Filomat 38:7 (2024), 2419–2425
https://doi.org/10.2298/FIL2407419T

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On ADS modules with the summand sum property

Özgür Taşdemir[a]

[a]*Trakya University, Faculty of Economics and Administrative Sciences, Department of Business Administration, Balkan Campus, Edirne, Türkiye*

Abstract. A module M is called *ADS*, if for any decomposition $M = K \oplus L$ of M and any complement N of K in M, $M = K \oplus N$. A module is said to have the *summand sum property* if the sum of any two direct summands is a direct summand. In this note, we investigate ADS modules with the summand sum property.

1. Introduction

The objective of this article is to examine the family of ADS modules having the summand sum property. Firstly, Fuchs [11] introduced ADS property for Abelian groups. In [7], Burgess and Raphael define that a module M is an *ADS module* if for any decomposition $M = K \oplus L$ of M and any complement N of K in M, $M = K \oplus N$. Later, ADS modules were studied by many authors [1, 21–23, 25, 26]. In [25], Takil Mutlu studied ADS modules having the summand intersection property (SIP), and the author calls M is an *SA module* if it is ADS having SIP (see also [26–28]). In this paper, we study on ADS modules with SSP. In Example 2.1, we provide some examples to show that the family of ADS modules and the family of SSP modules are different, and we define that a module M is *SSA* if M is ADS module having SSP. Firstly we remark that, SA modules are SSA (Example 2.4(1)). Some counter examples are provided in Example 2.9 that there exists an SSA module which is not SA. An equivalent condition for SSA modules is given in Theorem 2.3 which tell us that a module M is SSA iff for any direct summand K of M and any complement N of any direct summand of M, we have $K + N \leq^{\oplus} M$. We have shown which module properties imply the SSA property: If a module is both injective and prime then it is SSA, and every (weak) duo module is SSA, and an extending module with SSP is an SSA module. A necessary condition for the equivalence of SA and SSA module families is given in Proposition 2.10 that the family of SA modules and the family of SSA modules coincide for D4-modules. It is proved in Proposition 2.12 that a direct summand of an SSA module is SSA. But the direct sum of two SSA modules need not to be SSA (Example 2.13). We give a condition as to when direct sum of SSA modules is SSA that if K and L are any two SSA modules over a ring R satisfying that $r(K) + r(L) = R$, then the R-module $K \oplus L$ is SSA (Theorem 2.14). In Theorem 2.15, we prove that if M is an extending module, then M is SSA iff the sum of two closed submodules of M is closed.

In the rest of the paper, we characterize semisimple rings, right V-rings, right hereditary rings, von Neumann regular rings and right SI-rings with using SSA modules. For example, we prove in Remark

2020 *Mathematics Subject Classification*. Primary 16D50; Secondary 16D70, 16D80.
Keywords. ADS module, summand sum property, semisimple ring, V-ring, hereditary ring, regular ring, SI-ring.
Received: 25 February 2022; Revised: 20 September 2023; Accepted: 30 October 2023
Communicated by Dijana Mosić
Email address: ozgurtasdemir@trakya.edu.tr (Özgür Taşdemir)

2.17 that a ring R is semisimple iff any R-module is SSA; or equivalently, any finitely generated R-module is SSA; or equivalently, the direct sum of any two SSA modules is SSA. A ring R is a right V-ring iff any finitely cogenerated R-module is SSA; or equivalently, any finitely copresented R-module is SSA. A ring R is right hereditary iff any injective R-module is SSA; or equivalently, any factor module of an injective R-module is SSA; or equivalently, the sum of two injective submodule of any R-module is SSA. We provide new characterizations of regular rings that a ring R is regular iff any principal right ideal of $M_2(R)$ is SSA; or equivalently, any principal right ideal of $M_2(R)$ generated by a diagonal matrix is SSA; or equivalently, any finitely generated submodule of a projective right R-module is a direct summand; or equivalently, any finitely generated submodule of a projective right R-module is SSA; or equivalently, any 2-generated submodule of a projective right R-module is SSA. At the end of the paper, we prove in Theorem 2.20 that a ring R is right SI iff R_R is nonsingular and all singular right R-modules are SSA.

In this paper, R always denotes an associative ring with unity, and every module is unital, we use M_R to denote a right R-module. $K \cong K'$ illustrates that there is an isomorphism between K and K'. The injective hull of M will be denoted by $E(M)$. $K \leq M$, $K \leq^{ess} M$ and $K \leq^{\oplus} M$ denote K is a submodule of M, is essential in M and a direct summand of M, respectively. Wilson [29] defines a module M has *SIP* if for any $K, L \leq^{\oplus} M$, $K \cap L \leq^{\oplus} M$. Garcia [12] calls a module M has *SSP* if for any $K, L \leq^{\oplus} M$, $K + L \leq^{\oplus} M$. A module M is called:

- *extending* if any submodule is essential in a direct summand.
- $C3$ *module* if for any $K, L \leq^{\oplus} M$ with $K \cap L = 0$, $K \oplus L \leq^{\oplus} M$.
- *quasi-continuous* if it is extending having $C3$.

Let R be ring and n a positive integer, $M_n(R)$ denote the ring of $n \times n$ matrices over R. For any terminology, which is not presented, we suggest to see [4, 10, 19].

Now, we recall two known results that we will use frequently throughout the paper:

Lemma 1.1. *[7, Proposition 1.1] Amodule M is ADS iff for any decomposition $M = K \oplus L$, K and L are mutually injective.*

Lemma 1.2. *[2, Theorem 8] A module M has SSP iff for any decomposition $M = K \oplus L$ and any homomorphism $\sigma : K \to L$, $\sigma(K) \leq^{\oplus} L$.*

2. ADS modules with SSP: SSA Modules

At the beginning of the section, we give some examples to illustrate that the family of ADS modules and the family of SSP modules are different.

Example 2.1. 1. *Let F be a field and R be the ring below*
$R = \left\{ \begin{bmatrix} u & z & 0 & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & v & t \\ 0 & 0 & 0 & u \end{bmatrix} : u, v, z, t \in F \right\}$*, and $M = R_R$. M does not have SSP by [15, Example 2.6]. Since M is an injective right R-module by [17, Example 16.19], M is an ADS module.*

2. *Let $M = \mathbb{Q} \oplus (\mathbb{Z}/\mathbb{Z}p)$ where p is prime. There is four direct summands as follows: $0 \oplus 0$, M, $0 \oplus (\mathbb{Z}/\mathbb{Z}p)$ and $\mathbb{Q} \oplus 0$. Thus, it is obvious that M has SSP. But M is not ADS by [25, Example 2.4].*

From above mentioned examples, naturally, one might pose the question that "what can be said of a module which is ADS with the summand sum property?". Hence;

Definition 2.2. *We call a module M is SSA if it is ADS having SSP. This abbreviation is initials of "summand", "sum" and "absolute" words.*

Our next theorem gives an equivalent condition for SSA modules.

Theorem 2.3. *The following conditions are equivalent for an R-module M:*

1. *M is an SSA module;*
2. *for any direct summand K of M and any complement N of any direct summand of M, $K + N \leq^{\oplus} M$;*

Proof. (1) ⇒ (2) Assume M is SSA. Let $K, K' \leq^{\oplus} M$ and N a complement of K' in M. Since M is ADS, $M = K' \oplus N$, and hence $N \leq^{\oplus} M$. Thus $K + N \leq^{\oplus} M$ because M has SSP.
(2) ⇒ (1) Assume M has the property above. Let $M = K \oplus L$ be an arbitrary decomposition of M, and N be a relative complement of K in M. Then $K \oplus N \leq^{ess} M$. By the hypothesis, $K \oplus N \leq^{\oplus} M$. Thus, $M = K \oplus N$, and hence M is ADS. Now, let $K, L \leq^{\oplus} M$. Then $M = L \oplus L'$ for some $L' \leq M$. By [24, Lemma 2.2], L is a relative complement of L' in M. By (2), $K + L \leq^{\oplus} M$, and hence M has SSP. □

Clearly, uniform, indecomposable and semisimple modules are SSA modules. Now, we give some non-trivial examples of SSA modules:

Example 2.4. 1. *Every SA module is SSA. To show this, let M be SA. Then M is a C3 module by [21, Theorem 2.7]. Then by [2, Lemma 19(1)], M has SSP, and hence M is SSA.*
2. *If R is injective and right Ore domain, then the right R-module $R \oplus R$ is SSA by [6, Proposition 4] and [2, Lemma 19(1)].*
3. *Any injective prime module is SSA by [15, Proposition 2.1] and [2, Lemma 19(1)].*

A submodule X of M is called *fully invariant*, if for any endomorphism ρ of M, $\rho(X) \subseteq X$. A module is called *(weak) duo* if every (direct summand) submodule is fully invariant. A module is weak duo iff its endomorphism ring is Abelian (see [8, Theorem 4.4]). Takil Mutlu proved in [25, Proposition 2.15] that if a module is extending, duo, PQ-injective, then M is SA. We can drop the extending and PQ-injective assumptions of this result. Now, we prove that every (weak) duo module is an SA module (and hence an SSA module):

Proposition 2.5. *Any (weak) duo module is an SSA module.*

Proof. Let M be a weak duo module. Then by [18, Theorem 5] M has both SIP and SSP. It remains to show that any weak duo module is ADS. For this, let a weak duo module $M = K \oplus L$ be a direct sum of submodules K and L of M. Let N be a complement of K in M. By [8, Theorem 4.4], any direct summand of M is uniquely complemented. Thus $K \oplus N \leq^{ess} M = K \oplus L$. Since K is uniquely complemented, $N \leq^{ess} L$. It implies that $N = L$, i.e., $M = K \oplus N$. Thus, M is ADS. □

Corollary 2.6. *If the direct sum $K \oplus L$ is weak duo, then K and L are mutually injective.*

Proof. It is clear by Proposition 2.5 and Lemma 1.1. □

By Proposition 2.5, we can also state that any commutative ring is SA.

Proposition 2.7. *If a module is extending with SSP, then it is an SSA module.*

Proof. It is routine. □

Converse of this proposition is not true, in general. There exists an SSA module which is not extending:

Example 2.8. *Let $\mathbb{F}$ be a field and V be an $\mathbb{F}$-vector space with dim $V_{\mathbb{F}} = 2$. Let $V = v_1\mathbb{F} \oplus v_2\mathbb{F}$ and $R = \begin{bmatrix} \mathbb{F} & V \\ 0 & \mathbb{F} \end{bmatrix} = \left\{ \begin{bmatrix} f & v \\ 0 & f \end{bmatrix} : f \in \mathbb{F},\ v \in V \right\}$ the trivial extension of $\mathbb{F}$ by V. Since R is indecomposable as an R-module, R_R is an SSA module. Take $I_1 = \left\{ \begin{bmatrix} 0 & v_1 f \\ 0 & 0 \end{bmatrix} : f \in \mathbb{F} \right\}$ and $I_2 = \left\{ \begin{bmatrix} 0 & v_2 f \\ 0 & 0 \end{bmatrix} : f \in \mathbb{F} \right\}$. I_1 is a complement of I_2 in R_R. But I_1 is not a direct summand of R_R. So, R_R is not an extending module.*

Example 2.4(1) and the following two examples indicate that the class of SA modules is a proper subclass of SSA modules.

Example 2.9. 1. *Let $\mathbb{F}$ be a field and $R = \begin{bmatrix} \mathbb{F} & \mathbb{F} \\ 0 & \mathbb{F} \end{bmatrix}$. $I = \begin{bmatrix} 0 & \mathbb{F} \\ 0 & \mathbb{F} \end{bmatrix}$ and $J = \begin{bmatrix} \mathbb{F} & \mathbb{F} \\ 0 & 0 \end{bmatrix}$ are right R-modules. Let $L = R/J$ and $K = I \oplus L$. K has SSP by [12, Remark on page 81]. It is showed in [2, Example 3] that K does not have SIP, and also it is showed in [25, Example 2.3] that K is an ADS module. Consequently, K is SSA but it is not SA.*

2. *Consider $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_{p^\infty}$ as a $\mathbb{Z}$-module. The authors showed in [15, Example 2.4(1)] that M is not SIP. Since $\mathbb{Z}_{p^\infty}$ is an injective $\mathbb{Z}$-module and any finite direct sum of injective modules is injective, M is injective. On the other hand, since $\mathbb{Z}$ is a right hereditary ring, M is an SSA module by Remark 2.17(3). But M is not SA.*

Clear from the definitions, every module with SSP satisfies $C3$, and it is known from [21] that every ADS module satisfies the $C3$ condition. From the above mentioned examples and propositions, we have the following implications for an R-module M:

$$M \text{ is (weak) duo} \Rightarrow M \text{ is SA} \Rightarrow M \text{ is SSA} \Rightarrow \begin{array}{c} M \textit{ has SSP} \\ \textit{or} \\ M \textit{ is ADS} \end{array} \Rightarrow M \text{ is } C3$$

M is called a *$D4$-module* if $K, L \leq^{\oplus} M$ with $M = K + L$ and $K \cong L$, then $K \cap L \leq^{\oplus} M$ [9]. The next proposition gives a condition as to when the family of SSA modules and the family of SA modules are equivalent.

Proposition 2.10. *If M is a $D4$-module, then M is an SA module iff M is an SSA module.*

Proof. The necessity is clear. The converse follows by [9, Example 2.7]. □

Corollary 2.11. *For any ring R, R is right SA-ring iff it is right SSA-ring.*

Proof. It is clear because any projective (free) R-module is a $D4$-module. □

Right SA-rings were studied by Takil Mutlu in [26].

Proposition 2.12. *Let M be SSA, and $X \leq^{\oplus} M$. Then N is also SSA.*

Proof. Let M be an SSA module, and $X \leq^{\oplus} M$. Then $M = X' \oplus X$ for some $X' \leq M$. Let $K \leq^{\oplus} X$, and N be a complement of a direct summand L in X. We want to show that $K + N \leq^{\oplus} X$. Clearly, $K \leq^{\oplus} M$. On the other hand, there exists a submodule L' of X such that $X = L \oplus L'$. Clearly $M = X' \oplus X = X' \oplus L \oplus L'$. Clearly, N is a complement of $X' \oplus L$ in M by [24, Lemma 2.2]. Since M is SSA, we have $K + N \leq^{\oplus} M$, and hence $K + N \leq^{\oplus} X$. So, X is SSA. □

Now, we give various examples which show that direct sum of two SSA modules need not to be an SSA module.

Example 2.13. 1. *Let $\mathbb{Z}$ denote the ring of integer. Since $\mathbb{Z}_{\mathbb{Z}}$ is indecomposable, $\mathbb{Z}_{\mathbb{Z}}$ is an SSA module. $(\mathbb{Z} \oplus \mathbb{Z})_{\mathbb{Z}}$ does not have SSP (see [2, Ex. 5]), and it is not ADS by Lemma 1.1. So, it is not SSA.*

2. *Let $\mathbb{Z}_4 := \mathbb{Z}/4\mathbb{Z}$. Since $(\mathbb{Z}_4)_{\mathbb{Z}_4}$ is indecomposable, $(\mathbb{Z}_4)_{\mathbb{Z}_4}$ is an SSA module. The injective $\mathbb{Z}_4$-module $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ is an ADS module but it does not have SSP. So, it is not an SSA module.*

3. *Let p be a prime integer. It is obvious that $\mathbb{Q}$ and $\mathbb{Z}/\mathbb{Z}p$ is an SSA module. It is showed in Example 2.1(2) that $M = \mathbb{Q} \oplus (\mathbb{Z}/\mathbb{Z}p)$ has SSP, but it is not ADS. So, it is not SSA.*

In the next result, for any $K \leq M$, $r(K)$ denotes the right annihilator of K in R. Now, we give a condition as to when direct sum of SSA modules to be SSA.

Theorem 2.14. *Let K and L be any two SSA modules over a ring R satisfying that $r(K) + r(L) = R$. Then, the R-module $K \oplus L$ is SSA.*

Proof. Let X be a direct summand of $K \oplus L$. By [15, Proposition 3.9], $X = K_1 \oplus L_1$ where $K_1 \leq K$ and $L_1 \leq L$. Clearly, $K_1 \leq^{\oplus} K$ and $L_1 \leq^{\oplus} L$. Now, let Z be a complement of a direct summand Y in $K \oplus L$. Then again by [15, Proposition 3.9], $Y = K_2 \oplus L_2$ and $Z = K_2' \oplus L_2'$ where $K_2, K_2' \leq K$ and $L_2, L_2' \leq L$. We want to show that $X + Z \leq^{\oplus} K \oplus L$. Then,

$$K_2' \cap K_2 = (K_2' \oplus L_2') \cap K_2 \leq (K_2' \oplus L_2') \cap (K_2 \oplus L_2) = Z \cap Y = 0,$$

$$L_2' \cap L_2 = (K_2' \oplus L_2') \cap L_2 \leq (K_2' \oplus L_2') \cap (K_2 \oplus L_2) = Z \cap Y = 0,$$

and hence we have $K_2' \cap K_2 = 0$ and $L_2' \cap L_2 = 0$. , So, K_2' is a complement of K_2 in K, and L_2' is a complement of L_2 in L. Since K and L are SSA modules, $K_1 + K_2' \leq^{\oplus} K$ and $L_1 + L_2' \leq^{\oplus} L$. Thus,

$$(K_1 + K_2') \oplus (L_1 + L_2') = (K_1 + L_1) + (K_2' + L_2') = X + Z \leq^{\oplus} K \oplus L.$$

So, $K \oplus L$ is an SSA module. □

Recall that any quasi-continuous module is ADS, but may not have SSP, please see Example 2.1(1). Thus, a quasi-continuous module may not be SSA. On the other hand, an SSA module satisfies $C3$ but may not be an extending module, see Example 2.8. So, an SSA module need not to be quasi-continuous.

Theorem 2.15. *Let M be an extending module. M is an SSA module iff the sum of two closed submodules of M is closed.*

Proof. Suppose C and C' be two closed submodules of M. Since M is an extending module, we have C, $C' \leq^{\oplus} M$. Since M has SSP, $C + C' \leq^{\oplus} M$. Thus, $C + C'$ is closed in M. For the converse, let $K, L \leq^{\oplus} M$. By the hypothesis, $K + L$ is a closed in M. We have $K + L \leq^{\oplus} M$ as M is extending. Then M is quasi-continous. Thus, M is ADS by [7], and hence M is SSA. □

Theorem 2.16. *Let $M = M_1 \oplus M_2$ be an R-module, and $f : M_1 \to M_2$ a homomorphism, $E(M) = E_1 \oplus E_2$, where E_1 is injective hull of $Im(f)$, and $\pi : E(M) \to E_1$ projection map. If M is SSA, then $\pi(M) \subseteq^{\oplus} M$.*

Proof. Let M be SSA, and $\sigma : M_1 \to M_2$ be a homomorphism. Now $Im(\sigma) \leq^{\oplus} M$ because M has SSP. The submodule $E_2 \cap M$ is a complement of $Im(\sigma)$ in M. Let'see this: if N is a complement containing $E_2 \cap M$ in M and $n \in N$, it can be written that $n = e_1 + e_2$, $e_i \in E_i$. If $e_1 = 0$, we have $n = e_2 \in E_2 \cap M$. If $e_1 \neq 0$, then there is $r \in R$ with $0 \neq re_1 \in Im(\sigma)$. Then $rn = re_1 + re_2$ and $re_1 \in Im(\sigma) \cap N = 0$, and this contradicts to our assumption. Hence, $M = Im(\sigma) \oplus (E_2 \cap M)$ and $\pi(M) = Im(\sigma)$ because M is SSA. □

The rest of the paper, we characterize some well-studied rings with using SSA modules. As an analogue of [5, Corollary 3.3], it is easy to prove the fact that if $M \oplus E(M)$ is SSA, then M is an injective module. With using this fact and using similar techniques in [3, 5], it is not difficult to prove the next four results in Remark 2.17, thus, we give them without proofs.

Remark 2.17. *Let R be an arbitrary ring.*

(1) R is semisimple iff any R-module is SSA iff any finitely generated R-module is SSA iff any 2-generated R-module is SSA iff the direct sum of any two SSA modules is SSA iff any submodule of a projective R-module is SSA iff any submodule of $R \oplus R$ is SSA iff any submodule of an injective R-module is SSA iff any SSA R-module is injective.

(2) R is a right V-ring (that is, any simple right R-module is injective [30]) iff any finitely cogenerated R-module is SSA iff any finitely copresented R-module is SSA.

(3) R is right hereditary (that is any factor module of an injective module is injective [30, 39.16]) iff any injective R-module is SSA iff any factor module of an injective R-module is SSA iff the sum of two injective submodule of any R-module is SSA.

(4) R is a von Neumann regular ring iff any principal right ideal of $M_2(R)$ is SSA iff any principal right ideal of $M_2(R)$ generated by a diagonal matrix is SSA iff any finitely generated submodule of a projective right R-module is SSA iff any 2-generated submodule of a projective right R-module is SSA.

As a consequence of Remark 2.17(3), we can state the following two corollaries.

Corollary 2.18. *Let M be an arbitrary module over a right hereditary ring R. Then $E(M)$ is an SSA R-module.*

Corollary 2.19. *Let M be an arbitrary $\mathbb{Z}$-module (i.e., Abelian group). Then, $E(M)$ is an SSA module.*

There exists a module over a right hereditary ring which is not SSA, please see Example 2.13(1).

R is called a *right SI-ring* if all singular right R-modules are injective; or equivalently, all singular right R-modules are semisimple [14, Proposition 3.1].

Theorem 2.20. *The next conditions are equivalent for a ring R:*

(1) R is a right SI-ring;
(2) R is right nonsingular and all singular right R-modules are SA;
(3) R is right nonsingular and all singular right R-modules are SSA;
(4) R is right nonsingular and all singular right R-modules have SIP;
(5) R is right nonsingular and all singular right R-modules have SSP.

Proof. (1) $\Rightarrow$ (2) First note that any right SI-ring is right nonsingular by [10, p.127]. If R is a right SI-ring, then all singular right R-modules are injective, and hence ADS. Then by [14, Proposition 3.1], all singular right R-modules is semisimple, and hence SA.
(2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) Clear.
(4) $\Rightarrow$ (1) Let A be a singular right R-module. By [13, Proposition 1.22(b) and 1.23(c)], $E(A) \oplus E(E(A)/A)$ is singular. Then, by the hypothesis, $E(A) \oplus E(E(A)/A)$ has SIP. Let $\rho : E(A) \to E(A)/A$ be the canonical epimorphism, and $i : E(A)/A \to E(E(A)/A)$ be the inclusion map. Clearly, $Ker(io\rho) = Ker(\rho)$ since i is monomorphism. Then $Ker(io\rho) = Ker(\rho) = A \leq^{\oplus} E(A)$ by [16, Proposition 1]. Thus, A is injective. Consequently, R is right SI.
(5) $\Rightarrow$ (1) We consider an arbitrary singular R-module A. By above facts, $A \oplus E(A)$ is singular. By the hypothesis, $A \oplus E(A)$ has SSP. Let $i : A \to E(A)$ be an injection map. Then $A \cong i(A) \leq^{\oplus} E(A)$ by Lemma 1.2. Thus, A is injective, and hence R is right SI. □

Acknowledgement. The author is grateful to the referee(s) for careful review and for improving the presentation of the paper with their useful suggestions and comments. Special thanks to Prof. Dijana Mosić.

References

[1] A. Alahmadi, S. K. Jain, A. Leroy, *ADS modules*, J. Algebra **352(1)** (2012), 215-222.
[2] M. Alkan, A. Harmanci,*On summand sum and summand intersection property of modules*, Turkish J. Math. **26(2)** (2002), 131–148.
[3] I. Amin, Y. Ibrahim, M. F. Yousif, *C3-modules*, Algebra Colloq. **22(4)** (2015), 655-670.
[4] F. W. Anderson, K. R. Fuller, *Rings and Categories of Modules*, New York, Springer-Verlag, 1974.
[5] S. E. Atani, K. Khoramdel, S. D. P. Hesari, *C3-modules*, Demonstr. Math. **49(3)** 2016, 282-292.
[6] G. F. Birkenmeier, F. Karabacak, A. Tercan, *When is the SIP (SSP) property inherited by free modules*, Acta Math. Hungar. **112(1-2)** (2006), 103-106.
[7] W. D. Burgess, R. Raphael, *On modules with the absolute direct summand property*, Ring Theory, 137-148, Granville, OH, 1992, World Sci. Publ., River Edge, 1993.
[8] G. Călugăreanu, P. Schultz, *Modules with abelian endomorphism rings*, Bull. Aust. Math. Soc. **82(1)** (2010), 99-112.
[9] N. Ding, Y. Ibrahim, M. Yousif, Y. Zhou, *D4-modules*, J. Algebra Appl. **16(09)** (2017), 1750166.
[10] N. V. Dung, D. V. Huynh, P. F. Smith, R. Wisbauer, *Extending Modules*, (Pitman RN Mathematics 313), Longman, Harlow, 1994.
[11] L. Fuchs, *Infinite Abelian Groups I*, Pure and Applied Mathematics, Academic Press, New York-London 1970.
[12] J. L. Garcia, *Properties of direct summands of modules*, Comm. Algebra, **17(1)** (1989) 73-92.
[13] K. R. Goodearl, *Ring theory: Nonsingular rings and modules*, (Vol. 33). CRC Press, 1976.
[14] K. R. Goodearl, *Singular torsion and the splitting properties*, No. 1-124. American Mathematical Soc. 1972.
[15] A. Hamdouni, A. Harmanci, A. C. Özcan, *Characterization of modules and rings by the summand intersection property and the summand sum property*, JP Jour. Algebra Number Theory and Appl. **5(3)** (2005), 469-490.
[16] J. Hausen, *Modules with the summand intersection property*, Comm. Algebra, **17(1)** (1989),135-148.

[17] T. Y. Lam, *Lectures on Modules and Rings*, Springer-Verlag, New York, 1999.
[18] A. Leghwel, M. T. Kosan, N. Agayev, A. Harmanci, *Duo modules and duo rings*, Far East J. Math. **20(3)** (2006), 341-346.
[19] S. H. Mohamed, B. J. Müller, *Continuous and Discrete Modules*, London Math. Soc. Lecture Note Series, **147** Cambridge, Cambridge Univ. Press, 1990.
[20] A. C. Ozcan, A. Harmanci, P. F. Smith, *Duo modules*, Glasg. Math. J. **48(3)** (2006), 533-545.
[21] T. C. Quynh, M. T. Koşan, *On ADS modules and rings*, Comm. Algebra **42(8)** (2014), 3541-3551.
[22] T. C. Quynh, M. T. Koşan, *Modules with ADS-(pre) envelopes and ADS-(pre) covers*, J. Algebra Appl. **17(5)** (2018), 1850096.
[23] T. C. Quynh, M. T. Koşan, J. Zemlicka, *Essentially ADS modules and rings*, Rings, modules and codes, Contemp. Math., 727, Amer. Math. Soc., Providence, RI, 223-236.
[24] P. F. Smith, A. Tercan, *Generalizations of CS-modules*, Comm. Algebra **21** (1993), 1809-1847.
[25] F. Takil Mutlu, *On Ads-modules with the SIP*, Bull. Iranian Math. Soc. **41(6)** (2015), 1355-1363.
[26] F. Takil Mutlu, *On matrix rings with the SIP and the Ads*, Turkish J. Math. **42(5)** (2018), 2657-2663.
[27] F. Takil Mutlu, *On free modules with the SA property*, AKU J. Sci. Eng. **21** (2021), 021001 236-249.
[28] Ö. Taşdemir, *Characterization of some well-known rings by using SA modules*, Eskisehir Technical University Journal of Science and Technology B-Theoretical Sciences, **7(1)** (2019), 75-80.
[29] G. V. Wilson, *Modules with the summand intersection property*, Comm. Algebra **14** (1986), 21-38.
[30] R. Wisbauer, *Foundations of Module and Ring Theory*, Gordon and Breach, Reading, 1991.