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Multi-parametric post quantum Simpson inequalities and their applications

Muhammad Raees[a], Muhammad Uzair Awan[b,*], Artion Kashuri[c], Matloob Anwar[a]

[a] *School of Natural Sciences, NUST, Islamabad, Pakistan*
[b] *Department of Mathematics, Government College University, Faisalabad, Pakistan*
[c] *Department of Mathematical Engineering, Polytechnic University of Tirana, Tirana 1001, Albania*

Abstract. In this paper, the authors established some very general analogues of Simpson type inequalities via newly defined two types of quantum integrals over a finite interval. Firstly, a multi-parameters identity is developed. Applying this generic identity as an auxiliary result, we derive several Simpson type quantum inequalities for those functions whose absolute value of first-order quantum derivatives are m-convex. The results of this study generate a family of interesting special cases. Finally, in order to show the efficiency of our results, several applications are obtained about the special means of different positive real numbers.

1. Introduction

The numerical integration and the numerical estimations of definite integrals is a vital piece of applied sciences. Simpson's rules are momentous among the numerical techniques. The procedure is credited to Thomas Simpson (1710-1761). Johannes Kepler worked on a similar estimation technique about a century ago, so the algorithm is some time called as Kepler's formula. Simpson's formula uses three-step Newton–Cotes quadrature rule, so estimations based on three steps quadratic kernel is sometimes termed as Newton type results. Following are the rules devised by Simpson.

1. Simpson's quadrature formula (Simpson's 1/3 rule)

$$\int_{\Theta_1}^{\Theta_2} f(\varrho)d\varrho \approx \frac{1}{6}\left[f(\Theta_1) + 4f\left(\frac{\Theta_1+\Theta_2}{2}\right) + f(\Theta_2)\right]. \tag{1}$$

2. Simpson's second quadrature formula or (Simpson's 3/8 rule)

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* Corresponding author: Muhammad Uzair Awan

Email addresses: fraeesqau1@gmail.com (Muhammad Raees), awan.uzair@gmail.com (Muhammad Uzair Awan), a.kashuri@fimif.edu.al (Artion Kashuri), matloob_t@yahoo.com (Matloob Anwar)

$$\int_{\Theta_1}^{\Theta_2} f(\varrho)d\varrho \approx \frac{1}{8}\left[f(\Theta_1) + 3f\left(\frac{2\Theta_1 + \Theta_2}{3}\right) + 3f\left(\frac{\Theta_1 + 2\Theta_2}{3}\right) + f(\Theta_2)\right]. \tag{2}$$

An exceptionally popular estimation relating to the above rules is called Simpson's inequality and is presented as follows:

Theorem 1.1. *Let $f : [\Theta_1, \Theta_2] \to \mathbb{R}$ be a fourth-order differentiable function on (Θ_1, Θ_2), where $\left\|f^{(4)}\right\|_\infty := \sup_{\varrho\in(\Theta_1,\Theta_2)} \left|f^{(4)}(\varrho)\right| < \infty$, then*

$$\left|\frac{1}{6}\left[f(\Theta_1) + 4f\left(\frac{\Theta_1 + \Theta_2}{2}\right) + f(\Theta_2)\right] - \frac{1}{\Theta_2 - \Theta_1}\int_{\Theta_1}^{\Theta_2} f(\varrho)d\varrho\right| \le \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (\Theta_2 - \Theta_1)^2. \tag{3}$$

Lately, many researchers have zeroed in on Simpson's type inequalities for differentiable classes of convex functions. In particular, a few mathematicians have chipped away at Simpson and Newton type results for convex mappings, since convexity hypothesis are powerful and solid strategy for tackling incredible number of issues which emerge in numerous part of applied sciences. For example, Dragomir *et al.* [1] introduced Simpson type inequalities alongside their applications to quadrature formula in numerical integration. After this beginning, Simpson type inequalities via s-convex functions were developed by Alomari *et al.* [2]. Sarikaya *et al.* [3] proved the variants of Simpson type inequalities dependent upon s-convexity. Kashuri *et al.* [4] obtained new Simpson type integral inequalities for s-convex functions.

On the other hand, Quantum Calculus (shortly, q-calculus) is the study of calculus without limits, where we can rediscover the classical results of ordinary calculus as special case by taking limit $q \to 1^-$. Origin of this theory is found in the work of Euler (1707-1783) in the tracks of Newton's infinite series, studied q-calculus. The theory became a hot area in the early 20th Century after the work of Jackson (1910) on defining an integral later known as the q-Jackson integral see previous studies, see [5–8]. In q-calculus, the classical derivative is replaced by the q-difference operator in order to deal with non-differentiable functions. Applications of q-calculus can be found in various branches of mathematics and physics, and the interested readers are should consider Sofonea paper [9] for a valuable information. Many integral inequalities well known in classical analysis such as Hölder inequality, power-mean inequality, Minkowski inequality, Hermite–Hadamard inequality and Ostrowski inequality, Cauchy–Bunyakovsky–Schwarz, Grüss, Grüss–Cebysev, and other integral inequalities have been developed utilizing q-calculus via classical convexity. The group of quantum integral inequalities related with various types of convex functions is an intriguing theme. Mohsin *et al.* [10] found some new quantum Hermite–Hadamard inequalities utilizing harmonic convexity property. Du *et al.* [11] established certain quantum estimates on the parametrized integral inequalities and derived some interesting applications as well. Erden *et al.* [12] obtain generalizations of some inequalities for convex functions via quantum integrals. Gauchman [13] derived some integral inequalities in q-calculus. Jhanthanam *et al.* [14] established some q-Hermite–Hadamard inequalities for differentiable convex functions. Liu *et al.* [15] obtained some quantum estimates of Hermite–Hadamard inequalities for convex functions. Noor *et al.* [16] established some quantum integral inequalities via preinvex functions. Noor *et al.* [17] found some quantum estimates for Hermite–Hadamard inequalities. Noor *et al.* [18] derived new bounds having Riemann type quantum integrals via strongly convex functions. Riahi *et al.* [19] found some complementary q-bounds via different classes of convex functions. Sudsutad *et al.* [20, 21] obtained some integral inequalities via fractional quantum calculus and also using convexity property of functions. Zhang *et al.* [22] established different types of quantum integral inequalities via (α, m)-convexity. Cortez *et al.* [23] found new quantum estimates of trapezium-type inequalities for generalized φ-convex functions. Cortez *et al.* [24] derived some inequalities using generalized convex functions in quantum analysis.

Motivation of these outcomes, particularly of their created in [25], we notice that it is feasible to treat quantum integral operators introduced in [26, 27] jointly to create some new Simpson type inequalities as been done in [25]. For this reason, we aim to achieve the following objectives:

1. To extend the identity given in Lemma 3.1.
2. To obtain some generalized trapezoid and Simpson type inequalities which would give some counter parts to the inequality (15).
3. To derive some applications about special means of real numbers using our main results.

2. Preliminaries

Throughout the remain paper, let $\mathcal{V}_m := \left[\Theta_1, \frac{\Theta_2}{m}\right] \subseteq \mathbb{R}$ with $0 \le \Theta_1 < \Theta_2$ be an interval, $\mathcal{V}_m^\circ$ be the interior of $\mathcal{V}_m$. Assume further that $\mathcal{V} := [\Theta_1, \Theta_2] \subseteq \mathbb{R}$ and $0 < q < 1$ be a constant.

This section is devoted to the basic and fundamental results in the q-calculus. We start by collating foundational results and definitions suitable for ongoing study.

Definition 2.1. *[28] Let $m \in (0, 1]$. A function $f : [0, c] \to \mathbb{R}, c > 0$ is called m-convex, if the inequality*

$$f(\Upsilon\Theta_1 + m(1 - \Upsilon)\Theta_2) \le \Upsilon f(\Theta_1) + m(1 - \Upsilon)f(\Theta_2), \tag{4}$$

is satisfied for all $\Theta_1, \Theta_2 \in [0, c]$ and $\Upsilon \in [0, 1]$.

Recall that the Jackson integral [7] from 0 to an arbitrary real number Θ is characterized as follows:

$$\int_0^\Theta f(\varsigma)\, d_q\varsigma = (1 - q)\Theta \sum_{\delta=0}^{\infty} q^\delta f(\Theta q^\delta), \tag{5}$$

provided the series on the right side converges absolutely. Moreover, he gave the integral for an arbitrary finite interval $[\Theta_1, \Theta_2]$ as

$$\int_{\Theta_1}^{\Theta_2} f(\varsigma)\, d_q\varsigma = \int_0^{\Theta_2} f(\varsigma)\, d_q\varsigma - \int_0^{\Theta_1} f(\varsigma)\, d_q\varsigma. \tag{6}$$

In [26], the authors while developing some classical inequalities in the quantum frame work studied the concept of q-differentiation and q-integration over the finite interval.

Definition 2.2. *For a continuous function $f : \mathcal{V} \to \mathbb{R}$ and $0 < q < 1$, then q_{Θ_1}-derivative of f at $\Theta \in \mathcal{V}$ is expressed by the quotient:*

$$_{\Theta_1}\mathbf{D}_q f(\Theta) = \frac{f(\Theta) - f(q\Theta + (1 - q)\Theta_1)}{(1 - q)(\Theta - \Theta_1)}, \quad \Theta \ne \Theta_1. \tag{7}$$

The function f is called q_{Θ_1}-differentiable on $\mathcal{V}$, if $_{\Theta_1}\mathbf{D}_q f(\Theta)$ exists for all $\Theta \in \mathcal{V}$. It is evident that

$$_{\Theta_1}\mathbf{D}_q f(\Theta_1) = \lim_{\Theta \to \Theta_1} {_{\Theta_1}\mathbf{D}_q f(\Theta)}. \tag{8}$$

If $\Theta_1 = 0$, then the q-derivative in classical sense [8] is obtained:

$$\mathbf{D}_q f(\Theta) = \frac{f(\Theta) - f(q\Theta)}{(1 - q)\Theta}. \tag{9}$$

Definition 2.3. *Let $f : \mathcal{V} \to \mathbb{R}$ be a continuous function and $q \in (0, 1)$. The definite q_{Θ_1}-integral of the function f is characterized by the expression*

$$\int_{\Theta_1}^{\alpha} f(\varsigma)\, {}_{\Theta_1}d_q\varsigma = (1-q)(\alpha-\Theta_1)\sum_{\delta=0}^{\infty} q^{\delta} f(q^{\delta}\alpha + (1-q^{\delta})\Theta_1), \quad \alpha \in \mathcal{V}. \tag{10}$$

In the same paper [26], they also proved the following q-Hölder inequality.

Theorem 2.4. *Let $f_1, f_2 : \mathcal{V} \to \mathbb{R}$ be two continuous functions. Then*

$$\int_{\Theta_1}^{y} \left|f_1(\Upsilon) f_2(\Upsilon)\right| {}_{\Theta_1}d_q\Upsilon \le \left(\int_{\Theta_1}^{y} \left|f_1(\Upsilon)\right|^{k_1} {}_{\Theta_1}d_q\Upsilon\right)^{\frac{1}{k_1}} \left(\int_{\Theta_1}^{y} \left|f_2(\Upsilon)\right|^{k_2} {}_{\Theta_1}d_q\Upsilon\right)^{\frac{1}{k_2}}, \tag{11}$$

holds for all $y \in \mathcal{V}$ and $k_1 > 1$ with $k_1^{-1} + k_2^{-1} = 1$.

In [27], the authors presented an analogous notion of q–derivatives and q–integrals by introducing the q^{Θ_2}–derivative and q^{Θ_2}–integrals over the finite real interval $\mathcal{V}$.

Definition 2.5. *Let $f : \mathcal{V} \to \mathbb{R}$ be a continuous function and $0 < q < 1$, then q^{Θ_2}–derivative of f at $\Theta \in \mathcal{V}$ is defined by the quotient:*

$${}^{\Theta_2}\mathbf{D}_q f(\Theta) = \frac{f(\Theta) - f(q\Theta + (1-q)\Theta_2)}{(1-q)(\Theta - \Theta_2)}, \quad \Theta \ne \Theta_2, \tag{12}$$

Definition 2.6. *Let $f : \mathcal{V} \to \mathbb{R}$ be a continuous function and $q \in (0,1)$. The definite q^{Θ_2}-integral of the function f is characterized by the expression:*

$$\int_{\beta}^{\Theta_2} f(\varsigma)\, {}^{\Theta_2}d_q\varsigma = (1-q)(\Theta_2-\beta)\sum_{\delta=0}^{\infty} q^{\delta} f(q^{\delta}\beta + (1-q^{\delta})\Theta_2), \quad \beta \in \mathcal{V}. \tag{13}$$

Remark 2.7. *It is of worth mentioning, that*

(i) *the derivatives ${}_{\Theta_1}\mathbf{D}_q f(\Theta)$ and ${}^{\Theta_2}\mathbf{D}_q f(\Theta)$ are not same for general functions defined over the finite real interval $\mathcal{V}$. Indeed, if we take $f(\Theta) = \Theta^2$, then*

$${}^{\Theta_2}\mathbf{D}_q f(\Theta) = (1+q)\Theta + (1-q)\Theta_2 \ne (1+q)\Theta + (1-q)\Theta_1 = {}_{\Theta_1}\mathbf{D}_q f(\Theta).$$

However,

$${}^{\Theta_2}\mathbf{D}_q f(\Theta) = f'(\Theta) = {}_{\Theta_1}\mathbf{D}_q f(\Theta)$$

provided that $q \to 1^-$.

(ii) *The q–integrals $\int_{k_1}^{\Theta_2} f(\Theta)\, {}^{\Theta_2}d_q\Theta$ and $\int_{k_1}^{\Theta_2} f(\Theta)\, {}_{k_1}d_q\Theta$ are different for general functions. For instance,*

$$\int_{k_1}^{\Theta_2} \Theta\, {}^{\Theta_2}d_q\Theta = \frac{\Theta_2 - k_1}{1+q}[k_1 + q\Theta_2] \ne \frac{\Theta_2 - k_1}{1+q}[k_1 q + \Theta_2] = \int_{k_1}^{\Theta_2} \Theta\, {}_{k_1}d_q\Theta.$$

Furthermore,

$$\int_{k_1}^{\Theta_2} \Theta\, {}^{\Theta_2}d_q\Theta = \frac{\Theta_2^2 - k_1^2}{2} = \int_{k_1}^{\Theta_2} \Theta\, {}_{k_1}d_q\Theta$$

subject to the condition that $q \to 1^-$.

It is also important to notice that the for an integer n, the quantum analogue is

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1}, \quad q \neq 1.$$

Clearly, the limiting value is n for $q \to 1^-$. For more details, see [8].

In [25], the authors presented the following lemma and establish associated Simpson type inequalities.

Lemma 2.8. *Assume that $f : \mathcal{V} \to \mathbb{R}$ be a q-differentiable function on $\mathcal{V}^\circ$ and $0 < q < 1$. Also, suppose that ${}^{\Theta_2}D_q f$ is continuous and integrable on $\mathcal{V}$, then*

$$\begin{aligned}(\Theta_2 - \Theta_1) & \int_{\Theta_1}^{\Theta_2} \Phi(\Upsilon)\, {}^{\Theta_2}D_q f(\Upsilon\Theta_1 + (1-\Upsilon)\Theta_2)\, d_q\Upsilon \\ &= \frac{1}{\Theta_2 - \Theta_1} \int_{\Theta_1}^{\Theta_2} f(\wp)\, {}^{\Theta_2}d_q\wp - \frac{1}{6}\left[f(\Theta_1) + 4f\left(\frac{\Theta_1 + \Theta_2}{2}\right) + f(\Theta_2)\right],\end{aligned} \tag{14}$$

where

$$\Phi(\Upsilon) := \begin{cases} q\Upsilon - \frac{1}{6}, & \textit{if} \quad 0 \leq \Upsilon < \frac{1}{2}, \\ q\Upsilon - \frac{5}{6}, & \textit{if} \quad \frac{1}{2} \leq \Upsilon \leq 1. \end{cases}$$

Theorem 2.9. *Let $f : \mathcal{V} \to \mathbb{R}$ be a continuous and q-differentiable function on $\mathcal{V}^\circ$ and $0 < q < 1$. If $\left|{}^{\Theta_2}D_q f\right|$ is convex and integrable on $\mathcal{V}$, then*

$$\begin{aligned}&\left|\frac{1}{\Theta_2 - \Theta_1} \int_{\Theta_1}^{\Theta_2} f(\wp)\, {}^{\Theta_2}d_q\wp - \frac{1}{6}\left[f(\Theta_1) + 4f\left(\frac{\Theta_1 + \Theta_2}{2}\right) + f(\Theta_2)\right]\right| \\ &\leq (\Theta_2 - \Theta_1)\left\{[M_1(q) + M_2(q)]\left|{}^{\Theta_2}D_q f(\Theta_1)\right| + [M_3(q) + M_4(q)]\left|{}^{\Theta_2}D_q f(\Theta_2)\right|\right\},\end{aligned} \tag{15}$$

where

$$M_1(q) := \int_0^{\frac{1}{2}} \left|q\wp - \frac{1}{6}\right| \wp\, d_q\wp = \begin{cases} \frac{1-2q-2q^2}{24(1+q)(1+q+q^2)}, & \textit{if} \quad 0 < q < \frac{1}{3}, \\ \frac{18q^2+18q-7}{216(1+q)(1+q+q^2)}, & \textit{if} \quad \frac{1}{3} \leq q < 1, \end{cases} \tag{16}$$

$$M_2(q) := \int_0^{\frac{1}{2}} \left|q\wp - \frac{1}{6}\right| (1-\wp)\, d_q\wp = \begin{cases} \frac{1-4q^3}{24(1+q)(1+q+q^2)}, & \textit{if} \quad 0 < q < \frac{1}{3}, \\ \frac{36q^3+12q^2+12q+1}{216(1+q)(1+q+q^2)}, & \textit{if} \quad \frac{1}{3} \leq q < 1, \end{cases} \tag{17}$$

$$M_3(q) := \int_{\frac{1}{2}}^{1} \left|q\wp - \frac{5}{6}\right| \wp\, d_q\wp = \begin{cases} \frac{15-6q-6q^2}{24(1+q)(1+q+q^2)}, & \textit{if} \quad 0 < q < \frac{5}{6}, \\ \frac{18q^2+18q-7}{216(1+q)(1+q+q^2)}, & \textit{if} \quad \frac{5}{6} \leq q < 1, \end{cases} \tag{18}$$

$$M_4(q) := \int_{\frac{1}{2}}^{1} \left|q\wp - \frac{5}{6}\right| (1-\wp)\, d_q\wp = \begin{cases} \frac{-5+8q+8q^2-8q^3}{24(1+q)(1+q+q^2)}, & \textit{if} \quad 0 < q < \frac{5}{6}, \\ \frac{12q^2+12q+5}{216(1+q)(1+q+q^2)}, & \textit{if} \quad \frac{5}{6} \leq q < 1. \end{cases} \tag{19}$$

3. Main Results

Before giving our main results, for the sake of comparison, we re-produce an identity proved in [29].

Lemma 3.1. *Let $f : \mathcal{V} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathcal{V}^\circ$. If $f' \in L_1(\mathcal{V})$ (the set of all integrable functions on $\mathcal{V}$), where $\Theta_1, \Theta_2 \in \mathcal{V}$, then the following identity holds:*

$$f\left(\frac{\Theta_1+\Theta_2}{2}\right) - \frac{1}{\Theta_2-\Theta_1}\int_{\Theta_1}^{\Theta_2} f(\delta)\,d\delta = \frac{\Theta_2-\Theta_1}{4}\left[\int_0^1 \Upsilon f'\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_1\right)d\Upsilon - \int_0^1 \Upsilon f'\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_2\right)d\Upsilon\right]. \tag{20}$$

We are ready to prove our main results. We begin with a multi-parameters identity for q_{Θ_1} and q^{Θ_2}-differentiable functions which provides some useful identities for Simpson and trapezoid type inequalities.

Lemma 3.2. *Let $f : \mathcal{V} \to \mathbb{R}$ be a q_{Θ_1}– and q^{Θ_2}–differentiable function on $\mathcal{V}^\circ$ with $q \in (0,1)$. If ${}_{\Theta_1}D_q f$ and ${}^{\Theta_2}D_q f$ are continuous and $q_{\Theta_1}-, q^{\Theta_2}$–integrable functions on $\mathcal{V}$, then for $U_1, U_2 \in \mathbb{R}$, the following identity holds:*

$$\Omega(U_1,U_2;q) = \frac{\Theta_2-\Theta_1}{4}\left[\int_0^1 (q\Upsilon - U_1)\,{}_{\Theta_1}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_1\right)d_q\Upsilon - \int_0^1 (q\Upsilon - U_2)\,{}^{\Theta_2}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_2\right)d_q\Upsilon\right], \tag{21}$$

where

$$\Omega(U_1,U_2;q) := \frac{U_1 f(\Theta_1)+U_2 f(\Theta_2)}{2} + \left(1-\frac{U_1+U_2}{2}\right)f\left(\frac{\Theta_1+\Theta_2}{2}\right) - \frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}} f(\tau)\,{}_{\Theta_1}d_q\tau + \int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2} f(\tau)\,{}^{\Theta_2}d_q\tau\right]. \tag{22}$$

Proof. By utilizing Definitions 2.2 and 2.3, we have

$$\begin{aligned}
&\int_0^1 (q\Upsilon - U_1)\,{}_{\Theta_1}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_1\right)d_q\Upsilon \\
&= \frac{2}{\Theta_2-\Theta_1}\int_0^1 (q\Upsilon - U_1)\left[\frac{f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_1\right) - f\left(q\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-q\Upsilon)\Theta_1\right)}{\Upsilon(1-q)}\right]d_q\Upsilon \\
&= \frac{2}{\Theta_2-\Theta_1}\left\{\int_0^1 q\left[\frac{f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_1\right) - f\left(q\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-q\Upsilon)\Theta_1\right)}{1-q}\right]d_q\Upsilon\right. \\
&\quad \left. - \int_0^1 U_1\left[\frac{f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_1\right) - f\left(q\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-q\Upsilon)\Theta_1\right)}{\Upsilon(1-q)}\right]d_q\Upsilon\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\Theta_2-\Theta_1}\left\{\left[q\sum_{\delta=0}^{\infty} q^{\delta} f\left(q^{\delta}\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-q^{\delta})\Theta_1\right)\right.\right.\\
&\quad \left.-\sum_{\delta=0}^{\infty} q^{\delta+1} f\left(q^{\delta+1}\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-q^{\delta+1})\Theta_1\right)\right]\\
&\quad \left.-U_1\left[\sum_{\delta=0}^{\infty} f\left(q^{\delta}\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-q^{\delta})\Theta_1\right)-\sum_{\delta=0}^{\infty} f\left(q^{\delta+1}\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-q^{\delta+1})\Theta_1\right)\right]\right\}\\
&= \frac{2}{\Theta_2-\Theta_1}\left\{q\sum_{\delta=0}^{\infty} q^{\delta} f\left(q^{\delta}\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-q^{\delta})\Theta_1\right)\right.\\
&\quad \left.-\sum_{\alpha=1}^{\infty} q^{\alpha} f\left(q^{\alpha}\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-q^{\alpha})\Theta_1\right)-U_1\left[f\left(\frac{\Theta_1+\Theta_2}{2}\right)-f(\Theta_1)\right]\right\}\\
&= \frac{2}{\Theta_2-\Theta_1}\left\{q\sum_{\delta=0}^{\infty} q^{\delta} f\left(q^{\delta}\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-q^{\delta})\Theta_1\right)\right.\\
&\quad \left.-\sum_{\delta=0}^{\infty} q^{\delta} f\left(q^{\delta}\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-q^{\delta})\Theta_1\right)+f\left(\frac{\Theta_1+\Theta_2}{2}\right)-U_1\left[f\left(\frac{\Theta_1+\Theta_2}{2}\right)-f(\Theta_1)\right]\right\}\\
&= \frac{2}{\Theta_2-\Theta_1}\left\{-(1-q)\sum_{\delta=0}^{\infty} q^{\delta} f\left(q^{\delta}\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-q^{\delta})\Theta_1\right)\right.\\
&\quad \left.+(1-U_1)f\left(\frac{\Theta_1+\Theta_2}{2}\right)+U_1 f(\Theta_1)\right\}\\
&= -\frac{4}{(\Theta_2-\Theta_1)^2}\frac{(1-q)(\Theta_2-\Theta_1)}{2}\sum_{\delta=0}^{\infty} q^{\delta} f\left(q^{\delta}\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-q^{\delta})\Theta_1\right)\\
&\quad +\frac{2(1-U_1)}{\Theta_2-\Theta_1}f\left(\frac{\Theta_1+\Theta_2}{2}\right)+\frac{2U_1}{\Theta_2-\Theta_1}f(\Theta_1)\\
&= -\frac{4}{(\Theta_2-\Theta_1)^2}\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}} f(\tau)\ {}_{\Theta_1}d_q\tau+\frac{2(1-U_1)}{\Theta_2-\Theta_1}f\left(\frac{\Theta_1+\Theta_2}{2}\right)+\frac{2U_1}{\Theta_2-\Theta_1}f(\Theta_1). \qquad (23)
\end{aligned}
$$

Similarly, by utilizing Definition 2.5, we get

$$
\begin{aligned}
&\int_0^1 (q\Upsilon-U_2)\ {}^{\Theta_2}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_2\right) d_q\Upsilon\\
&= -\frac{2}{\Theta_2-\Theta_1}\int_0^1 (q\Upsilon-U_2)\left[\frac{f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_2\right)-f\left(q\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-q\Upsilon)\Theta_2\right)}{\Upsilon(1-q)}\right] d_q\Upsilon\\
&= \frac{4}{(\Theta_2-\Theta_1)^2}\int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2} f(\tau)\ {}^{\Theta_2}d_q\tau-\frac{2(1-U_2)}{\Theta_2-\Theta_1}f\left(\frac{\Theta_1+\Theta_2}{2}\right)-\frac{2U_2}{\Theta_2-\Theta_1}f(\Theta_2). \qquad (24)
\end{aligned}
$$

Now by equalities (23) and (24), we obtain

$$
\begin{aligned}
&\int_0^1 (q\Upsilon - U_1)\, {}_{\Theta_1}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right) + (1-\Upsilon)\Theta_1\right) d_q\Upsilon \\
&\quad - \int_0^1 (q\Upsilon - U_2)\, {}^{\Theta_2}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right) + (1-\Upsilon)\Theta_2\right) d_q\Upsilon \\
&= \frac{2(2-[U_1+U_2])}{\Theta_2-\Theta_1} f\left(\frac{\Theta_1+\Theta_2}{2}\right) + \frac{2U_1}{\Theta_2-\Theta_1} f(\Theta_1) + \frac{2U_2}{\Theta_2-\Theta_1} f(\Theta_2) \\
&\quad - \frac{4}{(\Theta_2-\Theta_1)^2}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}} f(\tau)\, {}_{\Theta_1}d_q\tau + \int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2} f(\tau)\, {}^{\Theta_2}d_q\tau\right]. \qquad (25)
\end{aligned}
$$

Multiplying by factor $\frac{\Theta_2-\Theta_1}{4}$, we have the desired identity (21). □

Remark 3.3. *If we choose $U_1 = U_2 = 0$ and $q \to 1^-$, then we get identity* (20) *proved in [29]. This proves the compatibility of our identity* (21) *with existing literature.*

Corollary 3.4. *Consider Lemma 3.2. We have very interesting cases subject to the choices of parameter U_1 and U_2. For instance, if:*

1. $U_1 = U_2 = 0$, *then*

$$
\begin{aligned}
&f\left(\frac{\Theta_1+\Theta_2}{2}\right) - \frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}} f(\tau)\, {}_{\Theta_1}d_q\tau + \int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2} f(\tau)\, {}^{\Theta_2}d_q\tau\right] \\
&= \frac{\Theta_2-\Theta_1}{4}\left[\int_0^1 q\Upsilon\, {}_{\Theta_1}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right) + (1-\Upsilon)\Theta_1\right) d_q\Upsilon \right. \\
&\quad \left. - \int_0^1 q\Upsilon\, {}^{\Theta_2}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right) + (1-\Upsilon)\Theta_2\right) d_q\Upsilon\right], \qquad (26)
\end{aligned}
$$

which is quantum version of identity proved in Lemma 3.1.

2. $U_1 = U_2 = \frac{1}{3}$, *then*

$$
\begin{aligned}
&\frac{1}{3}\left[\frac{f(\Theta_1)+f(\Theta_2)}{2} + 2f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right] - \frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}} f(\tau)\, {}_{\Theta_1}d_q\tau + \int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2} f(\tau)\, {}^{\Theta_2}d_q\tau\right] \\
&= \frac{\Theta_2-\Theta_1}{4}\left[\int_0^1 \left(q\Upsilon - \frac{1}{3}\right) {}_{\Theta_1}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right) + (1-\Upsilon)\Theta_1\right) d_q\Upsilon \right. \\
&\quad \left. - \int_0^1 \left(q\Upsilon - \frac{1}{3}\right) {}^{\Theta_2}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right) + (1-\Upsilon)\Theta_2\right) d_q\Upsilon\right]. \qquad (27)
\end{aligned}
$$

3. $U_1 = \frac{1}{3}$ *and* $U_2 = \frac{2}{3}$, *then*

$$\frac{1}{6}\left[f(\Theta_1)+3f\left(\frac{\Theta_1+\Theta_2}{2}\right)+2f(\Theta_2)\right]-\frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}} f(\tau)\ {}_{\Theta_1}d_q\tau+\int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2} f(\tau)\ {}^{\Theta_2}d_q\tau\right]$$

$$= \frac{\Theta_2-\Theta_1}{4}\left[\int_0^1 (q\Upsilon-\frac{1}{3})\,{}_{\Theta_1}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_1\right)d_q\Upsilon\right.$$
$$\left.-\int_0^1 (q\Upsilon-\frac{2}{3})\,{}^{\Theta_2}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_2\right)d_q\Upsilon\right]. \tag{28}$$

4. $U_1 = \frac{1}{6}$ *and* $U_2 = \frac{5}{6}$, *then*

$$\frac{1}{12}\left[f(\Theta_1)+6f\left(\frac{\Theta_1+\Theta_2}{2}\right)+5f(\Theta_2)\right]-\frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}} f(\tau)\ {}_{\Theta_1}d_q\tau+\int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2} f(\tau)\ {}^{\Theta_2}d_q\tau\right]$$

$$= \frac{\Theta_2-\Theta_1}{4}\left[\int_0^1 (q\Upsilon-\frac{1}{6})\,{}_{\Theta_1}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_1\right)d_q\Upsilon\right.$$
$$\left.-\int_0^1 (q\Upsilon-\frac{5}{6})\,{}^{\Theta_2}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_2\right)d_q\Upsilon\right]. \tag{29}$$

5. $U_1 = \frac{1}{4}$ *and* $U_2 = \frac{3}{4}$, *then*

$$\frac{1}{8}\left[f(\Theta_1)+4f\left(\frac{\Theta_1+\Theta_2}{2}\right)+3f(\Theta_2)\right]-\frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}} f(\tau)\ {}_{\Theta_1}d_q\tau+\int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2} f(\tau)\ {}^{\Theta_2}d_q\tau\right]$$

$$= \frac{\Theta_2-\Theta_1}{4}\left[\int_0^1 (q\Upsilon-\frac{1}{4})\,{}_{\Theta_1}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_1\right)d_q\Upsilon\right.$$
$$\left.-\int_0^1 (q\Upsilon-\frac{3}{4})\,{}^{\Theta_2}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_2\right)d_q\Upsilon\right]. \tag{30}$$

6. $U_1 = \frac{1}{2} = U_2$, *then*

$$\frac{1}{4}\left[f(\Theta_1)+2f\left(\frac{\Theta_1+\Theta_2}{2}\right)+f(\Theta_2)\right]-\frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}} f(\tau)\ {}_{\Theta_1}d_q\tau+\int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2} f(\tau)\ {}^{\Theta_2}d_q\tau\right]$$

$$= \frac{\Theta_2-\Theta_1}{4}\left[\int_0^1 (q\Upsilon-\frac{1}{2})\,{}_{\Theta_1}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_1\right)d_q\Upsilon\right.$$
$$\left.-\int_0^1 (q\Upsilon-\frac{1}{2})\,{}^{\Theta_2}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_2\right)d_q\Upsilon\right]. \tag{31}$$

7. $U_1 = 1 = U_2$, *then*

$$\frac{f(\Theta_1)+f(\Theta_2)}{2} - \frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}} f(\tau)\ {}_{\Theta_1}d_q\tau + \int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2} f(\tau)\ {}^{\Theta_2}d_q\tau\right]$$

$$= \frac{\Theta_2-\Theta_1}{4}\left[\int_0^1 (q\Upsilon-1)\,{}_{\Theta_1}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_1\right)d_q\Upsilon \right.$$
$$\left. - \int_0^1 (q\Upsilon-1)\,{}^{\Theta_2}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_2\right)d_q\Upsilon\right]. \tag{32}$$

8. *If* $U_1 = U_2 = \frac{q}{1+q}$, *then*

$$\frac{1}{2(1+q)}\left[qf(\Theta_1)+2f\left(\frac{\Theta_1+\Theta_2}{2}\right)+qf(\Theta_2)\right] - \frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}} f(\tau)\ {}_{\Theta_1}d_q\tau + \int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2} f(\tau)\ {}^{\Theta_2}d_q\tau\right]$$

$$= \frac{\Theta_2-\Theta_1}{4}\left[\int_0^1 \left(q\Upsilon-\frac{q}{1+q}\right){}_{\Theta_1}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_1\right)d_q\Upsilon \right.$$
$$\left. - \int_0^1 \left(q\Upsilon-\frac{q}{1+q}\right){}^{\Theta_2}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_2\right)d_q\Upsilon\right]. \tag{33}$$

In the following Lemma 3.5, we give some useful q-integrals for further investigation.

Lemma 3.5. *For* $C \in \mathbb{R}$, *we have the following* q*-integrals:*

$$\mathcal{I}_1^q(C) := \int_0^1 \left|q\Upsilon - C\right| d_q\Upsilon = \begin{cases} \frac{q-C(1+q)}{1+q}, & \text{if } \frac{C}{q} \le 0, \\ \frac{2C^2-C(1+q)+q}{1+q}, & \text{if } 0 < \frac{C}{q} < 1, \\ \frac{C(1+q)-q}{1+q}, & \text{if } \frac{C}{q} \ge 1. \end{cases} \tag{34}$$

$$\mathcal{I}_2^q(C) := \int_0^1 \left|q\Upsilon - C\right| (1-\Upsilon)\, d_q\Upsilon = \begin{cases} \frac{q^3-Cq(1+q+q^2)}{[2]_q[3]_q}, & \text{if } \frac{C}{q} \le 0, \\ \frac{q^3-Cq(1+q+q^2)+2C^2(1+q+q^2)-2C^3}{[2]_q[3]_q}, & \text{if } 0 < \frac{C}{q} < 1, \\ \frac{Cq(1+q+q^2)-q^3}{[2]_q[3]_q}, & \text{if } \frac{C}{q} \ge 1. \end{cases} \tag{35}$$

$$\mathcal{I}_3^q(C) := \int_0^1 \left|q\Upsilon - C\right| \Upsilon\, d_q\Upsilon = \begin{cases} \frac{q(1+q)-C(1+q+q^2)}{[2]_q[3]_q}, & \text{if } \frac{C}{q} \le 0, \\ \frac{q(1+q)-C(1+q+q^2)+2C^3}{[2]_q[3]_q}, & \text{if } 0 < \frac{C}{q} < 1, \\ \frac{C(1+q+q^2)-q(1+q)}{[2]_q[3]_q}, & \text{if } \frac{C}{q} \ge 1. \end{cases} \tag{36}$$

$$\mathcal{I}_4^q(C) := \int_0^1 \left|q\Upsilon - C\right| \Upsilon(1-\Upsilon)\, d_q\Upsilon$$
$$= \begin{cases} \frac{q^4(1+q)-Cq^2(1+q+q^2+q^3)}{[2]_q[3]_q[4]_q}, & \text{if} \quad \frac{C}{q} \le 0, \\ \frac{2C^3(1+q+q^2+q^3)-2C^4(1+q)-Cq^2(1+q+q^2+q^3)+q^4(1+q)}{[2]_q[3]_q[4]_q}, & \text{if} \quad 0 < \frac{C}{q} < 1, \\ \frac{Cq^2(1+q+q^2+q^3)-q^4(1+q)}{[2]_q[3]_q[4]_q}, & \text{if} \quad \frac{C}{q} \ge 1. \end{cases} \tag{37}$$

$$\mathcal{I}_5^q(C) := \int_0^1 \left|q\Upsilon - C\right| \Upsilon(2-\Upsilon)\, d_q\Upsilon$$
$$= \begin{cases} \frac{q(1+2q+2q^2+3q^2+2q^4)-C(1+2q+4q^2+4q^3+3q^4+2q^5)}{[2]_q[3]_q[4]_q}, & \text{if} \quad \frac{C}{q} \le 0, \\ \frac{q(1+2q+2q^2+3q^2+2q^4)-C(1+2q+4q^2+4q^3+3q^4+2q^5)+4C^3(1+q+q^2+q^3)-2C^4(1+q)}{[2]_q[3]_q[4]_q}, & \text{if} \quad 0 < \frac{C}{q} < 1, \\ \frac{C(1+2q+4q^2+4q^3+3q^4+2q^5)-q(1+2q+2q^2+3q^2+2q^4)}{[2]_q[3]_q[4]_q}, & \text{if} \quad \frac{C}{q} \ge 1. \end{cases} \tag{38}$$

$$\mathcal{I}_6^q(C) := \int_0^1 \left|q\Upsilon - C\right| (2-\Upsilon)\, d_q\Upsilon$$
$$= \begin{cases} \frac{q(1+q+2q^2)-Cq(1+2q)(1+q+q^2)}{[2]_q[3]_q}, & \text{if} \quad \frac{C}{q} \le 0, \\ \frac{q(1+q+2q^2)-C(1+2q)(1+q+q^2)+4C^2(1+q+q^2)-2C^3}{[2]_q[3]_q}, & \text{if} \quad 0 < \frac{C}{q} < 1, \\ \frac{Cq(1+2q)(1+q+q^2)-q(1+q+2q^2)}{[2]_q[3]_q}, & \text{if} \quad \frac{C}{q} \ge 1. \end{cases} \tag{39}$$

$$\mathcal{I}_7^q(C) := \int_0^1 \left|q\Upsilon - C\right| (1-\Upsilon)(2-\Upsilon)\, d_q\Upsilon$$
$$= \begin{cases} \frac{q^3(2q^3+q^2+q+2)-Cq(2q^5+3q^4+5q^3+5q^2+3q+2)}{[2]_q[3]_q[4]_q[5]_q}, & \text{if} \quad \frac{C}{q} \le 0, \\ \frac{2C^4(1+q)-6C^3(1+q+q^2+q^3)+4C^2(q^5+2q^4+3q^3+3q^2+2q+1)}{[2]_q[3]_q[4]_q[5]_q} & \\ \quad + \frac{q^3(2q^3+q^2+q+2)-Cq(2q^5+3q^4+5q^3+5q^2+3q+2)}{[2]_q[3]_q[4]_q[5]_q}, & \text{if} \quad 0 < \frac{C}{q} < 1, \\ \frac{Cq(2q^5+3q^4+5q^3+5q^2+3q+2)-q^3(2q^3+q^2+q+2)}{[2]_q[3]_q[4]_q[5]_q}, & \text{if} \quad \frac{C}{q} \ge 1. \end{cases} \tag{40}$$

$$\mathcal{I}_8^q(C) := \int_0^1 \left|q\Upsilon - C\right| \Upsilon^2\, d_q\Upsilon = \begin{cases} \frac{q(1+q+q^2)-C(1+q+q^2+q^3)}{[3]_q[4]_q}, & \text{if} \quad \frac{C}{q} \le 0, \\ \frac{2C^4+q(1+q+q^2)-C(1+q+q^2+q^3)}{[3]_q[4]_q}, & \text{if} \quad 0 < \frac{C}{q} < 1, \\ \frac{C(1+q+q^2+q^3)-q(1+q+q^2)}{[3]_q[4]_q}, & \text{if} \quad \frac{C}{q} \ge 1. \end{cases} \tag{41}$$

$$\mathcal{I}_9^q(C) := \int_0^1 \left|q\Upsilon - C\right| (1-\Upsilon)^2 \, d_q\Upsilon$$
$$= \begin{cases} \frac{q^3(1+q^3)-Cq(1+q^2)(1+q+q^2+q^3)}{[2]_q[3]_q[4]_q}, & \text{if } \frac{C}{q} \le 0, \\ \frac{2(1+q)C^4-4C^3(1+q+q^2+q^3)+2C^2(1+q+q^2)(1+q+q^2+q^3)}{[2]_q[3]_q[4]_q} \\ +\frac{q^3(1+q^3)-Cq(1+q^2)(1+q+q^2+q^3)}{[2]_q[3]_q[4]_q}, & \text{if } 0 < \frac{C}{q} < 1, \\ \frac{Cq(1+q^2)(1+q+q^2+q^3)-q^3(1+q^3)}{[2]_q[3]_q[4]_q}, & \text{if } \frac{C}{q} \ge 1. \end{cases} \tag{42}$$

With the help of integrals evaluated in Lemma 3.5, we have following inequalities of Simpson type.

Theorem 3.6. *Let $0 \le \Theta_1 < \Theta_2$ and $m \in (0,1]$. Assume that $f : \mathcal{V}_m \to \mathbb{R}$ satisfies the assumptions of Lemma 3.2. In addition, if $\left|{}_{\Theta_1}D_q f\right|$ and $\left|{}^{\Theta_2}D_q f\right|$ are m-convex functions on $\mathcal{V}_m$, then*

$$\left|\Omega(U_1,U_2;q)\right| \le \frac{\Theta_2-\Theta_1}{4}\left\{\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right| \mathcal{I}_3^q(U_1) + \left|{}^{\Theta_2}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right| \mathcal{I}_3^q(U_2) + m\left(\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1}{m}\right)\right| \mathcal{I}_2^q(U_1) + \left|{}^{\Theta_2}D_q f\left(\frac{\Theta_2}{m}\right)\right| \mathcal{I}_2^q(U_2)\right)\right\}, \tag{43}$$

and

$$\left|\Omega(U_1,U_2;q)\right| \le \frac{\Theta_2-\Theta_1}{8}\left\{\left(\left|{}_{\Theta_1}D_q f(\Theta_2)\right| \mathcal{I}_3^q(U_1) + \left|{}^{\Theta_2}D_q f(\Theta_1)\right| \mathcal{I}_3^q(U_2)\right) + m\left(\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1}{m}\right)\right| \mathcal{I}_6^q(U_1), \left|{}^{\Theta_2}D_q f\left(\frac{\Theta_2}{m}\right)\right| \mathcal{I}_6^q(U_2)\right)\right\}, \tag{44}$$

where $\mathcal{I}_2^q(C)$, $\mathcal{I}_3^q(C)$, and $\mathcal{I}_6^q(C)$ are given in Lemma 3.5.

Proof. Using Lemma 3.2 and taking modulus on both sides of the identity (21), we have

$$\left|\Omega(U_1,U_2;q)\right| \le \frac{\Theta_2-\Theta_1}{4}\left[\int_0^1 \left|q\Upsilon - U_1\right| \left|{}_{\Theta_1}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right) + (1-\Upsilon)\Theta_1\right)\right| d_q\Upsilon + \int_0^1 \left|q\Upsilon - U_2\right| \left|{}^{\Theta_2}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right) + (1-\Upsilon)\Theta_2\right)\right| d_q\Upsilon\right]. \tag{45}$$

Since $\left|{}_{\Theta_1}D_q f\right|$ and $\left|{}^{\Theta_2}D_q f\right|$ are m-convex functions, we get

$$\int_0^1 \left|q\Upsilon - U_1\right| \left|{}_{\Theta_1}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right) + (1-\Upsilon)\Theta_1\right)\right| d_q\Upsilon$$
$$\le \left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right| \int_0^1 \left|q\Upsilon - U_1\right| \Upsilon \, d_q\Upsilon + m\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1}{m}\right)\right| \int_0^1 \left|q\Upsilon - U_1\right| (1-\Upsilon) \, d_q\Upsilon, \tag{46}$$

and

$$\int_0^1 \left|q\Upsilon - U_2\right| \left|{}^{\Theta_2}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right) + (1-\Upsilon)\Theta_2\right)\right| d_q\Upsilon$$

$$\leq \left|{}^{\Theta_2}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right| \int_0^1 \left|q\Upsilon - U_2\right| \Upsilon \, d_q\Upsilon + m \left|{}^{\Theta_2}D_q f\left(\frac{\Theta_2}{m}\right)\right| \int_0^1 \left|q\Upsilon - U_2\right| (1-\Upsilon) \, d_q\Upsilon. \tag{47}$$

Using inequalities (46) and (47) in (45) together with the q-integrals evaluated in (35) and (36), we obtain the desired inequality (43). For the next inequality (44), utilizing the m-convexity of $\left|{}_{\Theta_1}D_q f\right|$ and $\left|{}^{\Theta_2}D_q f\right|$, we have

$$\begin{aligned}
&\int_0^1 \left|q\Upsilon - U_1\right| \left|{}_{\Theta_1}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right) + (1-\Upsilon)\Theta_1\right)\right| d_q\Upsilon \\
&\leq \frac{1}{2}\left|{}_{\Theta_1}D_q f(\Theta_2)\right| \int_0^1 \left|q\Upsilon - U_1\right| \Upsilon \, d_q\Upsilon + m \left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1}{m}\right)\right| \int_0^1 \left(1 - \frac{\Upsilon}{2}\right) \left|q\Upsilon - U_1\right| \, d_q\Upsilon,
\end{aligned} \tag{48}$$

and

$$\begin{aligned}
&\int_0^1 \left|q\Upsilon - U_2\right| \left|{}^{\Theta_2}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right) + (1-\Upsilon)\Theta_2\right)\right| d_q\Upsilon \\
&\leq \frac{1}{2}\left|{}^{\Theta_2}D_q f(\Theta_1)\right| \int_0^1 \left|q\Upsilon - U_2\right| \Upsilon \, d_q\Upsilon + m \left|{}^{\Theta_2}D_q f\left(\frac{\Theta_2}{m}\right)\right| \int_0^1 \left|q\Upsilon - U_2\right| \left(1 - \frac{\Upsilon}{2}\right) d_q\Upsilon.
\end{aligned} \tag{49}$$

After using the inequalities (48) and (49) together with the q-integrals evaluated in (36) and (39), we get the desired inequality (44). □

Example 3.7. *Let $\Theta_1 = 0, \Theta_2 = 2, U_1 = \frac{1}{4}, U_2 = \frac{3}{4}$. Let $f(\Upsilon) = \Upsilon^2$. Suppose further that $q = \frac{1}{2}$. In that case ${}_0D_{\frac{1}{2}}\left(\Upsilon^2\right) = (\frac{3}{2})\Upsilon$ and ${}^2D_{\frac{1}{2}}\left(\Upsilon^2\right) = (\frac{3}{2})\Upsilon + 1$. Also $\int_0^1 \Upsilon^2 \, {}_0d_{\frac{1}{2}} = \frac{4}{7}$ and $\int_1^2 \Upsilon^2 \, {}^2d_{\frac{1}{2}}\Upsilon = \frac{40}{21}$.*
Then all the assumptions of Theorem 3.6 are satisfied. With the above settings of parameters, the left-side is

$$\begin{aligned}
&\left|\frac{U_1 f(0) + U_2 f(2)}{2} + \left(1 - \frac{U_1+U_2}{2}\right) f\left(\frac{0+2}{2}\right) - \frac{1}{2-0}\left[\int_0^1 f(\tau) \, {}_0d_{\frac{1}{2}}\tau + \int_1^2 f(\tau) \, {}^2d_{\frac{1}{2}}\tau\right]\right| \\
&= \left|\frac{0 + \frac{3}{4} \times 4}{2} + \left(1 - \frac{1}{2}\right) - \left[\frac{4}{7} + \frac{40}{21}\right]\right| \\
&= \frac{16}{21} \approx 0.7619.
\end{aligned} \tag{50}$$

Now the right side turns to be

$$\frac{1}{2}\left[\frac{3}{2} \times \frac{11}{84} + \frac{5}{2} \times \frac{3}{14} + 0 \times \frac{1}{28} + 4 \times \frac{17}{84}\right] \approx 0.77083. \tag{51}$$

So the example justify the result demonstrated in (43).

Corollary 3.8. *Under the conditions of Theorem 3.6, we have the following special cases:*
(a) $U_1 = U_2 = 0$, then

$$\left|f\left(\frac{\Theta_1+\Theta_2}{2}\right) - \frac{1}{\Theta_2 - \Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}} f(\tau) \, {}_{\Theta_1}d_q\tau + \int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2} f(\tau) \, {}^{\Theta_2}d_q\tau\right]\right|$$

$$\leq \frac{q(\Theta_2-\Theta_1)}{4[2]_q[3]_q}\left\{(1+q)\left[\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|+\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|\right]+mq^2\left(\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1}{m}\right)\right|+\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_2}{m}\right)\right|\right)\right\}. \quad (52)$$

(b) $U_1 = 1 = U_2$, *then*

$$\left|\frac{f(\Theta_1)+f(\Theta_2)}{2}-\frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}} f(\tau)\ {}_{\Theta_1}d_q\tau+\int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2} f(\tau)\ {}^{\Theta_2}d_q\tau\right]\right|$$
$$\leq \frac{\Theta_2-\Theta_1}{4[2]_q[3]_q}\left\{\left[\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|+\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|\right]+mq(1+q)\left(\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1}{m}\right)\right|+\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_2}{m}\right)\right|\right)\right\}. \quad (53)$$

(c) $U_1 = \frac{1}{4}$ *and* $U_2 = \frac{3}{4}$, *then following Simpson-like quantum integral inequality holds:*

$$\left|\frac{1}{8}\left[f(\Theta_1)+4f\left(\frac{\Theta_1+\Theta_2}{2}\right)+3f(\Theta_2)\right]-\frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}} f(\tau)\ {}_{\Theta_1}d_q\tau+\int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2} f(\tau)\ {}^{\Theta_2}d_q\tau\right]\right|$$
$$\leq \begin{cases} \frac{\Theta_2-\Theta_1}{16[2]_q[3]_q}\left\{(1-3q-3q^2)\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|+(3-q-q^2)\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|\right. \\ \quad \left.+m\left(q(1+q-3q^2)\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1}{m}\right)\right|+q(3+3q-q^2)\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_2}{m}\right)\right|\right)\right\}, & \text{if } 0<q\leq\frac{1}{4}, \\ \frac{\Theta_2-\Theta_1}{128[2]_q[3]_q}\left\{(24q^2+24q-7)\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|+8(3-q-q^2)\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|\right. \\ \quad \left.+m\left((24q^3-4q^2-4q+3)\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1}{m}\right)\right|+8q(3+3q-q^2)\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_2}{m}\right)\right|\right)\right\}, & \text{if } \frac{1}{4}<q\leq\frac{3}{4}, \\ \frac{\Theta_2-\Theta_1}{128[2]_q[3]_q}\left\{(24q^2+24q-7)\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|+(8q^2+8q+3)\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|\right. \\ +m\left((24q^3-4q^2-4q+3)\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1}{m}\right)\right|\right. \\ \left.\left.+\ (8q^3+12q^2+12q+9)\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_2}{m}\right)\right|\right)\right\}, & \text{if } \frac{3}{4}<q<1. \end{cases}$$

(d) $U_1 = \frac{1}{3} = U_2$, *then the inequality* (43) *reduces to the following Simpson-type inequality:*

$$\left|\frac{1}{3}\left[\frac{f(\Theta_1)+f(\Theta_2)}{2}+2f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right]-\frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}} f(\tau)\ {}_{\Theta_1}d_q\tau+\int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2} f(\tau)\ {}^{\Theta_2}d_q\tau\right]\right|$$
$$\leq \begin{cases} \frac{\Theta_2-\Theta_1}{12[2]_q[3]_q}\left\{(1-2q-2q^2)\left(\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|+\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|\right)\right. \\ \quad \left.+m(1+q-2q^2)\left(\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1}{m}\right)\right|+\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_2}{m}\right)\right|\right)\right\}, & \text{if } 0<q\leq\frac{1}{3}, \\ \frac{\Theta_2-\Theta_1}{108[2]_q[3]_q}\left\{(18q^2+18q-7)\left(\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|+\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|\right)\right. \\ \quad \left.+m(18q^3-3q^2-3q+4)\left(\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1}{m}\right)\right|+\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_2}{m}\right)\right|\right)\right\}, & \text{if } \frac{1}{3}<q<1. \end{cases}$$

(e) $U_1 = \frac{q}{1+q} = U_2, q \in (0,1)$, *then the inequality* (43) *gives the following Simpson-like quantum integral inequality:*

$$\left|\frac{1}{2(1+q)}\left[qf(\Theta_1)+2f\left(\frac{\Theta_1+\Theta_2}{2}\right)+qf(\Theta_2)\right]-\frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}} f(\tau)\ {}_{\Theta_1}d_q\tau+\int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2} f(\tau)\ {}^{\Theta_2}d_q\tau\right]\right|$$

$$\leq \frac{q^2(\Theta_2-\Theta_1)}{4(1+q)^3[2]_q[3]_q}\left\{(1+4q+q^2)\left[\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|+\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|\right]\right.$$
$$\left.+m(2q^3+3q^2+1)\left[\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|+\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|\right]\right\}. \tag{54}$$

Theorem 3.9. *Let $0\leq\Theta_1<\Theta_2$ and $m\in(0,1]$. Assume that $f:\mathcal{V}_m\to\mathbb{R}$ satisfies the assumptions of Lemma 3.2. In addition, if $\left|{}_{\Theta_1}D_qf\right|^\kappa$ and $\left|{}^{\Theta_2}D_qf\right|^\kappa$ ($\kappa\geq1$), are m-convex functions on $\mathcal{V}_m$, then*

$$\left|\Omega(U_1,U_2;q)\right|\leq\frac{\Theta_2-\Theta_1}{4}\left\{\left(\mathcal{I}_1^q(U_1)\right)^{\frac{\kappa-1}{\kappa}}\left(\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^\kappa\mathcal{I}_3^q(U_1)\right.\right.$$
$$\left.+m\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^\kappa\mathcal{I}_2^q(U_1)\right)^{\frac{1}{\kappa}}+\left(\mathcal{I}_1^q(U_2)\right)^{\frac{\kappa-1}{\kappa}}$$
$$\left.\times\left(\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^\kappa\mathcal{I}_3^q(U_2)+m\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^\kappa\mathcal{I}_2^q(U_2)\right)^{\frac{1}{\kappa}}\right\}, \tag{55}$$

and

$$\left|\Omega(U_1,U_2;q)\right|\leq\frac{\Theta_2-\Theta_1}{4\sqrt[\kappa]{2}}\left\{\left(\mathcal{I}_1^q(U_1)\right)^{\frac{\kappa-1}{\kappa}}\left(\left|{}_{\Theta_1}D_qf(\Theta_2)\right|^\kappa\mathcal{I}_3^q(U_1)+m\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^\kappa\mathcal{I}_6^q(U_1)\right)^{\frac{1}{\kappa}}\right.$$
$$\left.+\left(\mathcal{I}_1^q(U_2)\right)^{\frac{\kappa-1}{\kappa}}\left(\left|{}^{\Theta_2}D_qf(\Theta_1)\right|^\kappa\mathcal{I}_3^q(U_2)+m\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^\kappa\mathcal{I}_6^q(U_2)\right)^{\frac{1}{\kappa}}\right\}. \tag{56}$$

and $\mathcal{I}_1^q(C),\mathcal{I}_2^q(U_1),\mathcal{I}_3^q(U_1)$, and $\mathcal{I}_6^q(U_1)$ are given in Lemma 3.5.

Proof. Consider Lemma 3.2, taking modulus on both sides of the identity (21), and applying the power-mean inequality, we have

$$\left|\Omega(U_1,U_2;q)\right|\leq\frac{\Theta_2-\Theta_1}{4}\left[\int_0^1\left|q\Upsilon-U_1\right|\left|{}_{\Theta_1}D_qf\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_1\right)\right|d_q\Upsilon\right.$$
$$\left.+\int_0^1\left|q\Upsilon-U_2\right|\left|{}^{\Theta_2}D_qf\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_2\right)\right|d_q\Upsilon\right]$$
$$\leq\frac{\Theta_2-\Theta_1}{4}\left[\left(\int_0^1\left|q\Upsilon-U_1\right|d_q\Upsilon\right)^{1-\frac{1}{\kappa}}\right.$$
$$\times\left(\int_0^1\left|q\Upsilon-U_1\right|\left|{}_{\Theta_1}D_qf\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_1\right)\right|^\kappa d_q\Upsilon\right)^{\frac{1}{\kappa}}$$
$$+\left(\int_0^1\left|q\Upsilon-U_2\right|d_q\Upsilon\right)^{1-\frac{1}{\kappa}}$$
$$\left.\times\left(\int_0^1\left|q\Upsilon-U_2\right|\left|{}^{\Theta_2}D_qf\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right)+(1-\Upsilon)\Theta_2\right)\right|^\kappa d_q\Upsilon\right)^{\frac{1}{\kappa}}\right]$$

(By the m-convexity of $\left|{}_{\Theta_1}D_q f\right|^{\kappa}$ and $\left|{}^{\Theta_2}D_q f\right|^{\kappa}$, we get)

$$\leq \frac{\Theta_2-\Theta_1}{4}\left\{\left(\int_0^1 \left|q\Upsilon - U_1\right| d_q\Upsilon\right)^{1-\frac{1}{\kappa}}\left(\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}\int_0^1 \left|q\Upsilon - U_1\right|\Upsilon\, d_q\Upsilon\right.\right.$$
$$\left. + m\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\int_0^1 \left|q\Upsilon - U_1\right|(1-\Upsilon)\, d_q\Upsilon\right)^{\frac{1}{\kappa}} + \left(\int_0^1 \left|q\Upsilon - U_2\right| d_q\Upsilon\right)^{1-\frac{1}{\kappa}}$$
$$\times\left(\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}\int_0^1 \left|q\Upsilon - U_2\right|\Upsilon\, d_q\Upsilon\right.$$
$$\left.\left. + m\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\int_0^1 \left|q\Upsilon - U_2\right|(1-\Upsilon)\, d_q\Upsilon\right)^{\frac{1}{\kappa}}\right\}. \tag{57}$$

By utilizing the q-integrals evaluated in Lemma 3.5, we obtain the required inequality (55). For the next inequality by taking the modulus, we have

$$\left|\Omega(U_1,U_2;q)\right| \leq \frac{\Theta_2-\Theta_1}{4}\left[\int_0^1 \left|q\Upsilon - U_1\right|\left|{}_{\Theta_1}D_q f\left(\frac{\Upsilon}{2}\Theta_2+\left(1-\frac{\Upsilon}{2}\right)\Theta_1\right)\right| d_q\Upsilon\right.$$
$$\left. + \int_0^1 \left|q\Upsilon - U_2\right|\left|{}^{\Theta_2}D_q f\left(\frac{\Upsilon}{2}\Theta_1+\left(1-\frac{\Upsilon}{2}\right)\Theta_2\right)\right| d_q\Upsilon\right]$$

(By the power-mean inequality and m-convexity of $\left|{}_{\Theta_1}D_q f\right|^{\kappa}$ and $\left|{}^{\Theta_2}D_q f\right|^{\kappa}$, we get)

$$\leq \frac{\Theta_2-\Theta_1}{4}\left\{\left(\int_0^1 \left|q\Upsilon - U_1\right| d_q\Upsilon\right)^{1-\frac{1}{\kappa}}\left(\frac{1}{2}\left|{}_{\Theta_1}D_q f(\Theta_2)\right|^{\kappa}\int_0^1 \left|q\Upsilon - U_1\right|\Upsilon\, d_q\Upsilon\right.\right.$$
$$\left. + m\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\int_0^1 \left|q\Upsilon - U_1\right|\left(1-\frac{\Upsilon}{2}\right) d_q\Upsilon\right)^{\frac{1}{\kappa}}$$
$$+\left(\int_0^1 \left|q\Upsilon - U_2\right| d_q\Upsilon\right)^{1-\frac{1}{\kappa}}\left(\frac{1}{2}\left|{}^{\Theta_2}D_q f(\Theta_1)\right|^{\kappa}\int_0^1 \left|q\Upsilon - U_2\right|\Upsilon\, d_q\Upsilon\right.$$
$$\left.\left. + m\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\int_0^1 \left|q\Upsilon - U_2\right|\left(1-\frac{\Upsilon}{2}\right) d_q\Upsilon\right)^{\frac{1}{\kappa}}\right\}. \tag{58}$$

The required inequality (56) is thus obtained by utilizing again the q-integrals evaluated in Lemma 3.5. □

Corollary 3.10. *Under the conditions of Theorem 3.9, we have the following special cases:*
(a) $U_1 = U_2 = 0$, *then*

$$\left|f\left(\frac{\Theta_1+\Theta_2}{2}\right)-\frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}} f(\tau)\ {}_{\Theta_1}d_q\tau + \int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2} f(\tau)\ {}^{\Theta_2}d_q\tau\right]\right|$$

$$\leq \frac{(\Theta_2-\Theta_1)q(1+q+q^2)}{4[2]_q[3]_q\sqrt[\kappa]{q(1+q+q^2)}}\left\{\left(q(1+q)\left|_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}+mq^3\left|_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right.$$
$$\left.+\left(q(1+q)\left|^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}+mq^3\left|^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right\}. \quad (59)$$

(b) $U_1 = 1 = U_2$, then

$$\left|\frac{f(\Theta_1)+f(\Theta_2)}{2}-\frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}}f(\tau)\ _{\Theta_1}d_q\tau+\int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2}f(\tau)\ ^{\Theta_2}d_q\tau\right]\right|$$
$$\leq \frac{(\Theta_2-\Theta_1)(1+q+q^2)}{4[2]_q[3]_q\sqrt[\kappa]{1+q+q^2}}\left\{\left(\left|_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}+mq(1+q)\left|_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right.$$
$$\left.+\left(\left|^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}+mq(1+q)\left|^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right\}. \quad (60)$$

(c) $U_1 = \frac{q}{1+q} = U_2, q \in (0,1)$, then we have the following Simpson-like quantum integral inequality:

$$\left|\frac{1}{2(1+q)}\left[qf(\Theta_1)+2f\left(\frac{\Theta_1+\Theta_2}{2}\right)+qf(\Theta_2)\right]-\frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}}f(\tau)\ _{\Theta_1}d_q\tau+\int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2}f(\tau)\ ^{\Theta_2}d_q\tau\right]\right|$$
$$\leq \frac{(\Theta_2-\Theta_1)q^2(1+2q+2q^2+q^3)}{2(1+q)^3[2]_q[3]_q\sqrt[\kappa]{2q^2(1+2q+2q^2+q^3)}}$$
$$\times\left\{\left(q^2(q^2+4q+1)\left|_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}+mq^2(2q^3+3q^2+1)\left|_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right.$$
$$\left.+\left(q^2(q^2+4q+1)\left|^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}+mq^2(2q^3+3q^2+1)\left|^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right\}. \quad (61)$$

(d) $U_1 = \frac{1}{3}$ *and* $U_2 = \frac{2}{3}$*, then following Simpson-like quantum integral inequality holds:*

$$\left|\frac{1}{8}\left[f(\Theta_1)+4f\left(\frac{\Theta_1+\Theta_2}{2}\right)+3f(\Theta_2)\right]-\frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}} f(\tau)\ {}_{\Theta_1}d_q\tau+\int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2} f(\tau)\ {}^{\Theta_2}d_q\tau\right]\right|$$

$$\leq \begin{cases} \frac{\Theta_2-\Theta_1}{12[2]_q[3]_q}\left\{\frac{1-q-q^2-2q^3}{\sqrt[\kappa]{1-q-q^2-2q^3}}\left((1-2q-2q^2)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}\right.\right. \\ \left. + mq(1+q-2q^2)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}} + \frac{2+q+q^2-q^3}{\sqrt[\kappa]{2+q+q^2-q^3}} \\ \times\left((2-q-q^2)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}\right. \\ \left.\left. + mq(2+2q-q^2)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right\}, & \text{if} \quad 0<q\leq\frac{1}{3}, \\ \frac{\Theta_2-\Theta_1}{108[2]_q[3]_q}\left\{\frac{3(6q^3+5q^2+5q-1)}{\sqrt[\kappa]{3(6q^3+5q^2+5q-1)}}\left((18q^2+18q-7)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}\right.\right. \\ \left. + m(18q^3-3q^2-3q+4)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}} + \frac{9(2+q+q^2-q^3)}{\sqrt[\kappa]{9(2+q+q^2-q^3)}} \\ \times\left(9(2-q-q^2)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}\right. \\ \left.\left. + 9mq(2+2q-q^2)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right\}, & \text{if} \quad \frac{1}{3}<q\leq\frac{2}{3}, \\ \frac{\Theta_2-\Theta_1}{108[2]_q[3]_q}\left\{\frac{3(6q^3+5q^2+5q-1)}{\sqrt[\kappa]{3(6q^3+5q^2+5q-1)}}\left((18q^2+18q-7)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}\right.\right. \\ \left. + m(18q^3-3q^2-3q+4)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}} + \frac{3(3q^3+5q^2+5q+2)}{\sqrt[\kappa]{3(3q^3+5q^2+5q+2)}} \\ \times\left((9q^2+9q-2)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}\right. \\ \left.\left. + m(9q^3+6q^2+6q+8)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right\}, & \text{if} \quad \frac{2}{3}<q<1. \end{cases}$$

(e) $U_1 = \frac{1}{3} = U_2$, *then the following Simpson-like inequality is obtained:*

$$\left|\frac{1}{3}\left[\frac{f(\Theta_1)+f(\Theta_2)}{2}+2f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right]-\frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}} f(\tau)\ {}_{\Theta_1}d_q\tau+\int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2} f(\tau)\ {}^{\Theta_2}d_q\tau\right]\right|$$

$$\leq \begin{cases} \frac{(\Theta_2-\Theta_1)(1-q-q^2-2q^3)}{12[2]_q[3]_q\sqrt[\kappa]{1-q-q^2-2q^3}}\left\{\left((1-2q-2q^2)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^\kappa\right.\right. \\ \quad \left.+\ mq(1+q-2q^2)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^\kappa\right)^{\frac{1}{\kappa}} \\ \quad +\left((1-2q-2q^2)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^\kappa\right. \\ \left.\left.+\ mq(1+q-2q^2)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^\kappa\right)^{\frac{1}{\kappa}}\right\}, & \text{if} \quad 0<q\leq\frac{1}{3}, \\ \frac{(\Theta_2-\Theta_1)(6q^3+5q^2+5q-1)}{36[2]_q[3]_q\sqrt[\kappa]{3(6q^3+5q^2+5q-1)}}\left\{\left((18q^2+18q-7)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^\kappa\right.\right. \\ \quad \left.+\ m(18q^3-3q^2-3q+4)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^\kappa\right)^{\frac{1}{\kappa}} \\ \quad +\left((18q^2+18q-7)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^\kappa\right. \\ \left.\left.+\ m(18q^3-3q^2-3q+4)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^\kappa\right)^{\frac{1}{\kappa}}\right\}, & \text{if} \quad \frac{1}{3}<q<1. \end{cases}$$

Theorem 3.11. *Let* $0 \leq \Theta_1 < \Theta_2$ *and* $m \in (0,1]$. *Assume that* $f : \mathcal{V}_m \to \mathbb{R}$ *satisfies the assumptions of Lemma 3.2. In addition, if* $\left|{}_{\Theta_1}D_qf\right|^\kappa$ *and* $\left|{}^{\Theta_2}D_qf\right|^\kappa$ $(\kappa \geq 1)$, *are m-convex functions on* $\mathcal{V}_m$, *then*

$$\begin{aligned} \left|\Omega(U_1,U_2;q)\right| &\leq \frac{\Theta_2-\Theta_1}{4} \\ &\times\left\{\left(\mathcal{I}_2^q(U_1)\right)^{\frac{\kappa-1}{\kappa}}\left(\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^\kappa\mathcal{I}_4^q(U_1)+m\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^\kappa\mathcal{I}_9^q(U_1)\right)^{\frac{1}{\kappa}}\right. \\ &+\left(\mathcal{I}_3^q(U_1)\right)^{\frac{\kappa-1}{\kappa}}\left(\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^\kappa\mathcal{I}_8^q(U_1)+m\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^\kappa\mathcal{I}_4^q(U_1)\right)^{\frac{1}{\kappa}} \\ &+\left(\mathcal{I}_2^q(U_2)\right)^{\frac{\kappa-1}{\kappa}}\left(\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^\kappa\mathcal{I}_4^q(U_2)+m\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^\kappa\mathcal{I}_9^q(U_2)\right)^{\frac{1}{\kappa}} \\ &\left.+\left(\mathcal{I}_3^q(U_2)\right)^{\frac{\kappa-1}{\kappa}}\left(\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^\kappa\mathcal{I}_8^q(U_2)+m\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^\kappa\mathcal{I}_4^q(U_2)\right)^{\frac{1}{\kappa}}\right\}, \end{aligned} \tag{62}$$

and

$$\begin{aligned} \left|\Omega(U_1,U_2;q)\right| \leq \frac{\Theta_2-\Theta_1}{4\sqrt[\kappa]{2}}&\left\{\left(\mathcal{I}_2^q(U_1)\right)^{\frac{\kappa-1}{\kappa}}\left(\left|{}_{\Theta_1}D_qf(\Theta_2)\right|^\kappa\mathcal{I}_4^q(U_1)+m\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^\kappa\mathcal{I}_7^q(U_1)\right)^{\frac{1}{\kappa}}\right. \\ &+\left(\mathcal{I}_3^q(U_1)\right)^{\frac{\kappa-1}{\kappa}}\left(\left|{}_{\Theta_1}D_qf(\Theta_2)\right|^\kappa\mathcal{I}_8^q(U_1)+m\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^\kappa\mathcal{I}_5^q(U_1)\right)^{\frac{1}{\kappa}} \\ &+\left(\mathcal{I}_2^q(U_2)\right)^{\frac{\kappa-1}{\kappa}}\left(\left|{}_{\Theta_1}D_qf(\Theta_2)\right|^\kappa\mathcal{I}_4^q(U_2)+m\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^\kappa\mathcal{I}_7^q(U_2)\right)^{\frac{1}{\kappa}} \\ &\left.+\left(\mathcal{I}_3^q(U_2)\right)^{\frac{\kappa-1}{\kappa}}\left(\left|{}_{\Theta_1}D_qf(\Theta_2)\right|^\kappa\mathcal{I}_8^q(U_2)+m\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^\kappa\mathcal{I}_5^q(U_2)\right)^{\frac{1}{\kappa}}\right\}, \end{aligned} \tag{63}$$

where $\mathcal{I}_2^q(C) - \mathcal{I}_9^q(C)$ are given in Lemma 3.5.

Proof. Consider Lemma 3.2, taking modulus on both sides of the identity (21), and applying the power-mean inequality, we have

$$\int_0^1 \left|q\Upsilon - U_1\right| \left|{}_{\Theta_1}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right) + (1-\Upsilon)\Theta_1\right)\right| d_q\Upsilon$$

$$\leq \left(\int_0^1 \left|q\Upsilon - U_1\right| (1-\Upsilon)\, d_q\Upsilon\right)^{1-\frac{1}{\kappa}}$$

$$\times \left(\int_0^1 (1-\Upsilon)\left|q\Upsilon - U_1\right| \left|{}_{\Theta_1}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right) + (1-\Upsilon)\Theta_1\right)\right|^{\kappa} d_q\Upsilon\right)^{\frac{1}{\kappa}}$$

$$+ \left(\int_0^1 \left|q\Upsilon - U_1\right| \Upsilon\, d_q\Upsilon\right)^{1-\frac{1}{\kappa}} \left(\int_0^1 \Upsilon\left|q\Upsilon - U_1\right| \left|{}_{\Theta_1}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right) + (1-\Upsilon)\Theta_1\right)\right|^{\kappa} d_q\Upsilon\right)^{\frac{1}{\kappa}}$$

(By the m-convexity of $\left|{}_{\Theta_1}D_q f\right|^{\kappa}$, we get)

$$\leq \left(\mathcal{I}_2^q(U_1)\right)^{1-\frac{1}{\kappa}} \left(\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa} \int_0^1 (1-\Upsilon)\left|q\Upsilon - U_1\right| \Upsilon\, d_q\Upsilon\right.$$

$$\left. + m\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1}{m}\right)\right|^{\kappa} \int_0^1 \left|q\Upsilon - U_1\right| (1-\Upsilon)^2\, d_q\Upsilon\right)^{\frac{1}{\kappa}} + \left(\mathcal{I}_3^q(U_1)\right)^{1-\frac{1}{\kappa}}$$

$$\times \left(\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa} \int_0^1 \left|q\Upsilon - U_1\right| \Upsilon^2\, d_q\Upsilon\right.$$

$$\left. + m\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1}{m}\right)\right|^{\kappa} \int_0^1 \Upsilon\left|q\Upsilon - U_1\right| (1-\Upsilon)\, d_q\Upsilon\right)^{\frac{1}{\kappa}}. \tag{64}$$

Similarly,

$$\int_0^1 \left|q\Upsilon - U_2\right| \left|{}^{\Theta_2}D_q f\left(\Upsilon\left(\frac{\Theta_1+\Theta_2}{2}\right) + (1-\Upsilon)\Theta_2\right)\right| d_q\Upsilon$$

$$\leq \left(\int_0^1 \left|q\Upsilon - U_2\right| (1-\Upsilon)\, d_q\Upsilon\right)^{1-\frac{1}{\kappa}} \left(\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa} \int_0^1 (1-\Upsilon)\left|q\Upsilon - U_2\right| \Upsilon\, d_q\Upsilon\right.$$

$$\left. + m\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_2}{m}\right)\right|^{\kappa} \int_0^1 \left|q\Upsilon - U_2\right| (1-\Upsilon)^2\, d_q\Upsilon\right)^{\frac{1}{\kappa}} + \left(\int_0^1 \left|q\Upsilon - U_2\right| \Upsilon\, d_q\Upsilon\right)^{1-\frac{1}{\kappa}}$$

$$\times \left(\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa} \int_0^1 \left|q\Upsilon - U_2\right| \Upsilon^2\, d_q\Upsilon\right.$$

$$+ m\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\int_0^1 \Upsilon\left|q\Upsilon - U_2\right|(1-\Upsilon)\,d_q\Upsilon\Bigg)^{\frac{1}{\kappa}}. \tag{65}$$

Inequalities (64) and (65) together with (45) gives the inequality (62). For the next inequality (63), we observe that

$$\begin{aligned}
&\int_0^1 \left|q\Upsilon - U_1\right|\left|{}_{\Theta_1}D_q f\left(\frac{\Upsilon}{2}\Theta_2 + \left(1-\frac{\Upsilon}{2}\right)\Theta_1\right)\right| d_q\Upsilon \\
&\leq \left(\int_0^1 \left|q\Upsilon - U_1\right|(1-\Upsilon)\,d_q\Upsilon\right)^{1-\frac{1}{\kappa}}\left(\frac{\left|{}_{\Theta_1}D_q f(\Theta_2)\right|^{\kappa}}{2}\int_0^1 \Upsilon(1-\Upsilon)\left|q\Upsilon - U_1\right| d_q\Upsilon\right. \\
&\quad \left. + m\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\int_0^1 (1-\Upsilon)\left|q\Upsilon - U_1\right|\left(1-\frac{\Upsilon}{2}\right) d_q\Upsilon\right)^{\frac{1}{\kappa}} + \left(\int_0^1 \left|q\Upsilon - U_1\right|\Upsilon\,d_q\Upsilon\right)^{1-\frac{1}{\kappa}} \\
&\quad \times\left(\frac{\left|{}_{\Theta_1}D_q f(\Theta_2)\right|^{\kappa}}{2}\int_0^1 \Upsilon^2\left|q\Upsilon - U_1\right| d_q\Upsilon + m\left|{}_{\Theta_1}D_q f\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\int_0^1 \Upsilon\left|q\Upsilon - U_1\right|\left(1-\frac{\Upsilon}{2}\right) d_q\Upsilon\right)^{\frac{1}{\kappa}}.
\end{aligned} \tag{66}$$

Similarly to (66), we obtain

$$\begin{aligned}
&\int_0^1 \left|q\Upsilon - U_2\right|\left|{}^{\Theta_2}D_q f\left(\frac{\Upsilon}{2}\Theta_1 + \left(1-\frac{\Upsilon}{2}\right)\Theta_2\right)\right| d_q\Upsilon \\
&\leq \left(\int_0^1 \left|q\Upsilon - U_2\right|(1-\Upsilon)\,d_q\Upsilon\right)^{1-\frac{1}{\kappa}}\left(\frac{\left|{}^{\Theta_2}D_q f(\Theta_1)\right|^{\kappa}}{2}\int_0^1 \Upsilon(1-\Upsilon)\left|q\Upsilon - U_2\right| d_q\Upsilon\right. \\
&\quad \left. + m\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\int_0^1 (1-\Upsilon)\left|q\Upsilon - U_2\right|\left(1-\frac{\Upsilon}{2}\right) d_q\Upsilon\right)^{\frac{1}{\kappa}} + \left(\int_0^1 \left|q\Upsilon - U_2\right|\Upsilon\,d_q\Upsilon\right)^{1-\frac{1}{\kappa}} \\
&\quad \times\left(\frac{\left|{}^{\Theta_2}D_q f(\Theta_1)\right|^{\kappa}}{2}\int_0^1 \Upsilon^2\left|q\Upsilon - U_2\right| d_q\Upsilon + m\left|{}^{\Theta_2}D_q f\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\int_0^1 \Upsilon\left|q\Upsilon - U_2\right|\left(1-\frac{\Upsilon}{2}\right) d_q\Upsilon\right)^{\frac{1}{\kappa}}.
\end{aligned} \tag{67}$$

Thus (63) is the outcome of inequalities (66) and (67), which then completes the proof. □

Corollary 3.12. *Under the conditions of Theorem 3.11, we have the following special cases:*
(a) $U_1 = U_2 = 0$*, then*

$$\begin{aligned}
&\left|f\left(\frac{\Theta_1+\Theta_2}{2}\right) - \frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}} f(\tau)\,{}_{\Theta_1}d_q\tau + \int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2} f(\tau)\,{}^{\Theta_2}d_q\tau\right]\right| \\
&\leq \frac{\Theta_2-\Theta_1}{4[2]_q[3]_q[4]_q}\left\{\frac{q^3(1+q+q^2+q^3)}{\sqrt[\kappa]{q^3(1+q+q^2)}}\right.
\end{aligned}$$

$$\times\left[\left(q^4(1+q)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}+mq^3(1+q^3)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right.$$

$$\left.+\left(q^4(1+q)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}+mq^3(1+q^3)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right]$$

$$+\frac{q(1+2q+2q^2+2q^3+q^4)}{\sqrt[\kappa]{q(1+2q+2q^2+2q^3+q^4)}}\left[\left(q(1+2q+2q^2+q^3)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}\right.\right.$$

$$\left.+mq^4(1+q)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}+\left(q(1+2q+2q^2+2q^3)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}\right.$$

$$\left.\left.\left.+mq^4(1+q)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right]\right\}. \tag{68}$$

(b) $U_1 = 1 = U_2$, *then*

$$\left|\frac{f(\Theta_1)+f(\Theta_2)}{2}-\frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}}f(\tau)\ {}_{\Theta_1}d_q\tau+\int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2}f(\tau)\ {}^{\Theta_2}d_q\tau\right]\right|$$

$$\leq\frac{\Theta_2-\Theta_1}{4[2]_q[3]_q[4]_q}\left\{\frac{q(1+2q+2q^2+2q^3+q^4)}{\sqrt[\kappa]{q(1+2q+2q^2+2q^3+q^4)}}\left[\left(q^2(1+q)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}\right.\right.\right.$$

$$\left.+\ mq(1+q+q^2+2q^3+q^4)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}$$

$$\left.+\left(q^2(1+q)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}+\ mq(1+q+q^2+2q^3+q^4)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right]$$

$$+\ \frac{(1+q+q^2+q^3)}{\sqrt[\kappa]{(1+q+q^2+q^3)}}\left[\left((1+q)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}+mq^2(1+q)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right.$$

$$\left.\left.+\left((1+q)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}+\ mq^2(1+q)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right]\right\}. \tag{69}$$

(c) $U_1 = \frac{q}{1+q} = U_2, q \in (0,1)$, *then we have the following Simpson-like quantum integral inequality:*

$$\left|\frac{1}{2(1+q)}\left[qf(\Theta_1)+2f\left(\frac{\Theta_1+\Theta_2}{2}\right)+qf(\Theta_2)\right]\right.$$

$$\left.-\ \frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}}f(\tau)\ {}_{\Theta_1}d_q\tau+\int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2}f(\tau)\ {}^{\Theta_2}d_q\tau\right]\right|$$

$$\leq\frac{\Theta_2-\Theta_1}{4(q+1)^3[2]_q[3]_q[4]_q}\left\{\frac{2q^8+5q^7+5q^6+6q^5+5q^4+q^2}{\sqrt[\kappa]{2q^8+5q^7+5q^6+6q^5+5q^4+q^2}}\left[\left(Q_1(q)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}\right.\right.\right.$$

$$\left.+\ mQ_2(q)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}+\left(Q_1(q)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}\right.$$

$$\left.\left.+\ mQ_2(q)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right]+\frac{q^7+5q^6+6q^5+6q^4+5q^3+q^2}{\sqrt[\kappa]{q^7+5q^6+6q^5+6q^4+5q^3+q^2}}$$

$$\times\left[\left(Q_3(q)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}+mQ_1(q)\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right.$$
$$\left.+\left(Q_3(q)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}+mQ_1(q)\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right]\Bigg\}, \tag{70}$$

where

$$Q_1(q):=q^3(q^4+4q^3+2q^2-2q+1),\quad Q_2(q):=q^2(2q^6+4q^5+q^4+4q^3+6q^2+1),$$

and

$$Q_3(q):=q^6+4q^5+8q^4+4q^3+q^2.$$

(d) $U_1=\frac{1}{3}$ and $U_2=\frac{2}{3}$, then following Simpson-like quantum integral inequality holds:

$$\left|\frac{1}{6}\left[f(\Theta_1)+3f\left(\frac{\Theta_1+\Theta_2}{2}\right)+2f(\Theta_2)\right]-\frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}}f(\tau)\,{}_{\Theta_1}d_q\tau+\int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2}f(\tau)\,{}^{\Theta_2}d_q\tau\right]\right|$$

$$\leq\begin{cases}
\frac{\Theta_2-\Theta_1}{12[2]_q[3]_q[4]_q}\left\{\frac{q+2q^2-q^5-2q^6}{\sqrt[\kappa]{q+2q^2-q^5-2q^6}}\left(R_1\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}+mR_2\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right. \\
+\frac{1-q-3q^2-3q^3-4q^4-2q^5}{\sqrt[\kappa]{1-q-3q^2-3q^3-4q^4-2q^5}}\left(R_3\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}+mR_1\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}} \\
+\frac{2q+4q^2+3q^3+3q^4+q^5-q^6}{\sqrt[\kappa]{2q+4q^2+3q^3+3q^4+q^5-q^6}}\left(R_4\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}+mR_5\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}} \\
\left.+\frac{2+q-2q^4-q^5}{\sqrt[\kappa]{2+q-2q^4-q^5}}\left(R_6\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}+mR_4\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right\}, & \text{if}\quad 0<q\leq\frac{1}{3}, \\
\frac{\Theta_2-\Theta_1}{324[2]_q[3]_q[4]_q}\left\{\frac{3(18q^6+15q^5+12q^4+16q^3-2q^2+q+4)}{\sqrt[\kappa]{3(18q^6+15q^5+12q^4+16q^3-2q^2+q+4)}}\left(R_7\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}\right.\right. \\
\left.+mR_8\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}+\frac{3(18q^5+36q^4+29q^3+29q^2+11q-7)}{\sqrt[\kappa]{3(18q^5+36q^4+29q^3+29q^2+11q-7)}} \\
\times\left(R_9\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}+mR_7\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}} \\
+\frac{2q+4q^2+3q^3+3q^4+q^5-q^6}{\sqrt[\kappa]{2q+4q^2+3q^3+3q^4+q^5-q^6}}\left(R_4\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}\right. \\
\left.+mR_5\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}+\frac{2+q-2q^4-q^5}{\sqrt[\kappa]{2+q-2q^4-q^5}}\left(R_6\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}\right. \\
\left.\left.+mR_4\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right\}, & \text{if}\quad \frac{1}{3}<q\leq\frac{2}{3}, \\
\frac{\Theta_2-\Theta_1}{324[2]_q[3]_q[4]_q}\left\{\frac{3(18q^6+15q^5+12q^4+16q^3-2q^2+q+4)}{\sqrt[\kappa]{3(18q^6+15q^5+12q^4+16q^3-2q^2+q+4)}}\right. \\
\times\left(R_7\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}+mR_8\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}} \\
+\frac{3(18q^5+36q^4+29q^3+29q^2+11q-7)}{\sqrt[\kappa]{3(18q^5+36q^4+29q^3+29q^2+11q-7)}}\left(R_9\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}\right. \\
\left.+mR_7\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}} \\
+\frac{3(9q^6+15q^5+21q^4+29q^3+20q^2+14q+8)}{\sqrt[\kappa]{3(9q^6+15q^5+21q^4+29q^3+20q^2+14q+8)}}\left(R_{10}\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}\right. \\
\left.+mR_{11}\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}+\frac{3(9q^5+18q^4+16q^3+16q^2+7q-2)}{\sqrt[\kappa]{3(9q^5+18q^4+16q^3+16q^2+7q-2)}} \\
\left.\times\left(R_{12}\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^{\kappa}+mR_{10}\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^{\kappa}\right)^{\frac{1}{\kappa}}\right\}, & \text{if}\quad \frac{2}{3}<q<1,
\end{cases}$$

where

$$R_1 := q^2(1+q-2q^2-2q^3),\quad R_2 := q(1+q-q^2+2q^3+q^4-2q^5),$$
$$R_3 := 1-q-4q^2-4q^3-2q^4,\quad R_4 := q^2(2+2q-q^2-q^3),$$
$$R_5 := q(2+2q+q^2+4q^3+2q^4-q^5),\quad R_6 := 2+q-2q^2-2q^3-q^4,$$
$$R_7 := 54q^5+54q^4-21q^3-21q^2+4q+4,\quad R_8 := 54q^6-9q^5-18q^4+69q^3+15q^2-q+8,$$
$$R_9 := 54q^4+108q^3+108q^2+29q-25,\quad R_{10} := 27q^5+27q^4-6q^3-6q^2+16q+16,$$
$$R_{11} := 27q^6+18q^5+36q^4+66q^2+26q+8,\quad R_{12} := 27q^4+54q^3+54q^2+5q-22.$$

(e) $U_1 = \frac{1}{3} = U_2$, then the following Simpson-like inequality is obtained:

$$\left|\frac{1}{3}\left[\frac{f(\Theta_1)+f(\Theta_2)}{2}+2f\left(\frac{\Theta_1+\Theta_2}{2}\right)\right]-\frac{1}{\Theta_2-\Theta_1}\left[\int_{\Theta_1}^{\frac{\Theta_1+\Theta_2}{2}} f(\tau)\ {}_{\Theta_1}d_q\tau+\int_{\frac{\Theta_1+\Theta_2}{2}}^{\Theta_2} f(\tau)\ {}^{\Theta_2}d_q\tau\right]\right|$$

$$\leq\begin{cases}\frac{\Theta_2-\Theta_1}{12[2]_q[3]_q[4]_q}\left\{\frac{q+2q^2-q^5-2q^6}{\sqrt[\kappa]{q+2q^2-q^5-2q^6}}\left[\left(R_1\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^\kappa+mR_2\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^\kappa\right)^{\frac{1}{\kappa}}\right.\right.\\ \left.+\left(R_1\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^\kappa+mR_2\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^\kappa\right)^{\frac{1}{\kappa}}\right]\\ +\frac{1-q-3q^2-3q^3-4q^4-2q^5}{\sqrt[\kappa]{1-q-3q^2-3q^3-4q^4-2q^5}}\left[\left(R_3\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^\kappa+mR_1\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^\kappa\right)^{\frac{1}{\kappa}}\right.\\ \left.\left.+\left(R_3\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^\kappa+mR_1\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^\kappa\right)^{\frac{1}{\kappa}}\right]\right\}, & \text{if}\quad 0<q\leq\frac{1}{3},\\ \frac{\Theta_2-\Theta_1}{324[2]_q[3]_q[4]_q}\left\{\frac{3(18q^6+15q^5+12q^4+16q^3-2q^2+q+4)}{\sqrt[\kappa]{3(18q^6+15q^5+12q^4+16q^3-2q^2+q+4)}}\left[\left(R_7\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^\kappa\right.\right.\right.\\ \left.+\ mR_8\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^\kappa\right)^{\frac{1}{\kappa}}+\left(R_7\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^\kappa\right.\\ \left.\left.+\ mR_8\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^\kappa\right)^{\frac{1}{\kappa}}\right]+\frac{3(18q^6+15q^5+12q^4+16q^3-2q^2+q+4)}{\sqrt[\kappa]{3(18q^6+15q^5+12q^4+16q^3-2q^2+q+4)}}\\ \times\left[\left(R_9\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^\kappa+mR_7\left|{}_{\Theta_1}D_qf\left(\frac{\Theta_1}{m}\right)\right|^\kappa\right)^{\frac{1}{\kappa}}\right.\\ \left.\left.+\left(R_9\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_1+\Theta_2}{2}\right)\right|^\kappa+mR_7\left|{}^{\Theta_2}D_qf\left(\frac{\Theta_2}{m}\right)\right|^\kappa\right)^{\frac{1}{\kappa}}\right]\right\}, & \text{if}\quad \frac{1}{3}<q<1,\end{cases}$$

where R_i's are the same as given in case (d) for all $i = 1, 2, \ldots, 12$.

4. Applications to special means

Let us recall some useful special means of real numbers $0 < \Theta_1 < \Theta_2$.

- The arithmetic mean:
$$\mathcal{A}(\Theta_1,\Theta_2) := \frac{\Theta_1+\Theta_2}{2}.$$
- The geometric mean:
$$\mathcal{G}(\Theta_1,\Theta_2) := \sqrt{\Theta_1\Theta_2}.$$

- The logarithmic mean:

$$\mathcal{L}(\Theta_1,\Theta_2) := \frac{\Theta_2-\Theta_1}{\ln\Theta_2-\ln\Theta_1}.$$

- The generalized logarithmic mean:

$$£_\rho(\Theta_1,\Theta_2) := \left(\frac{\Theta_2^{\rho+1}-\Theta_1^{\rho+1}}{(\rho+1)(\Theta_2-\Theta_1)}\right)^{\frac{1}{\rho}},\ \rho\in\mathbb{R}\setminus\{-1,0\}.$$

Now, we are in position to obtain the following inequalities in terms of above special means of real numbers using our main results.

Proposition 4.1. *Let* $0<\Theta_1<\Theta_2$ *and* $0<q<1$*. Then for* $\mathcal{U}\in\mathbb{R}$*, we have*

$$\begin{aligned}&\Big|\mathcal{U}\mathcal{A}(\Theta_1,\Theta_2)+(1-\mathcal{U})\mathcal{A}^2(\Theta_1,\Theta_2)\\&-\left[\frac{3(1+3q+2q^2+2q^3)}{4[2]_q[3]_q}£_2^2(\Theta_1,\Theta_2)+\frac{1-q+2q^2-2q^3}{4[2]_q[3]_q}\mathcal{G}^2(\Theta_1,\Theta_2)\right]\Big|\\&\leq\frac{\Theta_2-\Theta_1}{4}\Big[(\mathcal{A}((3-q)\Theta_1,(1+q)\Theta_2)+\mathcal{A}((1+q)\Theta_1,(3-q)\Theta_2))\,\mathcal{I}_3^q(\mathcal{U})+4\mathcal{I}_2^q(\mathcal{U})\mathcal{A}(\Theta_1,\Theta_2)\Big].\end{aligned}\tag{71}$$

In general, for $n\geq 2$*, we get*

$$\begin{aligned}&\left|\mathcal{U}\mathcal{A}(\Theta_1^n,\Theta_2^n)+(1-\mathcal{U})\mathcal{A}^n(\Theta_1,\Theta_2)-\mathcal{D}_1(\Theta_1,\Theta_2;q)\right|\\&\leq\frac{\Theta_2-\Theta_1}{4}\Big[(n-1)\Big(£_{n-1}^{n-1}(\mathcal{A}((2-q)\Theta_1,q\Theta_2),\mathcal{A}(\Theta_1,\Theta_2))\\&+£_{n-1}^{n-1}(\mathcal{A}(q\Theta_1,(2-q)\Theta_2),\mathcal{A}(\Theta_1,\Theta_2))\Big)\mathcal{I}_3^q(\mathcal{U})+2n\mathcal{I}_2^q(\mathcal{U})\mathcal{A}(\Theta_1^{n-1},\Theta_2^{n-1})\Big],\end{aligned}\tag{72}$$

where

$$\mathcal{D}_1(\Theta_1,\Theta_2;q) := \frac{1-q}{2}\sum_{\delta=0}^{\infty}q^\delta\left[\left(\frac{q^\delta}{2}\Theta_2+\left(1-\frac{q^\delta}{2}\right)\Theta_1\right)^n+\left(\frac{q^\delta}{2}\Theta_1+\left(1-\frac{q^\delta}{2}\right)\Theta_2\right)^n\right].$$

Proof. Let consider Theorem 3.6 with $U_1=U_2=\mathcal{U}$. If we take $f(u)=u^2$ with $m=1$, then required inequality (71) is obtained by (43) while the inequality (43) reduces to (72), if we choose $f(u)=u^n$ for $n\geq 2$. □

Corollary 4.2. *Under the conditions of Proposition 4.1, letting* $q\to 1^-$*, we have the following special cases:*
(*a*) *If we take* $\mathcal{U}=0$ *and* $f(u)=u^n$*, then*

$$\left|\mathcal{A}^n(\Theta_1,\Theta_2)-£_n^n(\Theta_1,\Theta_2)\right|\leq\frac{n(\Theta_2-\Theta_1)}{12}\left\{2\mathcal{A}^{n-1}(\Theta_1,\Theta_2)+\mathcal{A}(\Theta_1^{n-1},\Theta_2^{n-1})\right\}.$$

(*b*) *If we choose* $\mathcal{U}=1$*, then*

$$\left|\mathcal{A}(\Theta_1^n,\Theta_2^n)-£_n^n(\Theta_1,\Theta_2)\right|\leq\frac{n(\Theta_2-\Theta_1)}{12}\left\{\mathcal{A}^{n-1}(\Theta_1,\Theta_2)+2\mathcal{A}(\Theta_1^{n-1},\Theta_2^{n-1})\right\}.\tag{73}$$

Proposition 4.3. *Let* $0<\Theta_1<\Theta_2$ *and* $U_1,U_2\in\mathbb{R}$*. Then*

$$\begin{aligned}&\left|\mathcal{A}(U_1\Theta_1^n,U_2\Theta_2^n)+\left(1-\frac{U_1+U_2}{2}\right)\mathcal{A}^n(\Theta_1,\Theta_2)-£_n^n(\Theta_1,\Theta_2)\right|\\&\leq\begin{cases}\frac{n(\Theta_2-\Theta_1)}{24}\Big\{(4-3(U_1+U_2)+2U_2^3)\mathcal{A}^{n-1}(\Theta_1,\Theta_2)\\\quad+(1-3U_1)\Theta_1^{n-1}+(1-3U_2+6U_2^2-2U_2^3)\Theta_2^{n-1}\Big\}, & \text{if}\quad U_1\leq 0<U_2<1,\\\frac{n(\Theta_2-\Theta_1)}{24}\Big\{(3(U_2-U_1)+2U_1^3)\mathcal{A}^{n-1}(\Theta_1,\Theta_2)\\\quad+(1-3U_1+6U_1^2-2U_1^3)\Theta_1^{n-1}+(3U_2-1)\Theta_2^{n-1}\Big\}, & \text{if}\quad 0<U_1<1\leq U_2.\end{cases}\end{aligned}$$

Proof. If we take $f(u) = u^n$ for $n \geq 2$ in Theorem 3.6 with $m = 1$ and letting $q \to 1^-$, we get the desired result. □

Proposition 4.4. *Let $0 < \Theta_1 < \Theta_2$ and $0 < q < 1$. Then for $\mathcal{U} \in \mathbb{R}$ and $\kappa \geq 1$, we have*

$$\begin{aligned}&\left|\mathcal{U}\mathcal{A}(\Theta_1^{-1}, \Theta_2^{-1}) + (1-\mathcal{U})\mathcal{A}^{-1}(\Theta_1, \Theta_2) - \mathcal{D}_2(\Theta_1, \Theta_2; q)\right| \\ &\leq \frac{\Theta_2 - \Theta_1}{2}\left(\mathcal{I}_1^q(\mathcal{U})\right)^{1-\frac{1}{\kappa}}\left\{\left(|\Delta_1|^\kappa \mathcal{I}_3^q(\mathcal{U}) + 2^{-\kappa}\Theta_1^{-2\kappa}\mathcal{I}_2^q(\mathcal{U})\right)^{\frac{1}{\kappa}}\right. \\ &\quad \left. + \left(|\Delta_2|^\kappa \mathcal{I}_3^q(\mathcal{U}) + 2^{-\kappa}\Theta_2^{-2\kappa}\mathcal{I}_2^q(\mathcal{U})\right)^{\frac{1}{\kappa}}\right\},\end{aligned} \tag{74}$$

where

$$\mathcal{D}_2(\Theta_1, \Theta_2; q) := \frac{1-q}{2}\sum_{\delta=0}^{\infty} q^\delta\left[\left(\frac{q^\delta}{2}\Theta_2 + \left(1 - \frac{q^\delta}{2}\right)\Theta_1\right)^{-1} + \left(\frac{q^\delta}{2}\Theta_1 + \left(1 - \frac{q^\delta}{2}\right)\Theta_2\right)^{-1}\right],$$

$$\Delta_1 := \pounds_{-2}^{-2}\left(\mathcal{A}^{-1}((2-q)\Theta_1, q\Theta_2), \mathcal{A}^{-1}(\Theta_1, \Theta_2)\right),$$

and

$$\Delta_2 := \pounds_{-2}^{-2}\left(\mathcal{A}^{-1}(q\Theta_1, (2-q)\Theta_2), \mathcal{A}^{-1}(\Theta_1, \Theta_2)\right).$$

Proof. Let consider Theorem 3.9 with $U_1 = U_2 = \mathcal{U}$. If we choose $f(u) = u^{-1}$ with $m = 1$, then (55) reduces to the desired inequality (74). □

Corollary 4.5. *Under the conditions of Proposition 4.4, letting $q \to 1^-$, we have the following special cases:*
(*a*) *If we take $\mathcal{U} = 0$, then*

$$\begin{aligned}&\left|\mathcal{A}^{-1}(\Theta_1, \Theta_2) - \mathcal{L}^{-1}(\Theta_1, \Theta_2)\right| \\ \leq\ &\frac{\Theta_2 - \Theta_1}{8\sqrt[\kappa]{3}}\left\{\left(2\mathcal{A}^{-2\kappa}(\Theta_1, \Theta_2) + \Theta_1^{-2\kappa}\right)^{\frac{1}{\kappa}} + \left(2\mathcal{A}^{-2\kappa}(\Theta_1, \Theta_2) + \Theta_2^{-2\kappa}\right)^{\frac{1}{\kappa}}\right\}.\end{aligned} \tag{75}$$

(*b*) *If we choose $\mathcal{U} = 1$, then*

$$\begin{aligned}&\left|\mathcal{A}(\Theta_1^{-1}, \Theta_2^{-1}) - \mathcal{L}^{-1}(\Theta_1, \Theta_2)\right| \\ \leq\ &\frac{\Theta_2 - \Theta_1}{8\sqrt[\kappa]{3}}\left\{\left(\mathcal{A}^{-2\kappa}(\Theta_1, \Theta_2) + 2\Theta_1^{-2\kappa}\right)^{\frac{1}{\kappa}} + \left(\mathcal{A}^{-2\kappa}(\Theta_1, \Theta_2) + 2\Theta_2^{-2\kappa}\right)^{\frac{1}{\kappa}}\right\}.\end{aligned} \tag{76}$$

Proposition 4.6. *Let $0 < \Theta_1 < \Theta_2$ and $U_1, U_2 \in \mathbb{R}$. Then for $\kappa \geq 1$, we have*

$$\begin{aligned}&\left|\mathcal{A}(U_1\Theta_1^{-1}, U_2\Theta_2^{-1}) + \left(1 - \frac{U_1 + U_2}{2}\right)\mathcal{A}^{-1}(\Theta_1, \Theta_2) - \mathcal{L}^{-1}(\Theta_1, \Theta_2)\right| \\ &\leq \begin{cases} \frac{\Theta_2-\Theta_1}{24}\left\{\frac{3-6U_1}{\sqrt[\kappa]{3(1-2U_1)}}\left((2-3U_1)\mathcal{A}^{-2\kappa}(\Theta_1, \Theta_2) + (1-3U_1)\Theta_1^{-2\kappa}\right)^{\frac{1}{\kappa}}\right. \\ + \frac{6U_2^2-6U_2+3}{\sqrt[\kappa]{6U_2^2-6U_2+3}}\left((2-3U_2+2U_2^3)\mathcal{A}^{-2\kappa}(\Theta_1, \Theta_2)\right. \\ \left.\left. + (1-3U_2+6U_2^2-2U_2^3)\Theta_2^{-2\kappa}\right)^{\frac{1}{\kappa}}\right\}, & \text{if} \quad U_1 \leq 0 < U_2 < 1, \\ \frac{\Theta_2-\Theta_1}{24}\left\{\frac{6U_1^2-6U_1+3}{\sqrt[\kappa]{6U_1^2-6U_1+3}}\left((2-3U_1+2U_1^3)\mathcal{A}^{-2\kappa}(\Theta_1, \Theta_2)\right.\right. \\ \left. + (1-3U_1+6U_1^2-2U_1^3)\Theta_1^{-2\kappa}\right)^{\frac{1}{\kappa}} \\ \left. + \frac{6U_2-3}{\sqrt[\kappa]{6U_2-3}}\left((3U_2-2)\mathcal{A}^{-2\kappa}(\Theta_1, \Theta_2) + (3U_2-1)\Theta_2^{-2\kappa}\right)^{\frac{1}{\kappa}}\right\}, & \text{if} \quad 0 < U_1 < 1 \leq U_2. \end{cases}\end{aligned}$$

Proof. If we take $f(u) = u^{-1}$ in Theorem 3.9 with $m = 1$ and letting $q \to 1^-$, we get the desired result. □

Proposition 4.7. *Let $0 < \Theta_1 < \Theta_2$ and $0 < q < 1$. Then for $\mathcal{U} \in \mathbb{R}$ and $\kappa \geq 1$, we have*

$$\begin{aligned}
&\left|\mathcal{U}\mathcal{A}(\lambda_1,\lambda_2) + (1-\mathcal{U})\mathcal{G}(\lambda_1,\lambda_2) - \mathcal{D}_3(\Theta_1,\Theta_2;q)\right| \\
&\leq \frac{\Theta_2-\Theta_1}{4} \\
&\quad \times \left\{\left(\mathcal{I}_2^q(\mathcal{U})\right)^{\frac{\kappa-1}{\kappa}} \left[\left(\left|\mathcal{L}\left(\mathcal{G}(\lambda_1^{2-q},\lambda_q^q),\mathcal{G}(\lambda_1,\lambda_2)\right)\right|^{\kappa} \mathcal{I}_4^q(\mathcal{U}) + \lambda_1^{\kappa}\mathcal{I}_9^q(\mathcal{U})\right)^{\frac{1}{\kappa}}\right.\right. \\
&\quad + \left.\left(\left|\mathcal{L}\left(\mathcal{G}(\lambda_1^q,\lambda_2^{2-q}),\mathcal{G}(\lambda_1,\lambda_2)\right)\right|^{\kappa} \mathcal{I}_4^q(\mathcal{U}) + \lambda_2^{\kappa}\mathcal{I}_9^q(\mathcal{U})\right)^{\frac{1}{\kappa}}\right] \\
&\quad + \left(\mathcal{I}_3^q(U_1)\right)^{\frac{\kappa-1}{\kappa}} \left[\left(\left|\mathcal{L}\left(\mathcal{G}(\lambda_1^{2-q},\lambda_q^q),\mathcal{G}(\lambda_1,\lambda_2)\right)\right|^{\kappa} \mathcal{I}_8^q(\mathcal{U}) + \lambda_1^{\kappa}\mathcal{I}_4^q(\mathcal{U})\right)^{\frac{1}{\kappa}}\right. \\
&\quad + \left.\left.\left(\left|\mathcal{L}\left(\mathcal{G}(\lambda_1^q,\lambda_2^{2-q}),\mathcal{G}(\lambda_1,\lambda_2)\right)\right|^{\kappa} \mathcal{I}_8^q(\mathcal{U}) + \lambda_2^{\kappa}\mathcal{I}_4^q(\mathcal{U})\right)^{\frac{1}{\kappa}}\right]\right\},
\end{aligned} \tag{77}$$

where

$$\mathcal{D}_3(\Theta_1,\Theta_2;q) := \frac{1-q}{2}\sum_{\delta=0}^{\infty} q^{\delta}\left[\lambda_2^{\frac{q^\delta}{2}}\lambda_1^{\left(1-\frac{q^\delta}{2}\right)} + \lambda_1^{\frac{q^\delta}{2}}\lambda_2^{\left(1-\frac{q^\delta}{2}\right)}\right],$$

and

$$e^{\Theta_1} = \lambda_1, \quad e^{\Theta_2} = \lambda_2.$$

Proof. Let consider Theorem 3.11 with $U_1 = U_2 = \mathcal{U}$ and $m = 1$. If we choose $f(u) = e^u$, then inequality (62) gives the required inequality (77). □

Corollary 4.8. *Under the conditions of Proposition 4.1, letting $q \to 1^-$, we have the following special cases:*
(a) If we take $\mathcal{U} = 0$, then

$$\begin{aligned}
|\mathcal{G}(\lambda_1,\lambda_2) - \mathcal{L}(\lambda_1,\lambda_2)| \leq \frac{\Theta_2-\Theta_1}{48}&\left\{\frac{2}{\sqrt[\kappa]{2}}\left[\left(\mathcal{G}^{\kappa}(\lambda_1,\lambda_2)+\lambda_1^{\kappa}\right)^{\frac{1}{\kappa}} + \left(\mathcal{G}^{\kappa}(\lambda_1,\lambda_2)+\lambda_2^{\kappa}\right)^{\frac{1}{\kappa}}\right]\right. \\
&+ \left.\frac{4}{\sqrt[\kappa]{4}}\left[\left(3\mathcal{G}^{\kappa}(\lambda_1,\lambda_2)+\lambda_1^{\kappa}\right)^{\frac{1}{\kappa}} + \left(3\mathcal{G}^{\kappa}(\lambda_1,\lambda_2)+\lambda_2^{\kappa}\right)^{\frac{1}{\kappa}}\right]\right\}.
\end{aligned} \tag{78}$$

(b) If we choose $\mathcal{U} = 1$, then

$$\begin{aligned}
|\mathcal{A}(\lambda_1,\lambda_2) - \mathcal{L}(\lambda_1,\lambda_2)| \leq \frac{\Theta_2-\Theta_1}{48}&\left\{\frac{4}{\sqrt[\kappa]{4}}\left[\left(\mathcal{G}^{\kappa}(\lambda_1,\lambda_2)+3\lambda_1^{\kappa}\right)^{\frac{1}{\kappa}} + \left(\mathcal{G}^{\kappa}(\lambda_1,\lambda_2)+3\lambda_2^{\kappa}\right)^{\frac{1}{\kappa}}\right]\right. \\
&+ \left.\frac{2}{\sqrt[\kappa]{2}}\left[\left(\mathcal{G}^{\kappa}(\lambda_1,\lambda_2)+\lambda_1^{\kappa}\right)^{\frac{1}{\kappa}} + \left(\mathcal{G}^{\kappa}(\lambda_1,\lambda_2)+\lambda_2^{\kappa}\right)^{\frac{1}{\kappa}}\right]\right\}.
\end{aligned} \tag{79}$$

Proposition 4.9. *Let* $0 < \Theta_1 < \Theta_2$ *and* $U_1, U_2 \in \mathbb{R}$. *Then for* $\kappa \geq 1$, *we have*

$$\left|\mathcal{A}(U_1\lambda_1, U_2\lambda_2) + \left(1 - \frac{U_1 + U_2}{2}\right)\mathcal{G}(\lambda_1, \lambda_2) - \mathcal{L}(\Theta_1, \Theta_2)\right|$$

$$\leq \begin{cases} \frac{\Theta_2-\Theta_1}{96}\left\{\frac{4(1-3U_1)}{\sqrt[\kappa]{4(1-3U_1)}}\left((2-4U_1)\mathcal{G}^\kappa(\lambda_1,\lambda_2) + (2-8U_1)\lambda_1^\kappa\right)^{\frac{1}{\kappa}}\right. \\ + \frac{4(2-3U_1)}{\sqrt[\kappa]{4(2-3U_1)}}\left(2(3-4U_1)\mathcal{G}^\kappa(\lambda_1,\lambda_2) + (2-4U_1)\lambda_1^\kappa\right)^{\frac{1}{\kappa}} \\ + \frac{4(1-3U_2+6U_2^2-2U_2^3)}{\sqrt[\kappa]{4(1-3U_2+6U_2^2-2U_2^3)}}\left((2-4U_2+8U_2^3-4U_2^4)\mathcal{G}^\kappa(\lambda_1,\lambda_2)\right. \\ \left. + (2-8U_2+24U_2^2-16U_2^3+4U_2^4)\lambda_2^\kappa\right)^{\frac{1}{\kappa}} + \frac{4(2-3U_2+2U_2^3)}{\sqrt[\kappa]{4(2-3U_2+2U_2^3)}} \\ \left.\times\left(2(3-4U_2+2U_2^4)\mathcal{G}^\kappa(\lambda_1,\lambda_2) + (2-4U_2+8U_2^3-4U_2^4)\lambda_2^\kappa\right)^{\frac{1}{\kappa}}\right\}, & \text{if } U_1 \leq 0 < U_2 < 1, \\ \frac{\Theta_2-\Theta_1}{96}\left\{\frac{4(1-3U_1+6U_1^2-2U_1^3)}{\sqrt[\kappa]{4(1-3U_1+6U_1^2-2U_1^3)}}\left((2-4U_1+8U_1^3-4U_1^4)\mathcal{G}^\kappa(\lambda_1,\lambda_2)\right.\right. \\ \left. + (2-8U_1+24U_1^2-16U_1^3+4U_1^4)\lambda_2^\kappa\right)^{\frac{1}{\kappa}} + \frac{4(2-3U_1+2U_1^3)}{\sqrt[\kappa]{4(2-3U_1+2U_1^3)}} \\ \times\left(2(3-4U_1+2U_1^4)\mathcal{G}^\kappa(\lambda_1,\lambda_2) + m(2-4U_1+8U_1^3-4U_1^4)\lambda_2^\kappa\right)^{\frac{1}{\kappa}} \\ + \frac{4(3U_2-1)}{\sqrt[\kappa]{4(3U_2-1)}}\left((4U_2-2)\mathcal{G}^\kappa(\lambda_1,\lambda_2) + (8U_2-2)\lambda_1^\kappa\right)^{\frac{1}{\kappa}} \\ \left. + \frac{4(3U_2-2)}{\sqrt[\kappa]{4(3U_2-2)}}\left(2(4U_2-3)\mathcal{G}^\kappa(\lambda_1,\lambda_2) + (4U_2-2)\lambda_1^\kappa\right)^{\frac{1}{\kappa}}\right\}, & \text{if } 0 < U_1 < 1 \leq U_2. \end{cases}$$

Proof. If we take $f(u) = u^{-1}$ in Theorem 3.11 with $m = 1$ and letting $q \to 1^-$, we get the desired result. □

5. Conclusion

The current investigation brings into the spotlight some very general inequalities of Simpson type which also provide the error bounds of Hermite–Hadamard inequalities as special case. The first finding is about the establishment of a new multi-parameters identity comprises of the left and right quantum integrals and quantum derivatives. The identity is proved to be very rich in its nature as a family of new inequalities of Simpson type emerge as outcome. To validate the generalized nature of our results, some of the interesting cases have been discussed in details. An example is also given for the justification of our main results. Finally, several applications are obtained about the special means of different positive real numbers. Many other new applications can be developed in different fields but we omit here. Interested readers can use our results in order to see the efficiency of them. The most economical feature of this study is that, at a single effort, two analogous inequalities are obtained. We feel that this feature will encourage and inspire the researchers to discuss the same identity for different classes of convex functions. The case of two variables will prove to be more attractive task for the researchers working in the area of quantum and fractional integral inequalities.

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