee_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left\{\left(\int_0^1\left(\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right)^p dt\right)^{\frac{1}{p}}\right.$$

$$\left.\times\left[\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right]\right\}$$

$$=\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{\left(\psi_1^\alpha(\lambda,p)\right)^{\frac{1}{p}}}{2\curlyvee_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left\{\left(\frac{\left|f'(b)\right|^q+3\left|f'(a)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\}.$$

The second inequality of Theorem 3.4 can be acquired immediately by letting $\vartheta_1=3\left|f''(a)\right|^q$, $\varrho_1=\left|f''(b)\right|^q$, $\vartheta_2=\left|f''(a)\right|^q$ and $\varrho_2=3\left|f''(b)\right|^q$ and applying the inequality:

$$\sum_{k=1}^n(\vartheta_k+\varrho_k)^s\leq\sum_{k=1}^n\vartheta_k^s+\sum_{k=1}^n\varrho_k^s,\quad 0\leq s<1.$$

Thus, the proof of Theorem 3.4 is completed. □

Remark 3.5. *If Theorem 3.4 is evaluated as $\lambda=0$, then the following result is obtained:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{b-}^\alpha f\left(\frac{a+b}{2}\right)+J_{a+}^\alpha f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\frac{b-a}{4}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{\left|f'(b)\right|^q+3\left|f'(a)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\}$$

$$\leq\frac{b-a}{4}\left(\frac{4}{\alpha p+1}\right)^{\frac{1}{p}}\left\{\left|f'(a)\right|+\left|f'(b)\right|\right\},$$

which is given by Mohammed et al. in [16, Theorem 2].

Theorem 3.6. *Consider the existence of a differentiable mapping such that $f:[a,b]\to\mathbb{R}$ on (a,b) and $f'\in L_1[a,b]$.*

Let's also assume that the function $\left|f'\right|^q$ is convex on $[a,b]$ with $q \geq 1$. Then, the following inequality is established:

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha)}{2\curlyvee_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathcal{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \left(\frac{b-a}{2}\right)^{\alpha+1}\frac{(\varphi_1(\alpha,\lambda))^{1-\frac{1}{q}}}{2\curlyvee_\lambda\left(\alpha,\frac{b-a}{2}\right)}$$

$$\times\left\{\left(\frac{(\varphi_1(\alpha,\lambda)+\varphi_2(\alpha,\lambda))}{2}\left|f'(b)\right|^q+\frac{(\varphi_1(\alpha,\lambda)-\varphi_2(\alpha,\lambda))}{2}\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{(\varphi_1(\alpha,\lambda)-\varphi_2(\alpha,\lambda))}{2}\left|f'(b)\right|^q+\frac{(\varphi_1(\alpha,\lambda)+\varphi_2(\alpha,\lambda))}{2}\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right\}.$$

Here, $\varphi_1(\alpha,\lambda)$ is described as in (16) and

$$\varphi_2(\alpha,\lambda) = \int_0^1 t\left(\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right)dt$$

$$= \left(\frac{2}{b-a}\right)^{\alpha+1}\left[\curlyvee_\lambda\left(\alpha+1,\frac{b-a}{2}\right)-\frac{2}{b-a}\curlyvee_\lambda\left(\alpha+2,\frac{b-a}{2}\right)\right].$$

Proof. With the help of the power-mean inequality, we have

$$\left|\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha)}{2\curlyvee_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathcal{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq \left(\frac{b-a}{2}\right)^{\alpha+1}\frac{1}{2\curlyvee_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left\{\left(\int_0^1\left|\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right|dt\right)^{1-\frac{1}{q}}\right.$$

$$\times\left[\left(\int_0^1\left|\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\int_0^1\left|\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}\right]\right\}.$$

Since $\left|f'\right|^q$ is convex on $[a,b]$, we establish

$$\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha)}{2\Gamma_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathcal{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{1}{2\curlyvee_\lambda\left(\alpha,\frac{b-a}{2}\right)}\left\{\left(\int_0^1\left(\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right)dt\right)^{1-\frac{1}{q}}\right.$$

$$\times\left[\left(\int_0^1\left|\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right|\left(\frac{1+t}{2}\left|f'(b)\right|^q+\frac{1-t}{2}\left|f'(a)\right|^q\right)dt\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\int_0^1\left|\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right|\left(\frac{1+t}{2}\left|f'(a)\right|^q+\frac{1-t}{2}\left|f'(b)\right|^q\right)dt\right)^{\frac{1}{q}}\right]\right\}.$$

With this calculation, the proof ends. □

Corollary 3.7. *If $\lambda=0$ in Theorem 3.6, then the following inequality is obtained*

$$\left|\frac{f(a)+f(b)}{2}-\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J_{b-}^\alpha f\left(\frac{a+b}{2}\right)+J_{a+}^\alpha f\left(\frac{a+b}{2}\right)\right]\right|$$

$$\leq\frac{b-a}{2^{\frac{1}{q}+2}(\alpha+1)}\left[\left(\frac{(2\alpha+3)\left|f'(b)\right|^q+\left|f'(a)\right|^q}{\alpha+2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{\left|f'(b)\right|^q+(2\alpha+3)\left|f'(a)\right|^q}{\alpha+2}\right)^{\frac{1}{q}}\right].$$

Remark 3.8. *If we choose $\alpha=1$ and $\lambda=0$ in Theorem 3.6, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right|$$

$$\leq\frac{b-a}{8}\left[\left(\frac{5\left|f'(b)\right|^q+\left|f'(a)\right|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(b)\right|^q+5\left|f'(a)\right|^q}{6}\right)^{\frac{1}{q}}\right],$$

which is given by Budak et al. in paper [2, Remark 5.6].

4. Midpoint-type Inequality Involving Tempered Fractional Integrals

In this section, we use tempered fractional integrals to construct midpoint-type inequalities with the help of the differentiable convex functions. First, let's set up the following identity to acquire midpoint-type inequalities.

Lemma 4.1. *Consider that $f:[a,b]\to\mathbb{R}$ is a differentiable mapping on (a,b) such that $f'\in L_1[a,b]$. Then, the following equality holds:*

$$\frac{\Gamma(\alpha)}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathcal{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right) \tag{18}$$

$$=\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{1}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\int_0^1\left\{\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right\}$$

$$\times\left[f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)-f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right]dt.$$

Proof. By using the integration by parts, we have

$$I_3=\int_0^1\left\{\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right\}f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt \tag{19}$$

$$=\frac{2}{b-a}\left\{\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right\}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\Bigg|_0^1$$

$$+\frac{2}{b-a}\int_0^1 t^{\alpha-1}e^{-\lambda\left(\frac{b-a}{2}\right)t}f\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)dt$$

$$=-\left(\frac{2}{b-a}\right)^{\alpha+1}\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)f\left(\frac{a+b}{2}\right)+\left(\frac{2}{b-a}\right)^{\alpha+1}\Gamma(\alpha)\,J_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right).$$

Similarly,

$$I_4=\int_0^1\left\{\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right\}f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)dt \tag{20}$$

$$=\left(\frac{2}{b-a}\right)^{\alpha+1}\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)f\left(\frac{a+b}{2}\right)-\left(\frac{2}{b-a}\right)^{\alpha+1}\Gamma(\alpha)\,J_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right).$$

From (19) and (20), if we examine the following calculation

$$\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{[I_3-I_4]}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}$$

$$=\frac{\Gamma(\alpha)}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathcal{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right).$$

Thus, the proof of Lemma 4.1 is finished. □

Theorem 4.2. *Assume that $f:[a,b]\to\mathbb{R}$ is a differentiable mapping on (a,b) and $|f'|$ is convex on $[a,b]$. Under these conditions, the following inequality is derived:*

$$\left|\frac{\Gamma(\alpha)}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathcal{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{\varphi_3(\alpha,\lambda)}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left\{|f'(b)|+|f'(a)|\right\},$$

where

$$\varphi_3(\alpha,\lambda)=\int_0^1\left\{\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right\}dt=\left(\frac{2}{b-a}\right)^{\alpha+1}\curlyvee_{\lambda}\left(\alpha+1,\frac{b-a}{2}\right).$$

Proof. Let us take the absolute value of both sides of (18). Then, we obtain

$$\left|\frac{\Gamma(\alpha)}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathcal{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right| \tag{21}$$

$$\leq\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{1}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\int_0^1\left|\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|dt$$

$$+\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{1}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\int_0^1\left|\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|dt.$$

It is known that $\left|f'\right|$ is convex on $[a,b]$. It follows

$$\left|\frac{\Gamma(\alpha)}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathcal{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{1}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\int_{0}^{1}\left\{\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right\}$$

$$\times\left(\frac{1-t}{2}\left|f'(b)\right|+\frac{1+t}{2}\left|f'(a)\right|+\frac{1-t}{2}\left|f'(a)\right|+\frac{1+t}{2}\left|f'(b)\right|\right)dt$$

$$=\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{1}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\int_{0}^{1}\left\{\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right\}\left[\left|f'(b)\right|+\left|f'(a)\right|\right]dt.$$

Thus, the proof of Theorem 4.2 is finished. □

Remark 4.3. *If we set $\lambda=0$ in Theorem 4.2, then we have*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)\alpha}{4(\alpha+1)}\left[\left|f'(b)\right|+\left|f'(a)\right|\right],$$

which is given by Budak et al. in [5].

Theorem 4.4. *Suppose that $f:[a,b]\to\mathbb{R}$ is a differentiable mapping on (a,b), such that $f'\in L_1[a,b]$. In addition, suppose that $\left|f'\right|^q$ is convex on $[a,b]$ with $q>1$. Then the following inequalities can be written*

$$\left|\frac{\Gamma(\alpha)}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathcal{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{\left(\psi_2^p(\alpha,\lambda)\right)^{\frac{1}{p}}}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left\{\left(\frac{\left|f'(b)\right|^q+3\left|f'(a)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right\}$$

$$\leq\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{\left(4\psi_2^p(\alpha,\lambda)\right)^{\frac{1}{p}}}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left\{\left|f'(a)\right|+\left|f'(b)\right|\right\}.$$

Here, $\frac{1}{p}+\frac{1}{q}=1$ and

$$\psi_2^p(\alpha,\lambda)=\int_{0}^{1}\left(\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right)^p dt.$$

Proof. If we use Hölder's inequality in (21), then we get

$$\left|\frac{\Gamma(\alpha)}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathcal{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\frac{\left(\frac{b-a}{2}\right)^{\alpha+1}}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left(\int_0^1\left|\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\frac{\left(\frac{b-a}{2}\right)^{\alpha+1}}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left(\int_0^1\left|\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^q dt\right)^{\frac{1}{q}}$$

Using the convexity of $\left|f'\right|^q$ on $[a,b]$, we have the following

$$\left|\frac{\Gamma(\alpha)}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathcal{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{1}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left\{\left(\int_0^1\left(\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right)^p dt\right)^{\frac{1}{p}}\right.$$

$$\left.\times\left[\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3\left|f'(a)\right|^q+\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right]\right\}$$

$$=\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{\left(\psi_2^p(\alpha,\lambda)\right)^{\frac{1}{p}}}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[\left(\frac{\left|f'(b)\right|^q+3\left|f'(a)\right|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+3\left|f'(b)\right|^q}{4}\right)^{\frac{1}{q}}\right].$$

The second inequality of Theorem 4.4 can be acquired immediately by letting $\vartheta_1=3\left|f''(a)\right|^q$, $\varrho_1=\left|f''(b)\right|^q$, $\vartheta_2=\left|f''(a)\right|^q$ and $\varrho_2=3\left|f''(b)\right|^q$ and applying the inequality:

$$\sum_{k=1}^n(\vartheta_k+\varrho_k)^s\leq\sum_{k=1}^n\vartheta_k^s+\sum_{k=1}^n\varrho_k^s,\quad 0\leq s<1.$$

Thus, the proof of Theorem 4.4 is completed. □

Corollary 4.5. *If we consider $\lambda = 0$ in Theorem 4.4, then the following result is obtained*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\left(\frac{b-a}{4}\right)\left(\psi_2^p(\alpha,0)\right)^{\frac{1}{p}}\left\{\left(\frac{|f'(b)|^q+3|f'(a)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right\}$$

$$\leq\left(\frac{b-a}{4}\right)\left(4\psi_2^p(\alpha,0)\right)^{\frac{1}{p}}\left\{|f'(a)|+|f'(b)|\right\}.$$

Remark 4.6. *If we assign $\alpha = 1$ and $\lambda = 0$ in Theorem 4.4, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\left(\frac{b-a}{4}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left\{\left(\frac{|f'(b)|^q+3|f'(a)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{\frac{1}{q}}\right\}$$

$$\leq\left(\frac{b-a}{4}\right)\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left\{|f'(a)|+|f'(b)|\right\},$$

which is given in paper [10, Theorem 2.4].

Theorem 4.7. *Consider the existence of a differentiable mapping such that $f : [a, b] \to \mathbb{R}$ on (a, b) and $f' \in L_1[a, b]$. Let's also assume that the function $|f'|^q$ is convex on $[a, b]$ with $q \geq 1$. Then, the following inequality is established:*

$$\left|\frac{\Gamma(\alpha)}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathcal{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{(\varphi_3(\alpha,\lambda))^{1-\frac{1}{q}}}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}$$

$$\times\left\{\left(\frac{(\varphi_3(\alpha,\lambda)+\varphi_4(\alpha,\lambda))}{2}|f'(b)|^q+\frac{(\varphi_3(\alpha,\lambda)-\varphi_4(\alpha,\lambda))}{2}|f'(a)|^q\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{(\varphi_3(\alpha,\lambda)-\varphi_4(\alpha,\lambda))}{2}|f'(b)|^q+\frac{(\varphi_3(\alpha,\lambda)+\varphi_4(\alpha,\lambda))}{2}|f'(a)|^q\right)^{\frac{1}{q}}\right\}.$$

Here, $\varphi_3(\alpha,\lambda)$ is described as in Theorem 4.2. and

$$\varphi_4(\alpha,\lambda)=\int_0^1 t\left(\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right)dt$$

$$=\left(\frac{2}{b-a}\right)^{\alpha}\left[\frac{1}{2}\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)-\frac{2}{b-a}\curlyvee_{\lambda}\left(\alpha+1,\frac{b-a}{2}\right)+\left(\frac{2}{b-a}\right)^{2}\curlyvee_{\lambda}\left(\alpha+2,\frac{b-a}{2}\right)\right].$$

Proof. With the help of the power-mean inequality, we have

$$\left|\frac{\Gamma(\alpha)}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathcal{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{1}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left\{\left(\int_0^1\left|\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right|dt\right)^{1-\frac{1}{q}}\right.$$

$$\times\left[\left(\int_0^1\left|\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right|\left|f'\left(\frac{1-t}{2}a+\frac{1+t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\int_0^1\left|\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right|\left|f'\left(\frac{1+t}{2}a+\frac{1-t}{2}b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right]\right\}.$$

Since $\left|f'\right|^{q}$ is convex on $[a,b]$, we establish

$$\left|\frac{\Gamma(\alpha)}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left[\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)+\mathcal{J}_{b-}^{(\alpha,\lambda)}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq\left(\frac{b-a}{2}\right)^{\alpha+1}\frac{1}{2\curlyvee_{\lambda}\left(\alpha,\frac{b-a}{2}\right)}\left\{\left(\int_0^1\left(\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right)dt\right)^{1-\frac{1}{q}}\right.$$

$$\times\left[\left(\int_0^1\left|\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right|\left(\frac{1+t}{2}\left|f'(b)\right|^{q}+\frac{1-t}{2}\left|f'(a)\right|^{q}\right)dt\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\int_0^1\left|\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,1)-\curlyvee_{\lambda\left(\frac{b-a}{2}\right)}(\alpha,t)\right|\left(\frac{1+t}{2}\left|f'(a)\right|^{q}+\frac{1-t}{2}\left|f'(b)\right|^{q}\right)dt\right)^{\frac{1}{q}}\right]\right\}.$$

With this calculation, the proof ends. □

Corollary 4.8. *If $\lambda = 0$ in Theorem 4.7, then we have*

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{b-}^{\alpha}f\left(\frac{a+b}{2}\right)+J_{a+}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{b-a}{4(\alpha+1)}\left\{\left(\frac{(3\alpha+5)\left|f'(b)\right|^{q}+(\alpha+3)\left|f'(a)\right|^{q}}{4(\alpha+2)}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{(\alpha+3)\left|f'(b)\right|^{q}+(3\alpha+5)\left|f'(a)\right|^{q}}{4(\alpha+2)}\right)^{\frac{1}{q}}\right\}.$$

Remark 4.9. *If we select $\alpha = 1$ and $\lambda = 0$ in Theorem 4.7, then the following inequality holds:*

$$\left|\frac{1}{b-a}\int_{a}^{b}f(x)\,dx-f\left(\frac{a+b}{2}\right)\right|$$

$$\leq \frac{b-a}{8}\left[\left(\frac{2\left|f'(b)\right|^{q}+\left|f'(a)\right|^{q}}{3}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(b)\right|^{q}+2\left|f'(a)\right|^{q}}{3}\right)^{\frac{1}{q}}\right],$$

which is given in paper [7].

5. Conclusions

In the current research, we derive the new Hermite–Hadamard-type, trapezoid-type, and midpoint-type inequalities by making use of tempered fractional integrals. Convexity of the function, Hölder and power-mean inequalities are used in these inequalities. Furthermore, special choices of the variables in the theorems, generalizations of some articles, and new results were found. In the future, the authors may derive new inequalities of different fractional types related to these Hermite–Hadamard-type inequalities. Interested readers can also establish new inequalities using different kinds of convexities. These inequalities created are new as far as we know and according to the literature review. These inequalities will inspire new studies in various fields of mathematics.

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Conflict interests

The authors declare that they have no conflict interests.

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